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Krasnoselskii's Theorem, Integral Equations, Open Mappings, and Non-Uniqueness

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Abstract: We study Krasnoselskii's fixed point theorem on the sum of two operators restricted to the Banach space of continuous functions with the supremum norm. The work is based on "open mappings" in the sense that our mapping P maps a closed bounded convex set M into its interior M^o. We show that any fixed point of a mapping of the whole space must reside in M^o. This is very informative in case of non-uniqueness. We also extend a known transformation to hold for integral equations being the sum of a contraction and a compact map where the "forcing function" is the contraction. Several examples are given showing the construction of the unusually simple mapping sets.

Keywords: *Krasnoselskii's fixed points; open mappings; transformations; uniqueness.*

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1 Introduction

Much has been written about fixed point mappings which are either contractions or compact. But around 1954 Krasnoselskii studied a paper by Schauder on differential equations and concluded a variant of the idea that the inversion of a perturbed differential operator yields the sum of a contraction and a compact map. Embodied in that theorem are both Banach's contraction mapping principle and Schauder's second fixed point theorem. All three of these are conveniently found in the monograph by Smart [15]. Accordingly, Krasnoselskii offered the following fixed point theorem [15, p. 31].

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Theorem 1.1 *Let M be a closed convex non-empty subset of a Banach space $(\mathcal{B}, \|\cdot\|)$. Suppose that A and B map M into $\mathcal{B}$ and that*
(i) $x, y \in M \implies Ax + By \in M$,
(ii) A is continuous and maps bounded sets into compact sets.
(iii) B is a contraction mapping with constant $\alpha < 1$.
Then there exists $y \in M$ such that $Ay + By = y$.

We can get the idea from the survey paper by Park [14] that this has generated much interest. In fact, the literature on it is so large that we make no effort to survey it here.

It is noteworthy that investigators focus on the very general form of (i), but gloss over the common assumption that $(A + B) : M \to M$. Moreover, nothing is said about uniqueness or where another fixed point might reside. In real-world problems it can be a complete disaster if there is another fixed point and if it is in a set having fundamental features very different from those of points in M. We discuss these later.

We loosen (i) to just the mapping $(A+B)$ and then tighten it to $(A+B) : M \to M^o$, the interior of M. Then we restrict the space to continuous functions. This allows us to prove the result and be certain that any fixed point will reside in M. Hence, we will know that the fixed point has the general properties displayed in M. The next paragraph justifies the restriction of the space.

Normed spaces and even Banach spaces are very general and contain examples which have almost none of our intuitive properties in common. There is a classical example showing how wrong we can be when we see a counterexample to a conjecture when the counterexample involves sequence spaces, but our main interest lies in spaces of continuous functions with the supremum norm. A 1980 article in Smart [15, p. 39] states that there appears to be an open conjecture that [every shrinking mapping of the closed unit ball in a Banach space has a fixed point]. In fact, it is true for important spaces of continuous functions, but was shown to be false for sequence spaces in 1967. See MR3695827.

We believe that the same thing is at work here and it involves the property that a continuous function starting inside a Jordan curve cannot pass from the inside to the outside without explicitly crossing the curve. That is used frequently in stability theory of ordinary differential equations, but fails completely for corresponding results for difference equations in which the solution jumps over the boundary without touching it.

It is for these reasons that we believe that Krasnoselskii's theorem may be simple for Banach spaces of continuous functions $\phi : [0, E] \to \Re$ with the supremum norm denoted by $(\mathcal{B}, \|\cdot\|_{[0,E]})$ which typically concern a general class of integral equations represented as

$$x(t) = g(t, x(t)) + \int_0^t a(t, s) f(s, x(s))\, ds, \quad t \geq 0, \tag{$\mathcal{E}$}$$

in the aforementioned space. A special case of $(\mathcal{E})$ is

$$x(t) = g(t, x(t)) - \int_0^t C(t - s) f(s, x(s)) ds. \tag{1}$$

In the context of Krasnoselskii's theorem, $g(t, x)$ is B - the contraction, while the integral is A - the compact map. The natural mapping is then $P = B + A$ and we seek an appropriate closed bounded convex subset of $\mathcal{B}$. While we have greatly simplified (i), among other things we ask that $P : M \to M^o$ the interior of M and we also have asked that M be bounded. But this will yield far more than just a fixed point. With these conditions in hand we will use an old combination of Schaefer's theorem with Krasnoselskii's theorem and obtain a fixed point. With reference now to future remarks on non-uniqueness, we can be assured that any other solution will also reside in M which tells us also that all solutions are bounded.

