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Complete Symmetry and μ-Symmetry Analysis of the Kawahara-KdV Type Equation

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Abstract: The goal of this paper is complete analysis of the Kawahara-KdV type equation using the ordinary symmetry and μ-symmetry methods. In other words, the Lie symmetry, symmetry reduction, differential invariant and conservation laws for the Kawahara-KdV type equation are provided. And in the second part the μ-symmetry, order reduced equations, Lagrangian and μ-conservation laws for the Kawahara-KdV type equation are presented.

Keywords: *Lie symmetry; μ-symmetry; Kawahara-KdV type equation; symmetry reduction; differential invariant; conservation law; order reduced equations; Lagrangian; variational problem; μ-conservation law.*

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1 Introduction

The symmetry method is a powerful tool of differential geometry for accurate analysis of a mathematical model as a description of a system in many areas of applied mathematics and physics. Dispersive wave equations arise in many areas when the third order derivative in the KdV (Korteweg de Vries) equation approaches zero. It is necessary to take account of the higher order effect of dispersion in order to balance the nonlinear effect.

The Kawahara-KdV equation, modified Kawahara-KdV equation and Kawahara-KdV type equation, respectively, are given as:

$$u_t + uu_x + u_{xxx} - \gamma_1 u_{xxxxx} = 0, \quad u_t + 3u^2 u_x + u_{xxx} - \gamma_2 u_{xxxxx} = 0,$$
$$u_t + u_x + uu_x + u_{xxx} - \gamma u_{xxxxx} = 0, \tag{1}$$

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where $\gamma, \gamma_1, \gamma_2 \in \mathbb{R}^+$. When the cubic KdV type equation is weak, a lot of physical phenomena are described by the Kawahara-KdV type equations [6]. Especially, the Kawahara-KdV type equation as a specific form of the Benney-Lin equation describes the one-dimensional evolution problems. The λ-symmetries method is a special method for order reduction of ODEs. In 2004, Gaeta and Morando developed this method to a μ-symmetries method for PDEs, where $\mu = \lambda_i dx^i$ is a horizontal one-form on first order jet space $(J^{(1)}M, \pi, M)$ and also μ is a compatible. The concepts of variational problem and conservation law and their relationship with λ-symmetries of ODEs were presented by Muriel, Romero and Olver (2006). More precisely, they have extended the formulation of Nother's theorem for λ-symmetry of ODEs. Continuing this trend, in 2007, Cicogna and Gaeta generalized the results obtained by Muriel, Romero and Olver in the case of λ-symmetries for ODEs to the case of μ-symmetries for PDEs.

The outline of this paper is as follows. Section 2 is devoted to the Lie symmetry analysis, reduction and differential invariant of equations (1). We will find all conservation laws for equations (1) in Section 3. In Section 4, we compute the μ-symmetry and order reduction of equations (1). Section 5 deals with the Lagrangian of equations (1) in potential form. Finally, in the last section, μ-conservation laws of equations (1) are obtained.

2 Lie Symmetry Analysis, Reduction and Differential Invariant of the Kawahara-KdV Type Equation

The symmetry group of a system of differential equations is the largest local group of transformations acting on the independent and dependent variables of the system with the property that it transforms solutions of the system to other solutions [8].

First of all, we obtain the vector fields of equations (1) as follows: $\mathbf{v}_1 = \partial_x(space\,translation)$, $\mathbf{v}_2 = -\partial_t(time\,translation)$, $\mathbf{v}_3 = t\partial_x + \partial_u(Galilean\,boost)$. The commutation relations between vector fields is given by Table 2.

$[\mathbf{v}_i, \mathbf{v}_j]$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_1$	0	0	0
$\mathbf{v}_2$	0	0	$-\mathbf{v}_1$
$\mathbf{v}_3$	0	$\mathbf{v}_1$	0

Table 1: The commutator table of equations (1)

Note that the Lie algebra g is solvable , because $\mathrm{g}^{''} = [\mathrm{g}', \mathrm{g}'] = 0 \subset \mathrm{g}' = [\mathrm{g}, \mathrm{g}] =< \mathbf{v}_1 >\subset \mathrm{g}$. The one-parameter groups $G_1 : (x + \epsilon, t, u)$, $G_2 : (x, t - \epsilon, u)$ and $G_3 : (\epsilon t + x, t, u + \epsilon)$ are generated by $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, respectively, so that the entries give the transformed point $exp(\epsilon \mathbf{v}_i)(x, t, u) = (\tilde{x}, \tilde{t}, \tilde{u})$. Since each group G_i is a symmetry group, this fact implies that if $u = f(x, t)$ is a solution of equations (1), so are the functions $u_1 = f(x - \epsilon, t)$, $u_2 = f(x, t + \epsilon)$ and $u_3 = f(x - \epsilon t, t) + \epsilon$.

