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On the Stabilization of Infinite Dimensional Bilinear Systems with Unbounded Control Operator

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Abstract: In this paper, we deal with regional stabilization of infinite dimensional bilinear system evolving in a spatial domain Ω with unbounded control operator. It consists in studying the asymptotic behaviour of such a system in a subregion ω of Ω. Hence, we give sufficient conditions to obtain weak and strong stabilization on ω. An example and simulations are presented.

Keywords: *infinite dimensional bilinear systems; unbounded control operator; regional stabilization.*

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1 Introduction

Bilinear systems constitute an important subclass of nonlinear systems. The nonlinearity in mathematical models appears in the multiplication of state and control in the dynamical process. Bilinear systems model several phenomena in nature and in industry, e.g. the mass action law in chemistry, the transfer of heat by conduction convection in energetic systems, the generation of cells via cellular division, and the dynamics of the blood's organs in biology [4]. Yet, the modeling may give rise to an unbounded control operator which allows us to describe some interesting phenomena when the control is acting in regions or on a boundary or when the measure is taken at some sensing point. The problem of feedback stabilization of distributed systems has been studied in many works along with various types of controls [1–3, 5].

The question of regional stabilization for linear systems was tackled and developed by Zerrik and Ouzahra [7], and consists in studying the asymptotic behaviour of a distributed

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system only within a subregion ω interior or in the boundary of its evolution domain Ω. The principal reason for introducing this notion is that it makes sense for the usual concept of stabilization taking into account the spacial variable and then it becomes closer to real world problems, where one wishes to stabilize a system on a critical subregion of its geometrical domain. Regional stabilization of bilinear systems with bounded control operator has been considered by Zerrik and Ouzahra [6]. Many approaches were used to characterize different kinds of stabilization, and mainly the control which achieves the stabilization minimizing a given functional cost.

In this paper, we examine regional stabilization of infinite dimensional bilinear systems with unbounded control operator. The paper is organized as follows : in Section 2, we discuss different kinds of regional stabilization, and we give sufficient conditions to achieve weak and strong stabilization of such a system on ω. Finally, an example and simulations are given to illustrate the efficiency of the obtained results.

2 Regional Stabilization

2.1 Considered system and notations

Define an open, regular set $\Omega \subset \mathbb{R}^n$ $(n = 1, 2, ...)$ and consider the bilinear system

$$\begin{cases} \dot{z}(t) = Az(t) + v(t)Bz(t), \\ z(0) = z_0, \end{cases} \tag{1}$$

where $A : \mathcal{D}(A) \subset H \to H$ generates a strongly continuous semigroup of contractions $S(t)_{t\geq 0}$ on a Hilbert space $H := L^2(\Omega)$, endowed with norm and scalar product denoted, respectively, by $\|.\|$ and $\langle .,. \rangle$, $v(.) \in L^2(0, \infty)$ denotes the control function and B is an unbounded linear operator from H into itself, positive and self-adjoint but bounded from a subspace $V \subset H$ to some large space V' such that $H \subset V'$. Identify H with its dual H' so that $V \hookrightarrow H \hookrightarrow V'$, and $\langle h, w\rangle_{V',V} = \langle h, w\rangle$ $\forall h, w \in H$.
We suppose that the state $z(t) \in V$. The solution of system (1) is a solution of the equation

$$z(t) = S(t)z_0 + \int_0^t v(s)S(t-s)Bz(s)ds. \tag{2}$$

Let ω be an open subregion of Ω and Lebesgue non-null measure, $\chi_\omega : L^2(\Omega) \to L^2(\omega)$ the restriction operator to ω, while χ_ω^* denotes the adjoint operator given by

$$\chi_\omega^* y(x) = \begin{cases} y(x), & \text{if } x \in \omega, \\ 0, & \text{if } x \in \Omega \backslash \omega. \end{cases}$$

Denote $i_\omega = \chi_\omega^* \chi_\omega$ and suppose that
(H_1) $\langle i_\omega Ay, y\rangle \leq 0, \ \forall y \in \mathcal{D}(A)$;
(H_2) $By \in H$, $\forall y \in V$;
(H_3) $\langle i_\omega By, y\rangle_{V',V} \geq 0$, $\forall y \in V$.