Non-anticipative: The entire paper rests on the following two paragraphs.

We will assume throughout that we are dealing with Volterra operators [8, p. 84]. For any pair of functions $x, y \in \mathcal{B}$ if $x(s) = y(s)$ on $0 \le s \le t \le E$ then the operator V satisfies $(Vx)(t) = (Vy)(t)$. When the operator P is the natural operator defined by $(\mathcal{E})$ and the integral has, at $t = 0$, constant value for any function $x \in \mathcal{B}$, say zero, then $(Px)(0) = g(0, x(0))$ for any function $x \in \mathcal{B}$. In particular, if $\phi \in \mathcal{B}$ is a fixed point so that $P\phi = \phi$, then it is true that

$$(P\phi)(0) = \phi(0) = g(0, \phi(0)),$$

and the last equality is an algebraic relation in $\phi(0)$. Since g is a contraction, this has a unique solution $\phi(0)$ independent of which fixed point of P is under discussion. As we have asked no smoothness conditions on f, there may be many fixed points of P but they all start at this single $\phi(0)$. Frequently we can solve that algebraic relation explicitly for $\phi(0)$.

Looking ahead, we will be finding a closed convex bounded set M with the property that P maps M into its interior. If ψ belongs to M and has the property that $\phi(0) = \psi(0)$, then $P\psi$ is in the interior of M, so the distance from $\phi(0)$ to the complement of M is positive.

2 The Location of a Fixed Point

Let $(\mathcal{B}, \|\cdot\|_{[0,E]})$ be the Banach space of continuous $\phi : [0, E] \to \Re$ with the supremum norm. If $B, A : \mathcal{B} \to \mathcal{B}$ are the operators defined, respectively, by g and the integral in $(\mathcal{E})$, then we want A to be compact in the sense that it is continuous and maps bounded subsets of $\mathcal{B}$ into compact sets, while B is a contraction. The central idea here is that if P_λ is the non-anticipative mapping $P_\lambda : \mathcal{B} \to \mathcal{B}$ defined by

$$P_\lambda \phi := \lambda B(\phi/\lambda) + \lambda A\phi, \quad 0 < \lambda \le 1, \tag{2}$$

then conditions of the following theorem can be verified if for every $\phi \in \mathcal{B}$ the number $(P_\lambda \phi)(0)$ is completely known. This is readily seen in $(\mathcal{E})$ which we later explain in detail.

Throughout the paper, for a positive number K we denote by M_K the closed ball of center 0 and radius K in the Banach space $\mathcal{B}$, i.e.,

$$M_K := \{x \in \mathcal{B} : ||\phi|| \le K\}.$$

Theorem 2.1 *Let the conditions on A, B, and P_λ of (2) hold. Assume that there is a $K > 0$ such that the unique fixed point x_0 of $g(0, x) = x$ belongs to $(-K, K)$, and, for each $\lambda \in (0, 1]$ the mapping P_λ of (2) satisfies $P_\lambda M_K \subset M_K^o := (M_K)^o$. Then $P\phi = \phi$ has a solution in M_K^o. Moreover, any solution ϕ of $P\phi = \phi$ resides in M_K^o.*

Proof. We firstly prove that all fixed points of P (if any) reside in M_K^o. Recall that any fixed point of P starts from the unique solution of the equation $g(0, x) = x$. We now come to the idea introduced in [5]. Let ϕ be a fixed point of P which does not reside in M_K^o. As $\phi(0) = x_0 \in (-K, K)$, we see that there exists $T \in (0, E]$ such that either $\phi(T) = K$ or $\phi(T) = -K$, while $-K < \phi(t) < K$, $t \in [0, T)$, say it holds $\phi(T) = K$. Then for the function

$$\phi_T(t) := \begin{cases} \phi(t), & t \in [0, T), \\ K, & T \le t \le E, \end{cases}$$

