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Boundedness Results for a New Hyperchaotic System and Their Application in Chaos Synchronization

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Abstract: In this paper, we attempt to investigate the boundedness of a new hyperchaotic system using the combination of the Lyapunov stability theory with the comparison principle method. Furthermore, explicit estimation of the two-dimensional parabolic ultimate bound with respect to x-z is established. Finally, a linear feedback approach with one input is used to realize the global synchronization of two four-dimesional hyperchaotic systems. Some numerical simulations are also used to verify the effectiveness and correctness of the proposed scheme.

Keywords: *4D hyperchaotic system; boundedness of solutions; Lyapunov stability; chaos synchronization; comparison principle method.*

Mathematics Subject Classification (2010): 65P20, 65P30, 65P40.

1 Introduction

Hyperchaotic systems are dissipative nonlinear dynamical systems with more than one positive Lyapunov exponent. The Lyapunov exponent of a chaotic system is a measure of the divergence of points which are initially very close and this can be used to quantify chaotic systems. So, the hyperchaos may be more useful in some fields such as communication encryption, and so forth.

An important paradigm of a 3-D chaotic system was discovered by Lorenz [7] while he was studying a 3-D weather model. Subsequently, many chaotic systems have attracted tremendous research interest, and many chaotic and hyperchaotic systems have been presented.

Chaotic systems are ultimately bounded. Thus, the phase portraits of the systems will be ultimately trapped in some compact sets. The ultimate boundedness of a chaotic

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system is very important for the study of the qualitative behavior of a chaotic system. In fact, except for the stability property, boundedness is also one of the foundational concepts of dynamical systems, which plays an important role in investigating the uniqueness of equilibrium, global asymptotic stability, global exponential stability, the existence of the periodic solution, its control and synchronization and so on. Furthermore, it can be applied in estimating the fractal dimensions of chaotic attractors, such as the Hausdorff dimension and the Lyapunov dimension of chaotic attractors [3].

Ultimate bound estimation of chaotic systems is a difficult yet interesting mathematical question. At present, several works on this topic were realized for some 3D and 4D dynamical systems, see [2, 4–6, 8–12, 14, 15].

Recently, Chen Hai-tao, Chen Di-yi and Ma Xiao-yi [1] introduced the following new system

$$\begin{cases} x^{'} = \alpha\left(y - x\right), \\ y' = \gamma x - xz - y, \\ z' = xy - \beta z, \\ w' = -x - \alpha w, \end{cases} \tag{1}$$

where $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ is a vector parameter. When $\alpha = 5$, $\beta = 0.7$, $\gamma = 26$, system (1) has a hyperchaotic attractor. Fig. 1. shows the phase portraits of system (1).

In this paper, we firstly investigate the boundedness for this new hyperchaotic system using a combination of Lyapunov stability theory with the comparison principle method. In addition, the two-dimensional parabolic ultimate bound with respect to $x - z$ is established.

Synchronization of chaotic systems has become an important topic in nonlinear science not only for its importance in theory but also for its potential applications in various areas, for example, secure communication, chemical and biomedical science, life science, electromechanical engineering and so on. During the last decades, many methods have been successfully applied to chaos synchronization such as PC method, linear feedback control, adaptive control, backstepping design, active control and nonlinear control, etc.

In this paper, based on the bounds previously obtained, we use linear feedback control with one input to realize global synchronization between two identical hyperchaotic systems.

The rest of this paper is organized as follows. In Section 2, we study the boundedness of the hyperchaotic systems (1). In Section 3, the two-dimensional bound estimation with respect to $x - z$ is established. In Section 4, our outcomes are applied between the master system and the slave system to the study of completely chaos synchronization. In Section 5, numerical simulation is presented to show the effectiveness of our results. Section 6 is the conclusion of the paper.