Definition 2.1 System (1) is said to be
1. weakly stabilizable on ω, if $\chi_\omega z(t)$ tends to 0 weakly, as $t \to \infty$.
2. strongly stabilizable on ω, if $\chi_\omega z(t)$ tends to 0 strongly, as $t \to \infty$.

Remark 2.1 It is clear that we are only interested in the behaviour of system (1) on a subregion ω without constraints on the residual part $\Omega\backslash\omega$, and when $\omega = \Omega$ we retrieve the classical definition of stabilization.

2.2 Regional weak stabilization

The following result gives sufficient conditions for weak stabilization of system (1) on ω.

Theorem 2.1 *Let A generate a semigroup $S(t)$ of contractions on H, the assumptions $(H_1), (H_2)$ and (H_3) hold, B be compact from V to V', and if the condition*

$$\langle i_\omega BS(t)y, S(t)y\rangle_{V',V} = 0,\ \forall t \geq 0 \implies \chi_\omega y = 0, \tag{3}$$

is verified, then the control $v(t) = -\langle i_\omega Bz(t), z(t)\rangle_{V',V}$ weakly stabilizes system (1) on ω.

Proof. From hypothesis (H_1), we have

$$\frac{d}{dt}\|\chi_\omega z(t)\|^2 \leq 2v(t)\langle i_\omega Bz(t), z(t)\rangle_{V',V}. \tag{4}$$

In order to make the energy non increasing, a natural choice for the control is

$$v(t) = -\langle i_\omega Bz(t), z(t)\rangle_{V',V}.$$

Since A generates a semigroup of contractions, we have

$$\|z(t)\|^2 - \|z(0)\|^2 \leq -2\int_0^t \langle i_\omega Bz(s), z(s)\rangle_{V',V}\langle Bz(s), z(s)\rangle_{V',V} ds. \tag{5}$$

Due to (H_3) and the fact that B is positive, it follows that

$$\|z(t)\| \leq \|z_0\|. \tag{6}$$

From (2), (6), and Schwartz's inequality, we deduce

$$\|z(t) - S(t)z_0\| \leq \delta\|z_0\|\sqrt{T\lambda(0)}, \quad \forall t \in [0, T], \tag{7}$$

where $\delta = \|B\|_{\mathcal{L}(V,V')}$ and $\lambda(t) = \displaystyle\int_t^{t+T} |\langle i_\omega Bz(s), z(s)\rangle_{V',V}|^2 ds$.
For all $s \geq 0$, we have

$$\begin{aligned}\langle i_\omega BS(s)z_0, S(s)z_0\rangle_{V',V} &= -\langle i_\omega B(z(s) - S(s)z_0), S(s)z_0)\rangle_{V',V} \\ &- \langle i_\omega Bz(s), (z(s) - S(s)z_0)\rangle_{V',V} + \langle i_\omega Bz(s), z(s)\rangle_{V',V}.\end{aligned}$$

Since χ_ω is continuous, there exists a constant $C > 0$ such that

$$|\langle i_\omega BS(s)z_0, S(s)z_0\rangle_{V',V}| \leq 2C\delta\|z_0\|\|z(s) - S(s)z_0\| + |\langle i_\omega Bz(s), z(s)\rangle_{V',V}|. \tag{8}$$

Now, let $\Gamma(t)z_0 := z(t)$ define a non-linear semigroup of contractions on H and replacing z_0 by $\Gamma(t)z_0$ in (7) and (8), we have

$$\begin{aligned}|\langle i_\omega BS(s)\Gamma(t)z_0, S(s)\Gamma(t)z_0\rangle_{V',V}| &\leq 2C\delta^2\|z_0\|^2\sqrt{T\lambda(t)} \\ &+ |\langle i_\omega B\Gamma(s+t)z_0, \Gamma(s+t)z_0\rangle_{V',V}|, \quad \forall t, s \geq 0.\end{aligned} \tag{9}$$

Integrating (9) over the interval $[0, T]$ and using Schwartz's inequality, we arrive at

$$\int_0^T |\langle i_\omega BS(s)\Gamma(t)z_0, S(s)\Gamma(t)z_0\rangle_{V',V}|ds \leq M\sqrt{\lambda(t)}, \tag{10}$$

where $M = \left(2C\delta^2\|z_0\|^2 T^{\frac{3}{2}} + \sqrt{T}\right)$ is a non-negative constant depending on $\|z_0\|$ and T. By virtue of (4), we have

$$\int_0^{+\infty} |\langle i_\omega B\Gamma(s)z_0, \Gamma(s)z_0\rangle_{V',V}|^2 ds < +\infty.$$