For better cognition, we now try to classify the infinite set of solutions of equations (1). This is, in fact, the categorized orbits of the influence of groups. In general, for each $s-$parameter subgroup H of G, there is a family of group-invariant solutions ($s \leq p$) and it is not usually feasible to list all solutions via this method, because there are infinite number of $s-$parameter subgroups. Now we classify them according to the conjugacy property, and this is an effective method to find an optimal system of subgroup in terms of conjugacy in equivalent. This matter is equivalent to finding an optimal system of subalgebras, a list of subalgebras with the property that any other subalgebra is conjugate to one subalgebra in that list. Table 2 shows adjoint representation to compute.

$Ad(\exp(\varepsilon \mathbf{v}_i)\mathbf{v}_j)$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_1$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_2$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3 + \varepsilon \mathbf{v}_1$
$\mathbf{v}_3$	$\mathbf{v}_1$	$\mathbf{v}_2 - \varepsilon \mathbf{v}_1$	$\mathbf{v}_3$

Table 2: Adjoint representation table of equations (1)

Theorem 2.1 *An optimal system of one-dimensional Lie algebras of equations (1) is provided by* $a_2\mathbf{v}_2 + \mathbf{v}_3$ *and* $a_1\mathbf{v}_2$.

Proof. The adjoint representation was determined in Table 2, and the matrices M_i^ε of F_i^ε, $i = 1, 2, 3$, with respect to basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are

$$M_1^\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M_2^\varepsilon = \begin{pmatrix} 1 & 0 & -\varepsilon \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M_3^\varepsilon = \begin{pmatrix} 1 & \varepsilon & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we will make coefficients a_i as simple as possible, by acting these matrices on a vector field $\mathbf{V}$ alternatively. First, suppose that $a_3 \neq 0$, so we can assume that $a_3 = 1$, and by M_1^ε or M_2^ε, the coefficients of $\mathbf{v}_1$ vanish and $\mathbf{V}$ reduces to case 1. The second mode will be the same. □

Assume G acts projectably on M and Δ is a system of differential equation defined in it. By using the Lie-group method the number of independent variables can be reduced and the reduced system of differential equation is in quotient manifold M/G. If s denotes the dimension of the orbit of G, then there are precisely $(p - s)$ invariants which depend on x and play the role of independent variables $y = (y^1, ..., y^{p-s)})$ [7].

Now by integrating the characteristic equation, the invariants will be calculated. All results are coming in Table 2. In the following, differential invariants are computed. Let

operator	y	v	u	reduced equations
$\mathbf{v}_1$	t	u	$v(y)$	$v_y = 0$
$\alpha\, \mathbf{v}_2$	x	u	$v(y)$	$v_y + v\, v_y + v_{y^3} - \gamma\, v_{y^5} = 0$
$\mathbf{v}_3$	t	$x - t\, u$	$\frac{1}{t}(x - v(y))$	$1 - v_y = 0$
$\alpha\, \mathbf{v}_2 + \mathbf{v}_3$	$t^2 + 2\alpha\, x$	$t + \alpha\, u$	$\frac{1}{\alpha}(v(y) - t)$	$-1 + \alpha\, v_y + v\, v_y + \alpha^3 v_{y^3} - \alpha^5 v_{y^5} = 0$

Table 3: Reduction of equations (1).

us remind, if G is a symmetry group for a system with functionally differential invariants, then the system can be rewritten in terms of these invariants. Table 2 shows differential invariants of the equation (1) up to order 3.

vector field	up to the 3-rd order
$\mathbf{v}_1$	$t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{xxt}, u_{xtt}, u_{ttt}$
$\mathbf{v}_2$	$x, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{xxt}, u_{xtt}, u_{ttt}$
$\mathbf{v}_3$	$t,\ \frac{-x}{t}+u,\ \frac{x}{t}u_x+u_t,\ \frac{x}{t}u_{xx}+u_{xt},\ \frac{x^2}{t^2}u_{xx}+\frac{2x}{t^2}(xu_{xx}+tu_{xt})+u_{tt}, u_{xxx}, u_{xxt}, u_{xtt}, u_{ttt}$

Table 4: Differential invariants of invariant(1).