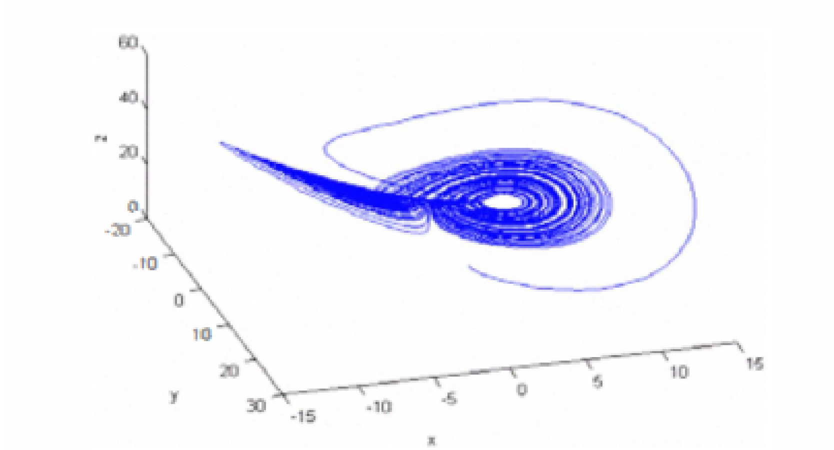

Fig. 1: Phase portrait of the system (1) in the $x-y-z$ space with parameters $\alpha = 5$, $\beta = 0.7$, $\gamma = 26$.

2 Bounds for Solutions of the New Hyperchaotic System

Lemma 2.1 *[5] Define a set*

$$\Gamma = \left\{ (x, y, z) / \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(z-c)^2}{c^2} = 1, \ a > 0, \ b > 0, \ c > 0 \right\} \tag{2}$$

and $G = x^2 + y^2 + z^2$, $H = x^2 + y^2 + (z-2c)^2$, $(x, y, z) \in \Gamma$. *Then we have*

$$\max_{(x,y,z)\in\Gamma} G = \max_{(x,y,z)\in\Gamma} H = \begin{cases} \dfrac{a^4}{a^2 - c^2}, \ a \geq b, \ a \geq \sqrt{2}c, \\ 4c^2, \ a < \sqrt{2}c, \ b < \sqrt{2}c, \\ \dfrac{b^4}{b^2 - c^2}, \ b > a, \ b \geq \sqrt{2}c. \end{cases} \tag{3}$$

Theorem 2.1 *For* $\alpha > 0$, $\beta > 0$, $\gamma > 0$ *the following set*

$$\Omega = \left\{ (x, y, z, w) / x^2 + y^2 + (z - \alpha - \gamma)^2 \leq R^2, \ w^2 \leq \frac{R^2}{\alpha^2} \right\} \tag{4}$$

is the bound for system (1), *where*

$$R^2 = \begin{cases} \dfrac{\beta^2 (\alpha + \gamma)^2}{4\alpha (\beta - \alpha)} \ , \ \textit{if} \ \beta \geq 2\alpha, \ \alpha \leq 1, \\ (\alpha + \gamma)^2 \ , \ \textit{if} \ \beta < 2\alpha, \ \beta < 2, \\ \dfrac{\beta^2 (\alpha + \gamma)^2}{4 (\beta - 1)} \ , \ \textit{if} \ \alpha > 1, \ \beta \geq 2. \end{cases} \tag{5}$$

Proof. Construct the following Lyapunov function

$$V(x, y, z) = x^2 + y^2 + (z - \alpha - \gamma)^2 . \tag{6}$$

Then, its time derivative along the orbits of system (1) is

$$\dot{V} = 2x\dot{x} + 2y\dot{y} + 2\left(z-\alpha-\gamma\right)\dot{z} = -2\alpha x^2 - 2y^2 - 2\beta\left(z-\frac{\alpha+\gamma}{2}\right)^2 + \beta\frac{\left(\alpha+\gamma\right)^2}{2}. \quad (7)$$

Therefore, $\dot{V}=0$, that means, the surface

$$\Gamma = \left\{ (x,y,z)\,/\frac{x^2}{\frac{\beta\left(\alpha+\gamma\right)^2}{4\alpha}} + \frac{y^2}{\frac{\beta\left(\alpha+\gamma\right)^2}{4}} + \frac{\left(z-\frac{\alpha+\gamma}{2}\right)^2}{\frac{\left(\alpha+\gamma\right)^2}{4}} = 1 \right\} \quad (8)$$

is an ellipsoid in three-dimensional space. Outside Γ, $\dot{V}<0$, while inside Γ, $\dot{V}>0$. Since the function $V = x^2+y^2+\left(z-\alpha-\gamma\right)^2$ is continuous on the closed set Γ, V can reach its maximum on the surface Γ. Next, we use Lemma 1 and obtain the optimal value of V on Γ.