From the Cauchy criterion, we deduce that

$$\lambda(t) \to 0, \ \text{ as } \ t \to \infty. \tag{11}$$

To show that $\chi_\omega z(t) \rightharpoonup 0$, as $t \to +\infty$, let us consider a sequence $(t_n) \subset \mathbb{R}$ such that $t_n \to \infty$.
From (6), there exists a subsequence $(t_{\varphi(n)})$ of (t_n) such that

$$\Gamma(t_{\varphi(n)})z_0 \rightharpoonup y \text{ in } V \text{ ,as } \ n \to \infty.$$

Using the continuity of χ_ω and since B is a compact operator from V to V', we have for all $t \geq 0$ that

$$S(t)\Gamma(t_{\varphi(n)})z_0 \rightharpoonup S(t)y \ \text{ in } V \ \text{ and } \ BS(t)\Gamma(t_{\varphi(n)})z_0 \rightharpoonup BS(t)y \ \text{ in } V', \ \text{ as } n \to \infty.$$

Then

$$\lim_{n\to\infty} \langle i_\omega BS(t)\Gamma(t_{\varphi(n)})z_0, S(t)\Gamma(t_{\varphi(n)}z_0)\rangle_{V',V} = \langle i_\omega BS(t)y, S(t)y\rangle_{V',V}.$$

By the dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_0^T |\langle i_\omega BS(t)\Gamma(t_{\varphi(n)})z_0, S(t)\Gamma(t_{\varphi(n)}z_0)\rangle_{V',V}|dt = \int_0^T |\langle i_\omega BS(t)y, S(t)y)\rangle_{V',V}|dt.$$

From (10) and (11), it follows that

$$\int_0^T |\langle i_\omega BS(t)y, S(t)y)\rangle_{V',V}|dt = 0.$$

and then

$$\langle i_\omega BS(t)y, S(t)y)\rangle_{V',V} = 0, \ \ \forall t \in [0, T].$$

Using (3), we have

$$\chi_\omega\Gamma(t_{\varphi(n)})z_0 \rightharpoonup 0 \ \text{ as } \ n \to +\infty. \tag{12}$$

On the other hand, it is clear that (12) holds for each subsequence $(t_{\phi(n)})$ of (t_n) such that $\chi_\omega\Gamma(t_{\phi(n)})z_0$ weakly converges in $L^2(\omega)$. This shows that for all $\varphi \in L^2(\omega)$, $\langle\chi_\omega\Gamma(t_n)z_0, \varphi\rangle \rightharpoonup 0$, as $n \to +\infty$. Hence $\chi_\omega\Gamma(t)z_0 \rightharpoonup 0$, as $t \to +\infty$, which completes the proof. $\square$

Remark 2.2 In the case $\omega = \Omega$, we retrieve the result established in [2] concerning the weak stabilisation of system (1) on the whole domain Ω.

2.3 Regional strong stabilization

The following result gives sufficient conditions for strong stabilization of system (1) on ω.

Theorem 2.2 *Let A generate a semigroup $S(t)$ of contractions on H, the assumptions $(H_1), (H_2), (H_3)$ hold, and assume that the condition*

$$\int_0^T |\langle i_\omega BS(t)y, S(t)y\rangle_{V',V}|dt \geq \mu \|\chi_\omega y\|^2_{L^2(\omega)}, \text{ for some } T, \mu > 0 \tag{13}$$

is verified, then the control $v(t) = -\langle i_\omega Bz(t), z(t)\rangle_{V',V}$ strongly stabilizes system (1) on ω with the following decay estimate

$$\|\chi_\omega z(t)\|_{L^2(\omega)} = O(t^{-1/2}), \quad \text{as } t \to +\infty.$$

