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Comparison of New Iterative Method and Natural Homotopy Perturbation Method for Solving Nonlinear Time-Fractional Wave-Like Equations with Variable Coefficients

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Abstract: In this paper, we present a comparison between the new iterative method (NIM) and the natural homotopy perturbation method (NHPM) for solving nonlinear time-fractional wave-like equations with variable coefficients. The two methods introduced an efficient tool for solving this type of equations. The results show that the NIM has an advantage over the NHPM because it takes less time and uses only the inverse operator to solve the nonlinear problems and there is no need to use any other inverse transform as in the case of NHPM. Numerical examples are presented to illustrate the efficiency and accuracy of the proposed methods.

Keywords: *nonlinear time-fractional wave-like equations, Caputo fractional derivative, new iterative method, natural homotopy perturbation method.*

Mathematics Subject Classification (2010): Primary 35L05, 35R11; Secondary 35A35, 26A33.

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1 Introduction

The fractional calculus which deals with derivatives and integrals of arbitrary orders plays a vital role in many fields of applied science and engineering [4]. Recently, nonlinear fractional partial differential equations are successfully applied to many mathematical models in mathematical biology, aerodynamics, rheology, diffusion, electrostatics, electrodynamics, control theory, fluid mechanics, analytical chemistry and so on.

Several analytical and numerical methods have been proposed to solve nonlinear fractional partial differential equations. The most commonly used ones are: the adomian decomposition method (ADM) [8] variational iteration method (VIM) [10], fractional difference method (FDM) [4], homotopy perturbation method (HPM) [3].

In this paper, the main objective is to introduce a comparative study of nonlinear time-fractional wave-like equations with variable coefficients by using the new iterative method (NIM) which uses only the inverse operator and the natural homotopy perturbation method (NHPM) which is a coupling of the natural transform and the homotopy perturbation method (HPM) using He's polynomials.

Consider the following nonlinear time-fractional wave-like equations:

$$\begin{aligned} D_t^\alpha v &= \sum_{i,j=1}^{n} F_{1ij}(X,t,v)\frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(v_{x_i}, v_{x_j}) \\ &\quad + \sum_{i=1}^{n} G_{1i}(X,t,v)\frac{\partial^p}{\partial x_i^p} G_{2i}(v_{x_i}) + H(X,t,v) + S(X,t) \end{aligned} \tag{1}$$

with the initial conditions

$$v(X,0) = a_0(X), v_t(X,0) = a_1(X), \tag{2}$$

where D_t^α is the Caputo fractional derivative operator of order α, $1 < \alpha \leq 2$.

Here $X = (x_1, x_2, ..., x_n)$, F_{1ij}, G_{1i} are nonlinear functions of X, t and v, F_{2ij}, G_{2i} are nonlinear functions of derivatives of v with respect to x_i and x_j, respectively. Also H, S are nonlinear functions and k, m, p are integers.

In the classical case, these types of equations are of considerable significance in various fields of applied sciences, mathematical physics, nonlinear hydrodynamics, engineering physics, biophysics, human movement sciences, astrophysics and plasma physics. These equations describe the evolution of erratic motions of small particles that are immersed in fluids, fluctuations of the intensity of laser light, velocity distributions of fluid particles in turbulent flows [7].

2 Basic Definitions

In this section, we give some basic definitions and important properties of fractional calculus theory and natural transform, which will be used in this paper.