$$V \le \max_{(x,y,z)\in\Gamma} V = R^2 = \begin{cases} \dfrac{\beta^2\left(\alpha+\gamma\right)^2}{4\alpha\left(\beta-\alpha\right)}, & \text{if } \beta\ge 2\alpha,\ \alpha\le 1, \\ \left(\alpha+\gamma\right)^2, & \text{if } \beta<2\alpha,\ \beta<2, \\ \dfrac{\beta^2\left(\alpha+\gamma\right)^2}{4\left(\beta-1\right)}, & \text{if } \alpha>1,\ \beta\ge 2. \end{cases} \quad (9)$$

Thus, we have

$$|x| \le R \quad (10)$$

and

$$w' = -x-\alpha w \le -\alpha w + R. \quad (11)$$

By the comparison principle, we obtain

$$w\left(t\right) \le \frac{R}{\alpha} + \left(w\left(t_0\right) - \frac{R}{\alpha}\right)e^{-\alpha(t-t_0)} \quad (12)$$

and

$$\lim_{t\to+\infty} w\left(t\right) \le \frac{R}{\alpha}. \quad (13)$$

Consequently, we get $w^2 \le \dfrac{R^2}{\alpha^2}$ as $t\to+\infty$. Summarizing the above, we have the main result that

$$\Omega = \left\{ (x,y,z,w)\,/x^2+y^2+\left(z-\alpha-\gamma\right)^2 \le R^2,\ w^2 \le \frac{R^2}{\alpha^2} \right\} \quad (14)$$

is the bound for the hyperchaotic systems (1). This completes the proof.

3 Estimate of the Two-Dimensional Parabolic Ultimate Bound with Respect to x–z

Theorem 3.1 *When $\beta<2\alpha$, the system* (1) *has the following two-dimensional parabolic ultimate bound*

$$z \ge \frac{x^2}{2\alpha}. \quad (15)$$

Proof. Define

$$V(t) = \frac{1}{2\alpha} x^2(t) - z(t).$$

Then, its time derivative along the orbits of system (1) is

$$\dot{V} = \frac{1}{\alpha} x\dot{x} - \dot{z} = -x^2 + \beta z.$$

Thus,

$$\dot{V} + \beta V = -x^2 + \beta z + \frac{\beta}{2\alpha} x^2 - \beta z = \left(\frac{\beta}{2\alpha} - 1\right) x^2.$$

When $\beta < 2\alpha$, we have

$$\dot{V} + \beta V \leq 0.$$

For any initial value$V(t_0) = V_0$, according to the comparison theorem, we have

$$V(t) \leq V_0 e^{-\beta(t-t_0)} \to 0\,(t \to \infty).$$

Thus,

$$\lim_{t\to\infty} V(t) = \lim_{t\to\infty} \left[\frac{1}{2\alpha} x^2(t) - z(t)\right] \leq 0.$$

So, we get that system orbits satisfy the parabolic ultimate bound

$$z \geq \frac{x^2}{2\alpha}.$$

This completes the proof.

4 The Application in Chaos Synchronization

In this section, we will use the results obtained in Section 2 to study chaos synchronization via linear feedback. For the master system (1), we construct another system called the slave system, which is designed as

$$\begin{cases} \dot{x}_1 = \alpha(y_1 - x_1), \\ \dot{y}_1 = \gamma x_1 - x_1 z_1 - y_1 - k(y_1 - y), \\ \dot{z}_1 = x_1 y_1 - \beta z_1, \\ \dot{w}_1 = -x_1 - \alpha w_1, \end{cases} \tag{16}$$

where x_1, y_1, z_1, w_1 are the state variables and $k > 0$ is the control. From Theorem 2.1, we obtain

$$|y| \leq R, \;\; |z| \leq R + \alpha + \gamma. \tag{17}$$

Theorem 4.1 *Systems* (1) *and* (16) *are globally and asymptotically synchronized when*

$$k > \frac{\beta(\alpha(\sigma+1) + R + 2\gamma)^2}{4\alpha\beta\sigma - R^2} - 1. \;\left(\sigma > \frac{R^2}{4\alpha\beta} > 0\right). \tag{18}$$