Proof. From (10) and (13), we deduce that

$$\beta\sqrt{\lambda(kT)} \geq \|\chi_\omega \Gamma(kT)z_0\|^2, \quad \forall k \geq 0, \tag{14}$$

where $\beta = \dfrac{1}{\mu}M$.
Integrating the following inequality

$$\frac{d}{dt}\|\chi_\omega \Gamma(t)z_0\|^2 \leq -2|\langle i_\omega B\Gamma(t)z_0, \Gamma(t)z_0\rangle_{V',V}|^2$$

from kT to $(k+1)T$, $(k \in \mathbb{N})$, and using (14), we obtain

$$\|\chi_\omega \Gamma(kT)z_0\|^2 - \|\chi_\omega \Gamma(kT+T)z_0\|^2 \geq 2\lambda(kT), \quad \forall k \geq 0.$$

It follows that

$$\beta^2\|\chi_\omega \Gamma(kT+T)z_0\|^2 - \beta^2\|\chi_\omega \Gamma(kT)z_0\|^2 \leq -2\|\chi_\omega \Gamma(kT)z_0\|^4, \quad \forall k \geq 0. \tag{15}$$

Let us introduce the sequence $s_k = \|\chi_\omega \Gamma(kT)z_0\|^2, \ \forall k \geq 0$.
From (15), we deduce that

$$\frac{s_k - s_{k+1}}{s_k^2} \geq \frac{2}{\beta^2}, \quad \forall k \geq 0.$$

Since the sequence (s_k) decreases, we get

$$\frac{s_k - s_{k+1}}{s_k . s_{k+1}} \geq \frac{2}{\beta^2}, \quad \forall k \geq 0,$$

so

$$\frac{1}{s_{k+1}} - \frac{1}{s_k} \geq \frac{2}{\beta^2}, \quad \forall k \geq 0.$$

We deduce that

$$s_k \leq \frac{s_0}{\frac{2s_0}{\beta^2}k + 1}, \quad \forall k \geq 0.$$

Finally, introducing the integer part $k = E(\frac{t}{T})$ and using the fact that $\|\chi_\omega \Gamma(t)z_0\|$ decreases, we deduce the estimate

$$\|\chi_\omega z(t)\| = O(t^{-1/2}), \quad \text{as } t \to +\infty.$$

□

3 Example and Simulations

Let us consider the system defined in $\Omega =]0, +\infty[$ by the following equation

$$\begin{cases} \dfrac{\partial z(x,t)}{\partial t} = -\dfrac{\partial z(x,t)}{\partial x} + v(t)b(x)z(x,t), & \Omega \times]0, +\infty[, \\ z(0,t) = 0, &]0, +\infty[, \\ z(x,0) = z_0(x), & \Omega, \end{cases} \tag{16}$$

where $H = L^2(\Omega)$, $b(x) \geq 0$, a.e on Ω, and $b(x) \geq c > 0$ a.e on ω, $Az = -\dfrac{\partial z}{\partial x}$ with the domain $\mathcal{D}(A) = \{z \in H^1(\Omega) \mid z(0) = 0\}$, and consider the operator $B : \mathcal{D}(B)(\subset L^2(\Omega)) \to L^2(\Omega)$ given by $Bz = b(x)z$. The operator B is unbounded on $L^2(\Omega)$. By the Sobolev embeddings $H^1(\Omega) \hookrightarrow L^2(\Omega)$, B is bounded from $H^1(\Omega)$ to $L^2(\Omega)$. Hence, the space V is given by $V = H^1(\Omega)$.

The operator A generates the following semi-group of contractions

$$(S(t)z_0)(x) = \begin{cases} z_0(x-t), & if\ x \geq t, \\ 0, & if\ x < t. \end{cases}$$

Let $\omega =]0, a[$, with $a > 0$ we have

$$\langle i_\omega Az, z\rangle \quad = \quad -\int_0^a z'(x)z(x)dx = -\frac{z^2(a)}{2} \leq 0,$$

so hypothesis (H_1) holds. For $T = 1$, we have

$$\int_0^1 \langle i_\omega b(x)S(t)z_0, S(t)z_0\rangle_{V',V} dt = \int_0^1 \int_0^{1-t} b(x)|z_0(x)|^2 dxdt \geq c\|\chi_\omega z_0\|^2.$$

Then, the control $v(t) = -\displaystyle\int_0^a b(x)|z(x,t)|^2 dx$ strongly stabilizes system (16) on ω.

We consider system (16) with $b(x) = \dfrac{1}{\sqrt{x(x^2+1)}}$ and $z_0(x) = \sin(\pi x)$.