Definition 2.1 [4] The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu$, $\mu \geq -1$ is defined as follows:

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi) d\xi, t > 0. \tag{3}$$

Definition 2.2 [4] The Caputo fractional derivative operator of order $n-1 < \alpha \leq n$, $n \in \mathbb{N}$ of a function $f \in C_{-1}^{n}$ is defined as follows:

$$D_t^{\alpha} f(t) = I_t^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} f^{(n)}, t > 0. \tag{4}$$

For the Riemann-Liouville fractional integral and Caputo fractional derivative, we have the following relation:

$$I_t^{\alpha} D_t^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0) \frac{t^k}{k!}, t > 0. \tag{5}$$

Definition 2.3 [1] The natural transform is defined over the set of functions $A = \left\{ f(t)/\exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty) \right\}$ by the following integral:

$$\mathcal{N}^{+}[f(t)] = R^{+}(s,u) = \frac{1}{u} \int_0^{+\infty} e^{-\frac{st}{u}} f(t) dt, s, u \in (0, \infty). \tag{6}$$

Definition 2.4 [6] The natural transform of the Caputo fractional derivative of order $n-1 < \alpha \leq n$, $n \in \mathbb{N}$ is defined as follows:

$$\mathcal{N}^{+}[D_t^{\alpha} f(t)] = R_{\alpha}^{+}(s,u) = \frac{s^{\alpha}}{u^{\alpha}} R^{+}(s,u) - \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{u^{\alpha-k}} f^{(k)}(0^{+}). \tag{7}$$

3 The New Iterative Method (NIM)

In this section, we introduce the new iterative method for solving equations (1) and (2).

Applying the inverse operator I_t^{α} on both sides of equation (1) and using (5), we get

$$\begin{aligned} v(X,t) &= \sum_{k=0}^{n-1} v^{(k)}(X,0) \frac{t^k}{k!} + I_t^{\alpha} \left(\sum_{i,j=1}^{n} F_{1ij}(X,t,v) \frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(v_{x_i}, v_{x_j}) \right. \\ &\quad \left. + \sum_{i=1}^{n} G_{1i}(X,t,v) \frac{\partial^p}{\partial x_i^p} G_{2i}(v_{x_i}) + H(X,t,v) \right) + I_t^{\alpha} \left(S(X,t) \right). \end{aligned} \tag{8}$$

Let

$$\begin{aligned} g(X,t) &= \sum_{k=0}^{n-1} v^{(k)}(X,0) \frac{t^k}{k!} + I_t^{\alpha} \left(S(X,t) \right), \\ N(v(X,t)) &= I_t^{\alpha} \left(\sum_{i,j=1}^{n} F_{1ij}(X,t,v) \frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(v_{x_i}, v_{x_j}) \right. \\ &\quad \left. + \sum_{i=1}^{n} G_{1i}(X,t,v) \frac{\partial^p}{\partial x_i^p} G_{2i}(v_{x_i}) + H(X,t,v) \right). \end{aligned} \tag{9}$$

Thus, (8) can be written in the following form:

$$v(X,t) = g(X,t) + N(v(X,t)), \tag{10}$$

where g is a known function, N is a nonlinear operator of v.

The nonlinear operator N can be decomposed in the same way as in [2].

So, the solution of equation (10) can be written in the following series form:

$$v(X,t) = \sum_{i=0}^{\infty} v_i(X,t) = g(X,t) + N\left(\sum_{i=0}^{\infty} v_i(X,t)\right). \tag{11}$$

4 The Natural Homotopy Perturbation Method (NHPM)

In this section, we describe the application of the natural homotopy perturbation method (NHPM) for equations (1) and (2). First we define

$$\begin{aligned} Nv &= \sum_{i,j=1}^{n} F_{1ij}(X,t,v)\frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(v_{x_i}, v_{x_j}), \\ Mv &= \sum_{i=1}^{n} G_{1i}(X,t,v)\frac{\partial^p}{\partial x_i^p} G_{2i}(v_{x_i}), Kv = H(X,t,v). \end{aligned} \tag{12}$$

Equation (1) is written in the form

$$D_t^\alpha v(X,t) = Nv(X,t) + Mv(X,t) + Kv(X,t) + S(X,t), t > 0. \tag{13}$$

Apply the natural transform on both sides of (13) and use (7), after that, we take the inverse natural transform, we obtain

$$v(X,t) = L(X,t) + \mathcal{N}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathcal{N}^+\left[Nv(X,t) + Mv(X,t) + Kv(X,t)\right]\right), \tag{14}$$

where $L(X,t)$ is a term arising from the source term and the prescribed initial conditions.