we see that $\phi_T \in M_K$ hence $P\phi_T \in M_K^o := \{x \in \mathcal{B} : ||\phi|| < K\}$, and so $||P\phi_T|| < K$. But $\phi_T(t) = \phi(t)$ for $t \in [0, T]$, thus, as P is non-anticipative, we take $P\phi_T(T) = P\phi(T) = K$ which implies $||P\phi_T|| \ge K$, a contradiction to $P\phi_T \in M_K^o$. Thus, all fixed points of P (if any) reside in M_K^o. Then, in view of $P = P_1$, existence of a solution of the equation $P\phi = \phi$ is yielded by the following result of Burton and Kirk [4] applied on the Banach space $\mathcal{B}$ and the fact that, by assumption, for any $\lambda \in (0, 1)$ we have $P_\lambda M \subset M_K^o$ which excludes (ii) in that theorem.

Theorem 2.2 *Let $(\mathcal{B}, \|\cdot\|)$ be a Banach space, $A, B : \mathcal{B} \to \mathcal{B}$, B be a contraction with constant $\alpha < 1$, and A be continuous with A mapping bounded sets into compact sets. Either*

(i) $x = \lambda B(x/\lambda) + \lambda Ax$ has a solution in $\mathcal{B}$ for $\lambda = 1$, or

(ii) the set of all solutions $\lambda \in (0, 1)$ (if any), is unbounded.

We now want to employ Theorem 2.1 to look at solutions to the equation

$$x(t) = g(t, x(t)) + \int_0^t a(t, s) f(s, x(s)) \, ds, \quad t \ge 0. \qquad (\mathcal{E})$$

We assume that $g, f : [0, \infty) \times \Re \to \Re$ are continuous with $g(t, z)$ being a contraction in its second variable z, i.e., there exists an $\alpha \in (0, 1)$ such that

$$|g(t, z_1) - g(t, z_2)| \le \alpha |z_1 - z_2|, \quad t \ge 0, \ \ z_1, z_2 \in \Re,$$

and that the kernel $a(t, s) : \{(t, s) : 0 < s < t\} \to \Re$ is absolutely integrable with respect to the second variable on $[0, t]$, for $t > 0$, and such that the function $\widetilde{a}$ defined by

$$\widetilde{a}(t) := \int_0^t |a(t, s)| \, ds, \quad t \ge 0,$$

is well defined and continuous for $t \ge 0$. Note that $a(t, s)$ may not be defined for $t = 0$ but we ask that $\underset{t \to 0+}{lim} \int_0^t |a(t, s)| \, ds \in \Re$, in other words mild singularities of a are allowed. Clearly, if a is continuous on the closed triangle sets $\{(t, s) : 0 \le s \le t\}$, then a is absolutely integrable and $\underset{t \to 0+}{lim} \int_0^t |a(t, s)| \, ds = 0$.

In each one of the following three theorems we focus on giving conditions ensuring the existence of a ball M in the Banach space $\mathcal{B}$ so that for each $\lambda \in (0, 1]$ the corresponding mapping P_λ defined in (2) satisfies $P_\lambda M \subset M^o$. Recall that we have also assumed that g is a contraction and that A is compact. Then Theorem 2.1 not only yields existence of (at least one) solution to $(\mathcal{E})$, but also ensures that all solutions reside in M^o. As E may vary and the set M in Theorem 2.1 is taken to be a ball, for the sake of clarity we adopt the following notation:

For an $E > 0$ we denote by $\mathcal{B}_E$ the Banach space of bounded continuous functions $\phi : [0, E] \to \mathbb{R}$ equipped with the usual supremum norm

$$\|\phi\|_E := \sup_{t \in [0, E]} |\phi(t)|,$$

(or, simply $||\phi||$ when E is fixed), and set

$$M_K := \{\phi \in \mathcal{B}_E : \|\phi\| \leq K\},$$

where K is a positive number. As $g(t,0)$ is continuous on $[0,E]$ for any $E>0$, we set

$$L_E := \sup_{t\in[0,E]} |g(t,0)|.$$

Clearly, L_E is a well defined nonnegative real number for any $E>0$.