3 Conservation Laws for the Kawahara-KdV Type Equation

Suppose that the Kawahara-KdV type equation is an isolated system, a particular measurable property of this system is called a conservation law which does not change as the system evolves over time. Consider $\Phi = (\Phi^1(x, u^{(n)}), ..., \Phi^p(x, u^{(n)}))$ is a p-tuple of smooth functions on $J^{(n)}M$. In characteristic form, a local conservation law is

$$Div\Phi = D_1\Phi^1(x, u^{(n)}) + ... + D_n\Phi^n(x, u^{(n)}) = Q.\Delta, \quad Q = (Q_1, ..., Q_L),$$

where $\Phi^i s$ and Q are the fluxes and characteristics of the conservation law. In this section, the conservation law is calculated by the multiplier method and also remind the Euler operator with respect to U^j is $E_{U^j} = \frac{\partial}{\partial U^j} - D_i\frac{\partial}{\partial U^j} + \cdots + (-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial U^j_{i_1\cdots i_s}} + \cdots$.

The next theorem shows that the range of Div is a subset of the Euler operator's kernel.

Theorem 3.1 *The equations $E_{U^j}F(x, U, \partial_U, \cdots, \partial_U^s) \equiv 0$, $j = 1, \cdots, q$ hold for arbitrary $U(x)$ if and only if $F(x, U, \partial_U, \cdots, \partial_U^s)$ is in the range of Div [7, 8].*

Theorem 3.2 *The set of equations $E_{U^j}(\Lambda_\nu(x, U, \partial_U, \cdots, \partial_U^r)\Delta_\nu(x, u^{(n)})) \equiv 0$, $j = 1, \cdots q$, holds for arbitrary functions $U(x)$, if and only if the set $\{\Lambda_\nu(x, U, \partial_U, \cdots, \partial_U^r)\}_{\nu=1}^l$ yields a local conservation law for the system [7, 8].*

Now, to find all local conservation law multipliers of the form $\Lambda = \xi(x, t, u)$, we have

$$E_U[\xi(x, t, U)(U_t + U_x + UU_x + U_{xxx} - \gamma U_{xxxxx})] \equiv 0,$$

where $U(x, t)$ are arbitrary functions. The solution of the determining system is $\xi = 1,\ U,\ t + tU - x$. In other words, $D_t\Psi + D_x\Phi \equiv \xi(U_t + U_x + UU_x + U_{xxx} - \gamma U_{xxxxx})$, that is, (Ψ, Φ) determines a nontrivial local conservation law of the system. Further, (Ψ, Φ) are calculated by using the homotopy operator and all results are shown in Table 3.

4 μ-Symmetry and Order Reduction for Kawahara-KdV Type Equation

Let D_i be a total derivative up to x^i, $\lambda_i : J^{(1)}M \longrightarrow \mathbb{R}$ and $\mu = \lambda_i dx^i$ be a horizontal one-form on first order jet space $(J^{(1)}M, \pi, M)$ and compatible, i.e. $D_i\lambda_j - D_j\lambda_i = 0$. Suppose $\Delta(x, u^{(k)}) = 0$ is a scalar PDE, involving p independent variables $x = (x^1, ..., x^p)$ and one dependent variable $u = u(x^1, ..., x^p)$ of order k.

Let $X = \sum_{i=1}^p \xi^i\partial_{x^i} + \varphi\partial_u$ be a vector field on M, $Y = X + \sum_{J=1}^k \Psi_J\,\partial_{u_J}$ be the μ-prolongation of X on jet space J^kM if $\Psi_{J,i} = (D_i+\lambda_i)\Psi_J - u_{J,m}(D_i+\lambda_i)\xi^m$, $(\Psi_0 = \varphi)$. Suppose $\mathcal{S}_\Delta \subset J^kM$ is a solution manifold for $\Delta = 0$ and $Y : \mathcal{S} \longrightarrow T\mathcal{S}$, then X is said to be a μ-symmetry for Δ. Generally, in this thread if $\mu = 0$, ordinary prolongation