Now we apply the homotopy perturbation method and the nonlinear terms can be decomposed in the same way as in [9], we get

$$\begin{aligned} \sum_{n=0}^{\infty} p^n v_n(X,t) &= L(X,t) + p\left[\mathcal{N}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathcal{N}^+\left[\sum_{n=0}^{\infty} p^n H_n(v) + \sum_{n=0}^{\infty} p^n K_n(v)\right.\right.\right. \\ &\left.\left.\left. + \sum_{n=0}^{\infty} p^n J_n(v)\right]\right)\right], \end{aligned} \tag{15}$$

where $H_n(v)$, $K_n(v)$ and $J_n(v)$ are He's polynomials [5].

By using the coefficient of the like powers of p in equation (15), the following approximations are obtained:

$$\begin{aligned} p^0 &: v_0(X,t) = L(X,t), \\ p^1 &: v_1(X,t) = \mathcal{N}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathcal{N}^+\left[H_0(v) + K_0(v) + J_0(v)\right]\right), \\ p^2 &: v_2(X,t) = \mathcal{N}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathcal{N}^+\left[H_1(v) + K_1(v) + J_1(v)\right]\right) \\ &\quad ... \end{aligned} \tag{16}$$

Hence, the solution of equations (1) and (2) is given by

$$v(X,t)=\sum_{n=0}^{\infty}v_n(X,t). \tag{17}$$

5 Illustrative Examples and Numerical Results

Example 5.1 Consider the 2-dimensional nonlinear time-fractional wave-like equation with variable coefficients

$$D_t^{\alpha}v=\frac{\partial^2}{\partial x\partial y}(v_{xx}v_{yy})-\frac{\partial^2}{\partial x\partial y}(xyv_xv_y)-v, t>0, 1<\alpha\leq 2, \tag{18}$$

with the initial conditions

$$v(x,y,0)=e^{xy}, v_t(x,y,0)=e^{xy}, (x,y)\in\mathbb{R}^2. \tag{19}$$

5.1 Application of the NIM

By applying the steps involved in NIM as presented in Section 3 to equations (18) and (19), we have

$$\begin{aligned} v_0 &= (1+t)\,e^{xy}, v_1=-\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)e^{xy},\\ v_2 &= \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)e^{xy}... \end{aligned} \tag{20}$$

So, the solution of equations (18) and (19) is

$$v(x,y,t)=\left(1+t-\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}-...\right)e^{xy}. \tag{21}$$

In the special case, $\alpha=2$, the series (21) has the closed form

$$v(x,y,t)=(\cos t+\sin t)\,e^{xy}. \tag{22}$$

5.2 Application of the NHPM

By applying the steps involved in NHPM as presented in Section 4 to equations (18) and (19), we have

$$\begin{aligned} p^0 &: v_0(x,y,t)=(1+t)e^{xy}, p^1: v_1(x,y,t)=-\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)e^{xy},\\ p^2 &: v_2(x,y,t)=\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)e^{xy}... \end{aligned} \tag{23}$$

Therefore, the solution of equations (18) and (19) can be expressed by

$$v(x,y,t)=\left(1+t-\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}-...\right)e^{xy}. \tag{24}$$