For $\lambda \in (0,1]$, we consider the mapping P_λ on $\mathcal{B}_E$ defined by $\phi \in \mathcal{B}_E$ implies

$$(P_\lambda \phi)(t) := \lambda g\left(t, \frac{\phi(t)}{\lambda}\right) + \lambda \int_0^t A(t,s) f(s,\phi(s))\,ds, \quad t \in [0,E]. \tag{3}$$

Then $P_\lambda \phi$ is continuous on $[0,E]$, so $P_\lambda \phi : \mathcal{B}_E \to \mathcal{B}_E$.

Our first result presents a limit condition posed on the kernel $a(t,s)$ ensuring that $P_\lambda M_K \subset M_K^o$ for properly chosen positive numbers E and K.

Theorem 2.3 *Assume that*

$$\lim_{t\to 0+} \int_0^t |a(t,s)|\,ds = 0. \tag{4}$$

Then there always exist $E>0$ and $K>0$ such that for the mapping P_λ defined by (3) we have $P_\lambda : M_K \to M_K^o$ for any $\lambda \in (0,1]$.

Proof. Let $\lambda \in (0,1]$ and consider an arbitrary $E_1 > 0$. As $L_{E_1} := \sup_{t\in[0,E_1]} |g(t,0)|$ is a nonnegative real number, we may consider a $K>0$ such that

$$\frac{L_{E_1}+1}{1-\alpha} < K. \tag{5}$$

By continuity of f on the compact set $S_1 := [0,E_1]\times[-K,K]$ we see that

$$m := \sup_{(t,z)\in S_1} |f(t,z)| \in \Re.$$

In view of (4), if necessary, we may choose an $E \in (0,E_1]$ such that

$$m\int_0^t |a(t,s)|\,ds < 1, \quad t\in[0,E]. \tag{6}$$

With the real numbers E and K defined above, we consider the Banach space $\mathcal{B}_E$ and the ball

$$M_K := \{\phi \in \mathcal{B}_E : \|\phi\| \leq K\},$$

and note that as $[0,E] \subset [0,E_1]$, we have

$$S := [0,E]\times[-K,K] \subset [0,E_1]\times[-K,K] = S_1,$$

so

$$|f(t,z)| \leq m \quad \text{for all} \quad (t,z) \in [0,E]\times[-K,K],$$

and

$$|f(t,\phi(t))| \leq m \quad \text{for all } \ t \in [0,E], \quad \phi \in M_K.$$

Now for $\phi \in M_K$ and $\lambda \in (0,1]$ we have for $t \in [0,E]$

$$\left| g\left(t, \frac{\phi(t)}{\lambda}\right) - g(t,0) \right| \leq \alpha \left| \frac{\phi(t)}{\lambda} \right| \Longrightarrow \left| g\left(t, \frac{\phi(t)}{\lambda}\right) \right| \leq |g(t,0)| + \alpha \left| \frac{\phi(t)}{\lambda} \right|,$$

so

$$\begin{aligned} |(P_\lambda \phi)(t)| &\leq \left| \lambda g\left(t, \frac{\phi(t)}{\lambda}\right) \right| + \lambda \int_0^t |a(t,s)| \, |f(s,\phi(s))| \, ds \\ &\leq \lambda \left[|g(t,0)| + \alpha \left| \frac{\phi(t)}{\lambda} \right| \right] + \lambda \int_0^t |a(t,s)| \, |f(s,\phi(s))| \, ds \\ &\leq L_{E_1} + \alpha\lambda \frac{|\phi(t)|}{\lambda} + \int_0^t |a(t,s)| \, m ds \\ &\leq L_{E_1} + \alpha \|\phi\| + m \int_0^t |a(t,s)| \, ds, \end{aligned}$$

from which, by the use of (6) we find

$$|(P_\lambda \phi)(t)| < L_{E_1} + \alpha K + 1, \ \ t \in [0,E], \quad \phi \in M_K.$$

Consequently, in view of (5) we have

$$\|P_\lambda \phi\| \leq L_{E_1} + \alpha K + 1 := K_0 < K, \quad \phi \in M_K,$$

and hence for any $\phi \in M_K$ it holds

$$P_\lambda \phi \in \left\{ \phi \in \mathcal{B}_E : \|\phi\| < K_0 + \frac{K - K_0}{2} \right\} \subset M_K^o,$$

i.e., $P_\lambda(M_K) \subset M_K^o$.