$\xi(x,t,u)$	Ψ	Φ
1	$\Psi = u + \frac{1}{2}u^2 + u_{x^2} - \gamma u_{x^4}$	$\Phi = u$
u	$\Psi = \frac{1}{2}u^2 + \frac{1}{3}u^3 + uu_{x^2} - \frac{1}{2}u_x u_x$ $+\gamma(-uu_{x^4} + u_x u_{x^3} + \frac{1}{2}u_{x^2}u_{x^3})$	$\Phi = \frac{1}{2}u^2$
$t + tu - x$	$\Psi = tu + tu^2 - xu - \frac{x}{2}u^2 + \frac{t}{3}u^3 + u_x$ $+tu_{x^2}(1+u) - xu_{x^2} - \gamma u_{x^4}(tu + t - x)$ $+\gamma t u_x u_{x^3} - \gamma \frac{t}{2}u_{x^2}u_{x^2} - \gamma u_{x^3} - \frac{t}{2}u_x u_x$	$\Phi = tu + \frac{t}{2}u^2 - xu$

Table 5: Conservation laws for equations (1).

and ordinary symmetry is going to happen. Suppose $\mu = \lambda_i dx^i$ is a horizontal 1-form and compatible on $\mathcal{S}_\Delta$ and X is a vector field on M, then the exponential vector field $V = \exp\big(\int \mu\big)X$ is a general symmetry for Δ if and only if X is a μ-symmetry for Δ.

Theorem 4.1 *Let Δ be a scalar PDE of order k for $u = u(x^1, ..., x^p)$, $X = \xi^i(\frac{\partial}{\partial x^i}) + \varphi(\frac{\partial}{\partial u})$ be a vector field on M, with characteristic $Q = \varphi - u_i\xi^i$ and Y be the μ-prolongation of order k of X. If X is a μ-symmetry for Δ, then $Y : \mathcal{S}_X \longrightarrow T\mathcal{S}_X$, where $\mathcal{S}_X \subset J^{(k)}M$ is the solution manifold for the system Δ_X made of Δ and of $E_J := D_J Q = 0$ for all J with $\mid J \mid = 0, 1, ..., k-1$. [4]*

The process of calculating μ-symmetries of a given equation $\Delta = 0$ of order n is similar to that for the ordinary symmetries. Generally, if X is a generic vector field acting in M, then its μ-prolongation Y of order n for a generic $\mu = \lambda_i dx^i$, acting in $J^{(n)}M$ and applying Y to Δ and the obtained expression to $\mathcal{S}_\Delta \subset J^{(n)}M$, the result will be Δ_* up to ξ, τ, φ and λ_i. If we require λ_i to be functions on $J^{(k)}M$, all the dependences on u_J will be explicit, and one obtains a system of determining equations. This system should be complemented with the compatibility conditions between the λ_i. If we determine a priori the form μ, we are left with a system of linear equation for ξ, τ, φ; similarly, if we fix a vector field X and try to find the μ for which it is a μ-symmetry of the given equation Δ, we have a system of quasilinear equations for the λ_i [4].

To continue the μ-symmetry analysis of equations (1), let $\mu = \lambda_1 dx + \lambda_2 dt$ be a horizontal one-form and with the compatibility condition $D_t\lambda_1 = D_x\lambda_2$ when $\Delta = 0$. Suppose $X = \xi\partial_x + \tau\partial_t + \varphi\partial_u$ is a vector field on M, in order to compute μ-prolongation of order 5 of X, we have $Y = X + \Psi^x\partial_{u_x} + \Psi^t\partial_{u_t} + \Psi^{xx}\partial_{u_{xx}} + ... + \Psi^{ttttt}\partial_{u_{ttttt}}$, where coefficients Y are as follows:

$$\begin{aligned}
\Psi^x &= (D_x + \lambda_1)\varphi - u_x(D_x + \lambda_1)\xi - u_t(D_x + \lambda_1)\tau\,, \\
\Psi^t &= (D_t + \lambda_2)\varphi - u_x(D_t + \lambda_2)\xi - u_t(D_t + \lambda_2)\tau\,, ...
\end{aligned}$$