- For $\omega =]0, 2[$, we have

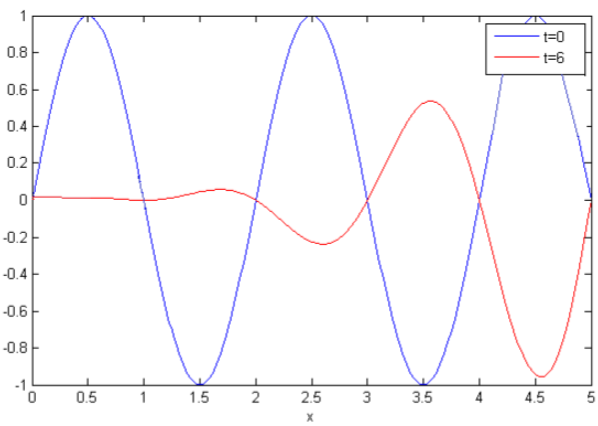

Figure 1: The stabilization on $\omega =]0, 2[$.

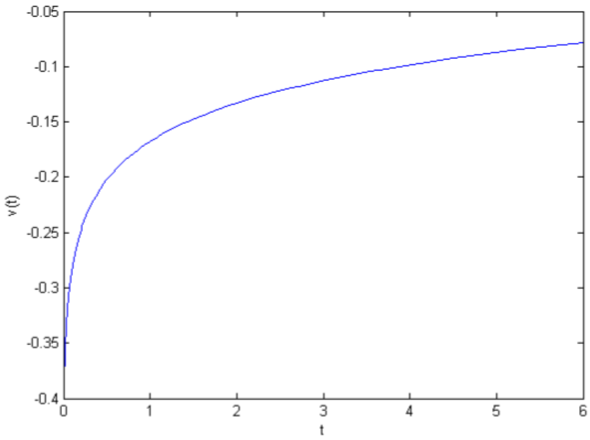

Figure 2: Control function.

Figure 1 shows that the system (16) is strongly stabilized on $\omega =]0, 2[$.

• For $\omega =]0, 3[$, we have

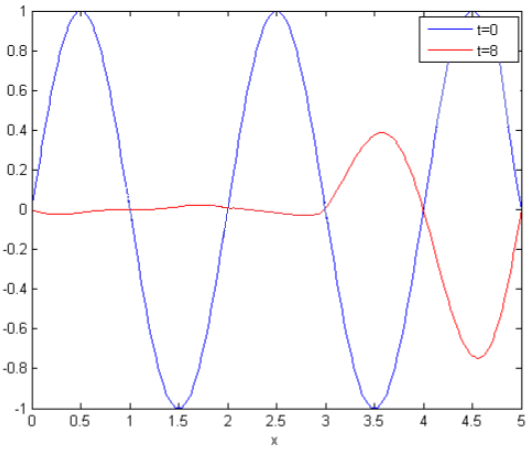

Figure 3: The stabilization on $\omega =]0, 3[$.

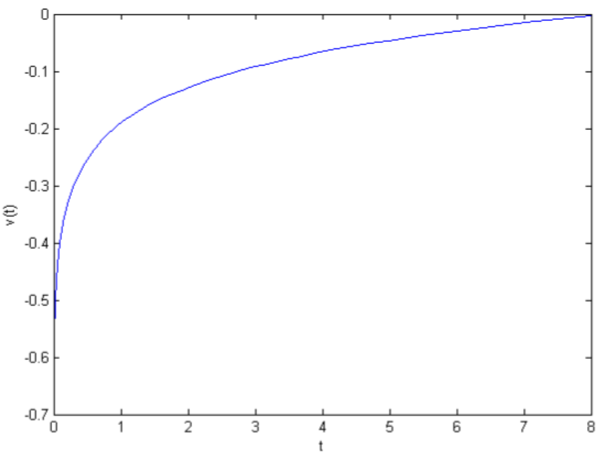

Figure 4: Control function.

Figure 3 shows that the system (16) is strongly stabilized on $\omega =]0, 3[$.

4 Conclusion

Regional stabilization of a class of infinite dimensional bilinear systems with unbounded control operator is considered. Under sufficient conditions, we give a control that ensures weak and strong regional stabilization. Questions are still open; this is, the case of boundary subregion.

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