As already mentioned, when a is continuous on the closed triangle sets $\{(s,t) : 0 \leq s \leq t\}, t > 0$, then (4) is automatically satisfied. However, if the kernel a has singularity at $t = 0$, the limit condition (4) may not be satisfied. Indeed, the function $a(t,s) : \{0 < s < t\}, \to \mathbb{R}$, defined by $a(t,s) = \frac{1}{\sqrt{t}}(t-s)^{-1/2}, 0 < s < t$, is absolutely integrable with respect to s on $[0,t]$ for any $t > 0$, also the function $\widetilde{a}(t) = \int_0^t |a(t,s)| \, ds$ may be well defined and continuous on $[0,t]$ by setting

$$\widetilde{a}(0) = \lim_{t \to 0+} \int_0^t a(t,s) \, ds = \lim_{t \to 0+} \frac{1}{\sqrt{t}} \int_0^t (t-s)^{-1/2} \, ds = \lim_{t \to 0+} \frac{1}{\sqrt{t}} 2\sqrt{t} = 2,$$

however, the limit condition (4) is not satisfied.

Example 2.1 Fractional kernels of the type

$$a(t,s) = (t-s)^{q-1}, \quad 0 < s < t,$$

with $q \in (0,1)$ do satisfy condition (4) since

$$\lim_{t \to 0+} \int_0^t (t-s)^{q-1} \, ds = \lim_{t \to 0+} \frac{t^q}{q} = 0,$$

and so Theorem 2.3 may be applied on the equation with fractional kernel

$$x(t) = g(t, x(t)) + \int_0^t (t-s)^{q-1} f(s, x(s))\, ds, \quad t \geq 0,$$

with $q \in (0,1)$, g, f continuous, and g being a contraction. In particular, Caputo type fractional equations can be considered as special cases of the above equation.

Theorem 2.3 yields the existence of a (sufficiently small) interval $[0, E]$ and a corresponding $K > 0$ so that it holds $P_\lambda M_K \subset M_K^o$. When f is bounded we may always find a $K_E > 0$ so that $P_\lambda M_{K_E} \subset M_{K_E}^o$ for any (arbitrarily large) $E > 0$, yet without assuming (4). In view of continuity of the function $\widetilde{a}(t) := \int_0^t |a(t,s)|\, ds$, we set

$$a_T := \sup_{0 \leq t \leq T} \widetilde{a}(t) = \sup_{0 \leq t \leq T} \int_0^t |a(t,s)|\, ds, \quad T > 0.$$

Theorem 2.4 *Assume that there exists an $m_f \geq 0$ with*

$$|f(t,z)| \leq m_f, \quad (t,z) \in [0,\infty) \times \Re.$$

Then for an arbitrary $E > 0$, there always exists a $K_E > 0$ so that for the mapping P_λ defined by (3) we have $P_\lambda M_{K_E} \subset M_{K_E}^o$ for any $\lambda \in (0,1]$.

Proof. Let $\lambda \in (0,1]$ and consider an arbitrary $E > 0$. Take $K_E > 0$ with

$$\frac{L_E + m_f a_E}{1-\alpha} < K_E. \tag{7}$$

Then for $\phi \in M_{K_E}$, $t \in [0,E]$, we have

$$\begin{aligned} |(P_\lambda \phi)(t)| &\leq \left| \lambda g\left(t, \frac{\phi(t)}{\lambda}\right) \right| + \lambda \int_0^t |a(t,s)|\, |f(s,\phi(s))|\, ds \\ &\leq L_E + \alpha |\phi(t)| + m_f \int_0^t |a(t,s)|\, ds \\ &\leq L_E + \alpha \|\phi\| + m_f \sup_{0 \leq t \leq E} \int_0^t |a(t,s)|\, ds \\ &\leq L_E + \alpha K_E + m_f a_E := \widehat{K}_E, \end{aligned}$$

so by (7) we take $\|(P_\lambda \phi)\| \leq \widehat{K}_E < K_E, \quad \phi \in M_{K_E}$, from which it follows that $P_\lambda M_{K_E} \subset M_{K_E}^o$.

Example 2.2 In relation to Example 2.1, as the function $f(t,x) = \sin^3 x + \frac{x}{x^2+1}$ is bounded, one can easily see that Theorem 2.4 applies to the equation

$$x(t) = g(t, x(t)) + \int_0^t (t-s)^{q-1} \left[\sin^3 x(s) + \frac{x(s)}{x^2(s)+1} \right] ds, \quad t \geq 0,$$

with $q \in (0,1)$, g continuous and contraction in x.