By applying Y to equations (1) and substituting $(1/\gamma)(u_t + u_x + uu_x + u_{xxx})$ for u_{xxxxx}, we obtain the following system:

$$\gamma\tau_{uuuuu} = 0, \quad \gamma\xi_{uuuuu} = 0, \quad 5\gamma\tau_u = 0, \quad ... \quad , 10\gamma(3\tau_{xu} + \tau\lambda_{1u} + 3\tau_u\lambda_1) = 0. \qquad (2)$$

For any choice of the type $\lambda_1 = D_x[f(x,t)] + g(x)$, $\lambda_2 = D_t[f(x,t)] + h(t)$, where $f(x,t)$, $g(x)$ and $h(t)$ are arbitrary functions and the functions λ_1 and λ_2 satisfy the compatibility condition. For instance, two cases studied to obtain the μ-symmetry and order reduction of equations (1) are as follows:

i) When $g(x) = 0$ and $h(t) = 0$, then by substituting the functions $\lambda_1 = D_x f(x,t)$ and $\lambda_2 = D_t f(x,t)$ into the system of (2) and solving that system, we deduce $\xi = (c_1 t + c_2)F(x,t), \tau = F(x,t)$ and $\varphi = c_1 F(x,t)$, where $f(x,t) = -\ln(F(x,t))$ and $F(x,t)$ is an arbitrary positive function and c_1 and c_2 are arbitrary constants. Then $X = \Big((c_1 t + c_2)\partial_x + \partial_t + c_1\partial_u\Big)F(x,t)$ is a μ-symmetry of equations (1) and corresponds to an ordinary symmetry $V = \exp\Big(\int D_x f(x,t)dx + D_t f(x,t)dt\Big)X$ of exponential type and order reduction of equations (1) is $Q = \varphi - \xi u_x - \tau u_t = \Big(c_1 - (c_1 t + c_2)u_x - u_t\Big)F(x,t)$.

ii) When $g(x) = 0$ and $h(t) = 1/(t + c_1)$, where c_1 is an arbitrary constant, then by substituting the functions $\lambda_1 = D_x f(x,t)$ and $\lambda_2 = D_t f(x,t) + 1/(t + c_1)$ into the system of (2) and solving them, we deduce $\xi = F(x,t), \tau = 0$, and $\varphi = 1/(t + c_1)\, F(x,t)$ where $f(x,t) = -\ln(F(x,t))$ and $F(x,t)$ is an arbitrary positive function. Then $X = \Big(\partial_x + 1/(t + c_1)\,\partial_u\Big)F(x,t)$ is a μ-symmetry of equations (1) and corresponds to an ordinary symmetry $V = \exp\Big(\int D_x f(x,t)dx + (D_t f(x,t) + 1/(t + c_1))dt\Big)X$ of exponential type. In this case reduction of equations (1) is $Q = \varphi - \xi u_x - \tau u_t = \Big(\frac{1}{t+c_1} - u_x\Big)F(x,t)$.

5 Lagrangian of the Kawahara-KdV Type Equation in Potential Form

In this section, we show that equations (1) do not admit a variational problem since they are of odd order, but equations (1) in potential form admit a variational problem.

Theorem 5.1 *Let $\Delta = 0$ be a system of differential equation. Then Δ is the Euler-Lagrange expression for some variational problem $\mathfrak{L} = \int L dx$, i.e. $\Delta = E(L)$ if and only if the Frechet derivative D_Δ is self-adjoint: $D^*_\Delta = D_\Delta$ [8].*

In this case, a Lagrangian for Δ can be explicitly constructed using the homotopy formula $L[u] = \int_0^1 u.\Delta[\lambda u]d\lambda$ and the Frechet derivative of $\Delta_{KK_u} : u_t + u_x + uu_x + u_{xxx} - \gamma u_{xxxxx} = 0$ is $D_{\Delta_{KK_u}} = D_t + (1 + u)D_x + D_x^3 - \gamma D_x^5 + u_x$. Obviously, it does not admit a variational problem since $D^*_{\Delta_{KK_u}} \neq D_{\Delta_{KK_u}}$. But the well-known differential substitution $u = v_x$ yields the related transformed Kawahara-KdV type equation as $\Delta_{KK_v} : v_{xt} + v_{xx} + v_x v_{xx} + v_{xxxx} - \gamma v_{xxxxxx} = 0$, that is called "the Kawahara-KdV type equation in potential form" and its Frechet derivative is $D_{\Delta_{KK_v}} = D_x D_t + v_{xx} D_x + (1 + v_x)D_x^2 + D_x^4 - \gamma D_x^6$, which is self-adjoint, i.e. $D^*_{\Delta_{KK_v}} = D_{\Delta_{KK_v}}$ and has a Lagrangian of the form