Taking $\alpha=2$ in equation (24), we obtain the exact solution as

$$v(x,y,t)=(\cos t+\sin t)\,e^{xy}. \tag{25}$$

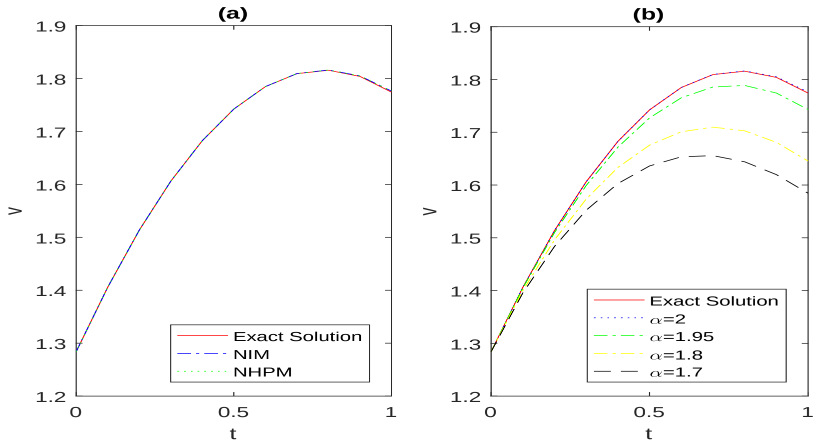

Figure 1: (a) The comparison of the 3−term approximate solution by NIM, NHPM and the exact solution, when $\alpha = 2$ and $x = y = 0.5$, (b) The behavior of the exact solution and the 3−term approximate solution by NIM and NHPM for different values of α when $x = y = 0.5$.

	$\|v_{exact} - v_{NIM}\|$	$\|v_{exact} - v_{NHPM}\|$	$\|v_{exact} - v_{NIM}\|$	$\|v_{exact} - v_{NHPM}\|$
$t/x, y$	0.5	0.5	0.7	0.7
0.1	1.8085×10^{-9}	1.8085×10^{-9}	2.2991×10^{-9}	2.2991×10^{-9}
0.3	1.3536×10^{-6}	1.3536×10^{-6}	1.7208×10^{-6}	1.7208×10^{-6}
0.5	2.9725×10^{-5}	2.9725×10^{-5}	3.7787×10^{-5}	3.7787×10^{-5}
0.7	2.2882×10^{-4}	2.2882×10^{-4}	2.9089×10^{-4}	2.9089×10^{-4}
0.9	1.0547×10^{-3}	1.0547×10^{-3}	1.3407×10^{-3}	1.3407×10^{-3}

Table 1: The absolute errors for differences between the exact solution and 3−term approximate solution by NIM and NHPM for Example 5.1, when $\alpha = 2$.

Example 5.2 Consider the following nonlinear time-fractional wave-like equation with variable coefficients

$$D_t^{\alpha} v = v^2 \frac{\partial^2}{\partial x^2}(v_x v_{xx} v_{xxx}) + v_x^2 \frac{\partial^2}{\partial x^2}(v_{xx}^3) - 18v^5 + v, t > 0, 1 < \alpha \leq 2, \tag{26}$$

with the initial conditions

$$v(x, 0) = e^x, v_t(x, 0) = e^x, x \in \,]0, 1[\,. \tag{27}$$

5.3 Application of the NIM

By applying the steps involved in NIM as presented in Section 3 to equations (26) and (27), we have

$$\begin{aligned} v_0 &= (1+t)\, e^x, v_1 = \left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right) e^x, \\ & v_2 = \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right) e^x ... \end{aligned} \tag{28}$$

So, the solution of equations (26) and (27) is

$$v(x,t) = \left(1 + t + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + ...\right) e^{x}. \tag{29}$$

In the special case, $\alpha = 2$, the series (29) has the closed form

$$v(x,t) = e^{x+t}. \tag{30}$$

5.4 Application of the NHPM

By applying the steps involved in NHPM as presented in Section 4 to equations (26) and (27), we have

$$\begin{aligned} p^{0} &: v_0(x,t) = (1+t)e^{x}, p^{1} : v_1(x,t) = \left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right) e^{x}, \\ p^{2} &: v_2(x,t) = \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right) e^{x} ... \end{aligned} \tag{31}$$

Therefore, the solution of equations (26) and(27) can be expressed by

$$v(x,t) = \left(1 + t + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + ...\right) e^{x}. \tag{32}$$