Proof. The complete synchronization error is defined by $e_1 = x_1 - x$, $e_2 = y_1 - y$, $e_3 = z_1 - z$, $e_4 = w_1 - w$. Then, the error dynamics is obtained as

$$\begin{cases} \dot{e}_1 = \alpha (e_2 - e_1), \\ \dot{e}_2 = (\gamma - z) e_1 - x e_3 - e_1 e_3 - (k+1) e_2, \\ \dot{e}_3 = y e_1 + x e_2 + e_1 e_2 - \beta e_3, \\ \dot{e}_4 = -e_1 - \alpha e_4. \end{cases} \tag{19}$$

Define the following Lyapunov function

$$V(e_1, e_2, e_3) = \sigma e_1^2 + e_2^2 + e_3^2,$$

where σ is a positive constant and $\sigma > \dfrac{R^2}{4\alpha\beta} > 0$. Then, its time derivative along the system (19) is

$$\begin{aligned} \frac{1}{2}\dot{V} &= \sigma e_1 \dot{e}_1 + e_2 \dot{e}_2 + e_3 \dot{e}_3 \\ &= \sigma e_1 (\alpha e_2 - \alpha e_1) + e_2 ((\gamma - z) e_1 - x e_3 - e_1 e_3 - (k+1) e_2) \\ &+ e_3 (y e_1 + x e_2 + e_1 e_2 - \beta e_3) \\ &= -\sigma\alpha e_1^2 - (k+1) e_2^2 - \beta e_3^2 + (\sigma\alpha + \gamma - z) e_1 e_2 + y e_1 e_3 \\ &\le -\sigma\alpha e_1^2 - (k+1) e_2^2 - \beta e_3^2 + (\alpha(\sigma+1) + R + 2\gamma) |e_1| |e_2| + R |e_1| |e_3| \\ &= -E^T P E, \end{aligned}$$

where

$$E = [|e_1|, |e_2|, |e_3|]^T, \; P = \begin{bmatrix} \sigma\alpha & -\dfrac{\alpha(\sigma+1) + R + 2\gamma}{2} & -\dfrac{R}{2} \\ -\dfrac{\alpha(\sigma+1) + R + 2\gamma}{2} & k+1 & 0 \\ -\dfrac{R}{2} & 0 & \beta \end{bmatrix},$$

which is positive definite when

$$\sigma > \frac{R^2}{4\alpha\beta} > 0, \; k > \frac{\beta (\alpha(\sigma+1) + R + 2\gamma)^2}{4\alpha\beta\sigma - R^2} - 1.$$

Thus, according to the Lyapunov function theory, it follows that

$$\lim_{t\to+\infty} |e_1| = 0, \lim_{t\to+\infty} |e_2| = 0, \lim_{t\to+\infty} |e_3| = 0. \tag{20}$$

In the following, we will prove $\lim_{t\to+\infty} e_4 = 0$. From (20), we have $\lim_{t\to+\infty} e_1 = 0$. Therefore, for any $\varepsilon > 0$, there is a sufficiently large $T > t_0$ such that, when $t \ge T$, we have $\left|\dfrac{e_1}{\alpha}\right| < \varepsilon$. So, for any $\varepsilon > 0$, when $t \ge T$, from (19), we have

$$\begin{aligned} e_4(t) &= e_4(t_0) e^{-\alpha(t-t_0)} + e^{-\alpha t} \int_{t_0}^{t} (-e_1) e^{\alpha\tau} d\tau \\ &\le e_4(t_0) e^{-\alpha(t-t_0)} + e^{-\alpha t} \int_{t_0}^{t} \alpha\varepsilon e^{\alpha\tau} d\tau \\ &= (e_4(t_0) - \varepsilon) e^{-\alpha(t-t_0)} + \varepsilon. \end{aligned}$$

Thus, if the initial value $e_4(t_0) > \varepsilon$ and $t \to +\infty$, we obtain

$$e_4(t) - \varepsilon \le (e_4(t_0) - \varepsilon)\, e^{-\alpha(t-t_0)} \to 0.$$