$$L[v] = \int_0^1 v.\Delta_{KK_v}[\lambda v]d\lambda = -\frac{1}{2}\Big(v_x v_t + v_x^2 + \frac{1}{3}v_x^3 - v_{xx}^2 + \gamma v_{xxx}^2\Big) + \mathrm{Div}P.$$

Hence, the Lagrangian of the Kawahara-KdV type equation in potential form Δ_{KK_v}, up to Div-equivalence is

$$\mathcal{L}_{\Delta_{KK_v}}[v] = -\frac{1}{2}\Big(v_x v_t + v_x^2 + \frac{1}{3}v_x^3 - v_{xx}^2 + \gamma v_{xxx}^2\Big)\,. \tag{3}$$

6 μ-Conservation Laws of the Kawahara-KdV Type Equation

A conservation law is a relation $\mathrm{Div}\,\mathbf{P} := \sum_{i=1}^{p} D_i P^i = 0$, where $\mathbf{P} = (P^1, \cdots, P^p)$ is a $p-$dimensional vector. Let $\mu = \lambda_i dx^i$ be a horizontal one-form and $D_i\lambda_j = D_j\lambda_i$.

A μ-conservation law is a relation as $(D_i+\lambda_i)P^i=0,$ where P^i is a vector and the $M-$vector P^i is called a μ-conserved vector.

Theorem 6.1 *Consider the $n-th$ order Lagrangian $\mathcal{L}=\mathcal{L}(x,u^{(n)})$ and the vector field X, then X is a μ-symmetry for $\mathcal{L}$, i.e. $Y[\mathcal{L}]=0$ if and only if there exists a $M-$vector P^i satisfying the μ-conservation law $(D_i+\lambda_i)P^i=0$.*

Suppose $\mathcal{L}=\mathcal{L}(x,t,u,u_x,...,u_t)$ is the first order Lagrangian and the vector field $X=\varphi\,(\partial/\partial u)$ is a μ-symmetry for $\mathcal{L}$, then the $M-$vector $P^i:=\varphi\,(\partial\mathcal{L}/\partial u_i)$ is a μ-conserved vector. Also, suppose $\mathcal{L}=\mathcal{L}(x,t,u,u_x,...,u_{tt})$ is the second order Lagrangian and the vector field $X=\varphi\,(\partial/\partial u)$ is a μ-symmetry for $\mathcal{L}$, then the $M-$vector $P^i:=\varphi(\partial\mathcal{L}/\partial u_i)+[(D_j+\lambda_j)\varphi](\partial\mathcal{L}/\partial u_{ij})-\varphi D_j(\partial\mathcal{L}/\partial u_{ij})$ is a μ-conserved vector. The $M-$vector P^i is obtained for the third order Lagrangian in the following theorem.

Theorem 6.2 *Consider the $3-$rd order Lagrangian $\mathcal{L}=\mathcal{L}(x,t,u,u_x,...,u_{ttt})$ and the vector field X, then $X=\varphi\,(\partial/\partial u)$ is a μ-symmetry for $\mathcal{L}$, i.e. $Y[\mathcal{L}]=0$ if and only if the $M-$vector $P^i:=\varphi\frac{\partial\mathcal{L}}{\partial u_i}+[(D_j+\lambda_j)\varphi]\frac{\partial\mathcal{L}}{\partial u_{ij}}-\varphi D_j\frac{\partial\mathcal{L}}{\partial u_{ij}}-(D_k+\lambda_k)\Big([(D_j+\lambda_j)\varphi]\frac{\partial\mathcal{L}}{\partial u_{jki}}-\varphi D_j\frac{\partial\mathcal{L}}{\partial u_{jki}}\Big)$ satisfies the μ-conservation law $(D_i+\lambda_i)P^i=0$.*