Taking $\alpha = 2$ in equation (32), we obtain the exact solution as

$$v(x,t) = e^{x+t}. \tag{33}$$

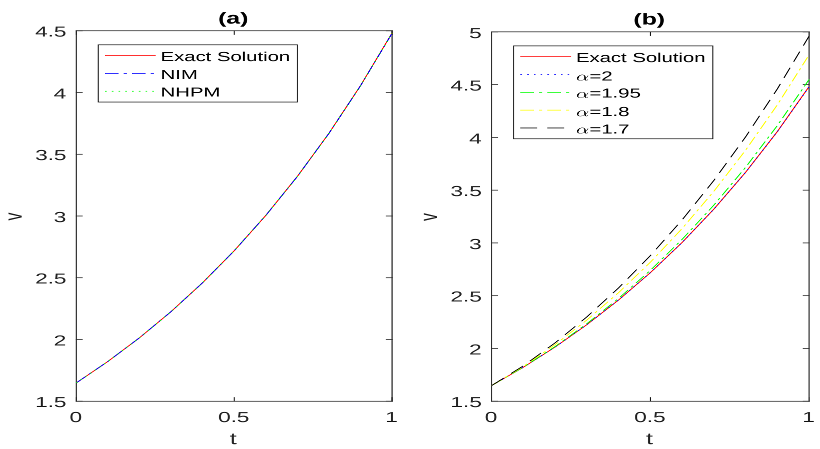

Figure 2: (a) The comparison of the 3−term approximate solution by NIM, NHPM and the exact solution, when $\alpha = 2$ and $x = 0.5$, (b) The behavior of the exact solution and the 3−term approximate solution by NIM and NHPM for different values of α when $x = 0.5$.

	$\|v_{exact} - v_{NIM}\|$	$\|v_{exact} - v_{NHPM}\|$	$\|v_{exact} - v_{NIM}\|$	$\|v_{exact} - v_{NHPM}\|$
t/x	0.5	0.5	0.7	0.7
0.1	2.323×10^{-9}	2.323×10^{-9}	2.8373×10^{-9}	2.8373×10^{-9}
0.3	1.7436×10^{-6}	1.7436×10^{-6}	2.1297×10^{-6}	2.1297×10^{-6}
0.5	3.8504×10^{-5}	3.8504×10^{-5}	4.7029×10^{-5}	4.7029×10^{-5}
0.7	2.9890×10^{-4}	2.9890×10^{-4}	3.6507×10^{-4}	3.6507×10^{-4}
0.9	1.3929×10^{-3}	1.3929×10^{-3}	1.7013×10^{-3}	1.7013×10^{-3}

Table 2: The absolute errors for differences between the exact solution and 3−term approximate solution by NIM and NHPM for Example 5.2, when $\alpha = 2$.

Example 5.3 Consider the following one-dimensional nonlinear time-fractional wave-like equation with variable coefficients

$$D_t^\alpha v = x^2 \frac{\partial}{\partial x}(v_x v_{xx}) - x^2 (v_{xx})^2 - v, t > 0, 1 < \alpha \leq 2, \tag{34}$$

with the initial conditions

$$v(x,0) = 0, v_t(x,0) = x^2, x \in \left]0,1\right[. \tag{35}$$

5.5 Application of the NIM

By applying the steps involved in NIM as presented in Section 3 to equations (34) and (35), we have

$$v_0 = tx^2, v_1 = -\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}x^2, v_2 = \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}x^2 ... \tag{36}$$

So, the solution of equations (34) and (35) is

$$v(x,t) = x^2 \left(t - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - ... \right). \tag{37}$$

In the special case, $\alpha = 2$, the series (37) has the closed form

$$v(x,t) = x^2 \sin t. \tag{38}$$

5.6 Application of the NHPM

By applying the steps involved in NHPM as presented in Section 4 to equations (34) and (35), we have

$$p^0 : v_0(x,t) = tx^2, p^1 : v_1(x,t) = -\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}x^2, p^2 : v_2(x,t) = \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}x^2 ... \tag{39}$$

Therefore, the solution of equations (34) and(35) can be expressed by

$$v(x,t) = x^2 \left(t - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - ... \right). \tag{40}$$