Also, we have

$$\begin{aligned} e_4(t) &= e_4(t_0)e^{-\alpha(t-t_0)} + e^{-\alpha t}\int_{t_0}^{t} (-e_1)\, e^{\alpha\tau} d\tau \\ &\ge e_4(t_0)e^{-\alpha(t-t_0)} - e^{-\alpha t}\int_{t_0}^{t} \alpha\varepsilon e^{\alpha\tau} d\tau \\ &= (e_4(t_0) + \varepsilon)\, e^{-\alpha(t-t_0)} - \varepsilon. \end{aligned}$$

Thus, if the initial value $e_4(t_0) < -\varepsilon$ and $t \to +\infty$, we get

$$e_4(t) + \varepsilon \le (e_4(t_0) + \varepsilon)\, e^{-\alpha(t-t_0)} \to 0.$$

Consequently, when the initial value $|e_4(t_0)| > \varepsilon$ and $t \to +\infty$, we have the distance $d\,(e_4(t), I) \to 0$, where $I = [-\varepsilon, \varepsilon]$. So, for any sufficiently small $\varepsilon > 0$, there is a sufficiently large $T > t_0$ such that, when $t > T$, we have $|e_4(t)| < \varepsilon$. By the definition of limit, we obtain

$$\lim_{t\to+\infty} e_4(t) = 0. \tag{21}$$

Summarizing the above, we have

$$\lim_{t\to+\infty} |e_1| = 0, \lim_{t\to+\infty} |e_2| = 0, \lim_{t\to+\infty} |e_3| = 0, \lim_{t\to+\infty} |e_4| = 0.$$

Finally, we conclude that the master system (1) and the slave system (16) are globally synchronized. This completes the proof.

5 Simulation Studies

In this section, using the MATLAB 7.4, some numerical simulations are presented. As initial conditions for the master and slave systems, we take $(1, -0.5, 3, 4)$ and $(-8, -1, -4, -1)$, respectively. When $\alpha = 5$, $\beta = 0.7$, $\gamma = 26$, it is easy to obtain $R = 31$, $\sigma > \dfrac{R^2}{4\alpha\beta} = 68.64$, according to Theorems 2.1 and 4.1. Choose $\sigma = 69$, $k = 26248$, then Fig. 2. shows the complete synchronization between systems (1) and (16).

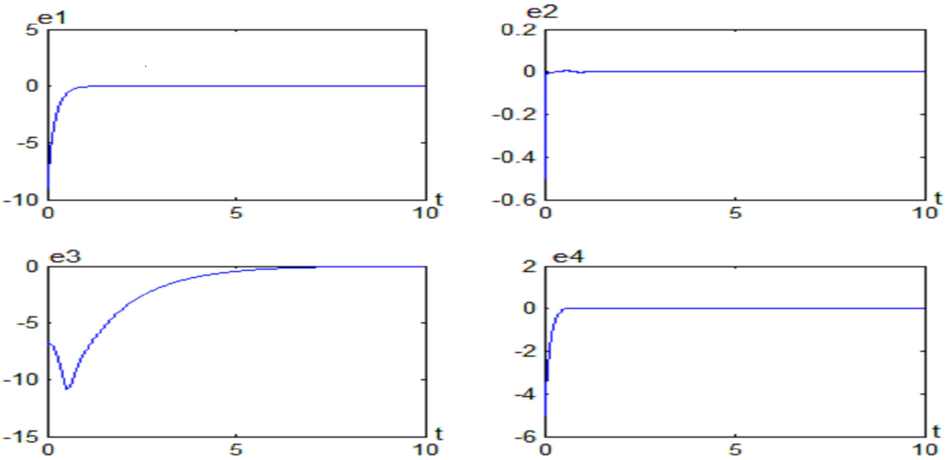

Fig. 2: Time evolution of synchronization errors e_1, e_2, e_3 and e_4 between the master system (1) and the slave system (16).

6 Conclusion

In this research work, the boundedness of a new hyperchaotic system has been investigated. Furthermore, the two-dimensional parabolic estimate with respect to $x-z$ for the new system is established. Finally, the result is applied to the chaos synchronization and numerical simulations are presented to show the effectiveness of the proposed scheme.

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