Proof. Let $X=\varphi\,(\partial/\partial u)$ be a μ-symmetry for $\mathcal{L}$, its $3-$rd order μ-prolongation is $Y=\varphi\frac{\partial}{\partial u}+[(D_x+\lambda_1)\varphi]\frac{\partial}{\partial u_x}+[(D_t+\lambda_2)\varphi]\frac{\partial}{\partial u_t}+...+[(D_t+\lambda_2)^3\varphi]\frac{\partial}{\partial u_{ttt}}$, then by applying Y on the Lagrangian $\mathcal{L}$, we have

$$
\begin{aligned}
Y[\mathcal{L}]&=\varphi\frac{\partial\mathcal{L}}{\partial u}+[(D_x+\lambda_1)\varphi]\frac{\partial\mathcal{L}}{\partial u_x}+[(D_t+\lambda_2)\varphi]\frac{\partial\mathcal{L}}{\partial u_t}+...+[(D_t+\lambda_2)^3\varphi]\frac{\partial\mathcal{L}}{\partial u_{ttt}}=\varphi\\
&\big(\frac{\partial\mathcal{L}}{\partial u}-D_x\varphi\frac{\partial\mathcal{L}}{\partial u_x}-D_t\varphi\frac{\partial\mathcal{L}}{\partial u_t}+D_x^2\varphi\frac{\partial\mathcal{L}}{\partial u_{xx}}+...-D_t^3\varphi\frac{\partial\mathcal{L}}{\partial u_{ttt}}\big)+(D_x+\lambda_1)\Big[\varphi\frac{\partial\mathcal{L}}{\partial u_x}+[(D_j\\
&+\lambda_j)\varphi]\frac{\partial\mathcal{L}}{\partial u_{xj}}-\varphi D_j\frac{\partial\mathcal{L}}{\partial u_{xj}}-(D_k+\lambda_k)\big([(D_j+\lambda_j)\varphi].\frac{\partial\mathcal{L}}{\partial u_{jkx}}-\varphi D_j\frac{\partial\mathcal{L}}{\partial u_{jkx}}\big)\Big]+(D_t+\lambda_2)\\
&\big[\varphi\frac{\partial\mathcal{L}}{\partial u_t}+[(D_j|+\lambda_j)\varphi]\frac{\partial\mathcal{L}}{\partial u_{tj}}-\varphi D_j\frac{\partial\mathcal{L}}{\partial u_{tj}}-(D_k+\lambda_k)\big([(D_j+\lambda_j)\varphi]\frac{\partial\mathcal{L}}{\partial u_{jkt}}-\varphi D_j\frac{\partial\mathcal{L}}{\partial u_{jkt}}\big)\big].
\end{aligned}
$$

We put $P^i:=\varphi\frac{\partial\mathcal{L}}{\partial u_i}+[(D_j+\lambda_j)\varphi]\frac{\partial\mathcal{L}}{\partial u_{ij}}-\varphi D_j\frac{\partial\mathcal{L}}{\partial u_{ij}}-(D_k+\lambda_k)\Big([(D_j+\lambda_j)\varphi]\frac{\partial\mathcal{L}}{\partial u_{jki}}-\varphi D_j\frac{\partial\mathcal{L}}{\partial u_{jki}}\Big)$. Then $Y[\mathcal{L}]=\varphi E(\mathcal{L})+(D_i+\lambda_i)P^i$, where E is the Euler-Lagrange operator. The Euler-Lagrange equations $E(\mathcal{L})$ vanishes, hence this reduces to $Y[\mathcal{L}]=(D_i+\lambda_i)P^i$. This shows that $Y[\mathcal{L}]=0$ implies $(D_i+\lambda_i)P^i=0$. $\square$

We consider the $3-$rd order Lagrangian (3) for the Kawahara-KdV type equation in potential form $\Delta_{KK_v}=v_{xt}+v_{xx}+v_xv_{xx}+v_{xxxx}-\gamma v_{xxxxxx}=E(\mathcal{L}_{\Delta_{KK_v}})$. Suppose $X=\varphi\partial_v$ is a vector field for $\mathcal{L}_{\Delta_{KK_v}}[v]$. Let $\mu=\lambda_1dx+\lambda_2dt$ be a horizontal one-form with the compatibility condition $D_t\lambda_1=D_x\lambda_2$ when $\Delta_{KK_v}=0$. In order to compute μ-prolongation of order 3 of X, we have $Y=\varphi\partial_v+\Psi^x\partial_{v_x}+\Psi^t\partial_{v_t}+\Psi^{xx}\partial_{v_{xx}}+...+\Psi^{ttt}\partial_{v_{ttt}}$, where coefficients Y are as follows:

$$\Psi^x=(D_x+\lambda_1)\varphi,\ \Psi^t=(D_t+\lambda_2)\varphi,\ \Psi^{xx}=(D_x+\lambda_1)\Psi^x,\ ...,\ \Psi^{ttt}=(D_t+\lambda_2)\Psi^{tt}.$$