Taking $\alpha = 2$ in equation (40), we obtain the exact solution as

$$v(x,t) = x^2 \sin t. \tag{41}$$

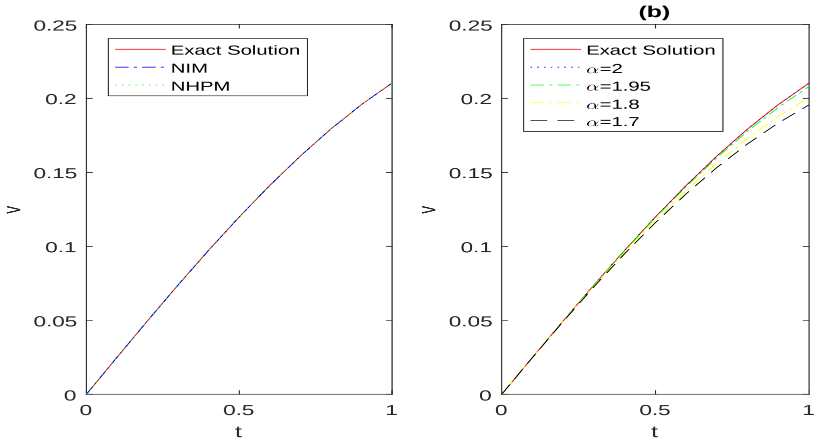

Figure 3: (a) The comparison of the 3−term approximate solution by NIM, NHPM and the exact solution, when $\alpha = 2$ and $x = 0.5$, (b) The behavior of the exact solution and the 3−term approximate solution by NIM and NHPM for different values of α when $x = 0.5$.

	$\lvert v_{exact} - v_{NIM}\rvert$	$\lvert v_{exact} - v_{NHPM}\rvert$	$\lvert v_{exact} - v_{NIM}\rvert$	$\lvert v_{exact} - v_{NHPM}\rvert$
t/x	0.5	0.5	0.7	0.7
0.1	4.9596×10^{-12}	4.9596×10^{-12}	9.7209×10^{-12}	9.7209×10^{-12}
0.3	1.0835×10^{-8}	1.0835×10^{-8}	2.1236×10^{-8}	2.1236×10^{-8}
0.5	3.8618×10^{-7}	3.8618×10^{-7}	7.5692×10^{-7}	7.5692×10^{-7}
0.7	4.0574×10^{-6}	4.0574×10^{-6}	7.9524×10^{-6}	7.9524×10^{-6}
0.9	2.346×10^{-5}	2.346×10^{-5}	4.5982×10^{-5}	4.5982×10^{-5}

Table 3: The absolute errors for differences between the exact solution and 3−term approximate solution by NIM and NHPM for Example 5.3, when $\alpha = 2$.

The numerical results (see Figures 1,2 and 3) affirm that when α approaches 2, our results approach the exact solutions. In Tables 1,2 and 3, the absolute errors obtained by NIM are the same as the results obtained by NHPM.

Remark 5.1 In general, the results obtained show that the method described by NIM is a very simple and easy method compared to the other methods and gives the approximate solution in the form of series, this series in closed form gives the corresponding exact solution of the given problem.

Remark 5.2 In this paper, we only apply three terms to approximate the solutions, if we apply more terms of the approximate solutions, the accuracy of the approximate solutions will be greatly improved.

6 Conclusion

In this paper, we have compared between the new iterative method (NIM) and the natural homotopy perturbation method (NHPM) for solving nonlinear time-fractional wave-like equations with variable coefficients. The two methods are powerful and efficient methods and both give approximations of higher accuracy and closed form solutions, if any. The comparison gives similar results and supplies quantitatively reliable results. It is worth mentioning that the NIM has an advantage over the NHPM because it takes less time and uses only the inverse operator to solve the nonlinear problems and there is no need to use any other inverse transform as in the case of NHPM. The two methods are powerful mathematical tools for solving other nonlinear fractional differential equations.

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