Thus, the μ-prolongation Y acts on the $\mathcal{L}_{\Delta_{KKv}}[v]$, and substituting $\Big(v_x^2+\frac{1}{3}v_x^3-v_{xx}^2+\gamma v_{xxx}^2\Big)/-v_x$ for v_t, we obtain the system as follows:

$$\varphi_{vv}=0\,,\quad(-1/6)\varphi_v=0\,,\quad ...,\quad \gamma(\varphi\lambda_{1v}+3\lambda_1\varphi_v+3\varphi_{xv})\ =0. \tag{4}$$

Suppose $\varphi=F(x,t)$, where $F(x,t)$ is an arbitrary positive function satisfying $\mathcal{L}_{\Delta_{KKv}}[v]=0$, then a special solution of the system (4) is given by $\lambda_1=-F_x(x,t)/F(x,t), \lambda_2=-F_t(x,t)/F(x,t)$, where $D_t\lambda_1=D_x\lambda_2$. Hence $X=F(x,t)\partial_v$ is a μ-symmetry for $\mathcal{L}_{\Delta_{KKv}}[v]$, then by Theorem 6.1, there exists a $M-$vector P^i satisfying the μ-conservation law $(D_i+\lambda_i)P^i=0$. Then by Theorem 6.2, the $M-$vector P^i is

$$P^1=-\frac{1}{2}F(x,t)\big(v_t+2v_x+v_x^2+2v_{xxx}-2\gamma v_{xxxxx}\big), P^2=-\frac{1}{2}F(x,t)v_x, \tag{5}$$

and $(D_x+\lambda_1)P^1+(D_t+\lambda_2)P^2=0$ is a μ-conservation law for the 3-rd order Lagrangian $\mathcal{L}_{\Delta_{KK_v}}[v]$. Therefore, the μ-conservation law for equations (1) in potential form $\Delta_{KKv}=E(\mathcal{L}_{\Delta_{KK_v}})$ is $D_xP^1+D_tP^2+\lambda_1P^1+\lambda_2P^2=0$, where P^1 and P^2 are the $M-$vectors P^i of (5).

Remark 6.1 The μ-conservation law for equations (1) in potential form Δ_{KK_v}, satisfies Noether's theorem for μ-symmetry, i.e. $(D_i+\lambda_i)P^i=QE(\mathcal{L}_{\Delta_{KK_v}})$.

We consider the Kawahara-KdV type equation in potential form $\Delta_{KK_v}=v_{xt}+v_{xx}+v_xv_{xx}+v_{xxxx}-\gamma v_{xxxxxx}=0$, or equivalently, $D_x(v_t+v_x+\frac{1}{2}v_x^2+v_{xxx}-\gamma v_{xxxxx})=0$, or $v_t+v_x+\frac{1}{2}v_x^2+v_{xxx}-\gamma v_{xxxxx}=f(t)$, where $f(t)$ is an arbitrary function. If we substitute $f(t)-v_x-\frac{1}{2}v_x^2-v_{xxx}+\gamma v_{xxxxx}$ by v_t and substitute u by v_x in the $M-$vector P^i of (5), then we obtain the $M-$vectors

$$P^1=-\frac{1}{2}F(x,t)\big(f(t)+u+\frac{1}{2}u^2+u_{xx}-\gamma u_{xxxx}\big), P^2=-\frac{1}{2}F(x,t)u. \tag{6}$$

Also, the μ-conservation law for equations (1) is $D_xP^1+D_tP^2+\lambda_1P^1+\lambda_2P^2=0$, where P^1 and P^2 are the $M-$vectors P^i of (6).

Remark 6.2 Equations (1) satisfy the characteristic form, i.e. $(D_i+\lambda_i)P^i=(D_x+\lambda_1)P^1+(D_t+\lambda_2)P^2=Q\Delta_{KK_u}$.

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