In the next result the strict condition of boundedness of f is removed and sufficient conditions yielding that for a given $E>0$ there exists a set $M_E \subset B_E$ so that $P_\lambda M_E \subset M_E^o$ are given.

Theorem 2.5 *Assume that for a given $E>0$, there exists $K_E>0$ such that*

$$L_E + a_E \left[\sup_{(s,z)\in[0,E]\times[-K_E,K_E]} |f(s,z)| \right] < (1-\alpha) K_E. \tag{8}$$

Then for the mapping P_λ defined by (3) we have $P_\lambda M_{K_E}) \subset M_{K_E}^o$ for any $\lambda \in (0,1]$.

Proof. Let $E>0$ be given and $K_E>0$ be such that (8) holds true, i.e.,

$$\frac{L_E}{(1-\alpha) K_E} + \frac{a_E}{(1-\alpha) K_E} \left[\sup_{(s,z)\in[0,E]\times[-K_E,K_E]} |f(s,z)| \right] < 1. \tag{9}$$

Then for $\lambda \in (0,1]$, $\phi \in M_{K_E}$, $t \in [0,E]$, we have

$$\begin{aligned} |(P_\lambda \phi)(t)| &\le \lambda \left| g\left(t, \frac{\phi(t)}{\lambda}\right) \right| + \lambda \int_0^t |a(t,s)| \, |f(s,\phi(s))| \, ds \\ &\le \lambda L_E + \lambda\alpha \left| \frac{\phi(t)}{\lambda} \right| + \lambda \int_0^t |a(t,s)| \left[\sup_{(s,z)\in[0,E]\times[-K_E,K_E]} |f(s,z)| \right] ds \\ &\le L_E + \alpha K_E + a_E \left[\sup_{(s,z)\in[0,E]\times[-K_E,K_E]} |f(s,z)| \right], \end{aligned}$$

thus

$$\|P_\lambda \phi\| \le L_E + \alpha K_E + a_E \left[\sup_{(s,z)\in[0,E]\times[-K_E,K_E]} |f(s,z)| \right] := \widetilde{K}_E.$$

In view of (9) we have $\widetilde{K}_E < K_E$ and so

$$P_\lambda \phi \in \left\{ \phi \in \mathcal{B}_E : \|\phi\| \le \widetilde{K}_E \right\} \subset (M_{K_E})^o,$$

i.e., $PM_{K_E} \subset M_{K_E}^o$.

Example 2.3 Consider the equation

$$\begin{aligned} x(t) &= g_0(t) + \frac{t}{2t+1} \sin x(t) \\ &+ \frac{1}{2(11^2 t + 1)} \int_0^t (t-s)^{-1/2} \left(x^2(s) + \sqrt{|x(s)|} \right) \sin s ds, \quad t \ge 0 \end{aligned}$$

with g_0 continuous and bounded by 1. Here we have

$$g(t,x) = g_0(t) + \frac{t}{2t+1} \sin x, \quad t \ge 0, \quad x \in \Re,$$

and

$$|g(t,x) - g(t,y)| = \frac{t}{2t+1} |\sin x - \sin y| \le \frac{1}{2} |x-y|,$$

so g is a contraction with $\alpha = 1/2$. Setting

$$a(t,s) = \frac{(t-s)^{-1/2}}{2(11^2 t + 1)}, \quad 0 < s < t,$$

we may see that the function $\widetilde{a}(t)$ is continuous and for any $t \in [0,1]$ we have

$$\int_0^t |a(t,s)|\, ds = \frac{1}{2(11^2 t + 1)} \int_0^t (t-s)^{-1/2}\, ds = \frac{2\sqrt{t}}{2(11^2 t + 1)} \le \frac{1}{2 \cdot 11},$$

thus, for any $T \ge 0$ it holds

$$a_T := \sup_{0 \le t \le T} \int_0^t |a(t,s)|\, ds \le \frac{1}{2 \cdot 11}.$$

Next, we let $f(s,x) = \left(x^2 + \sqrt{|x|}\right) \sin s$, $s \ge 0$, $x \in \Re$ and so, taking $K_E = 3$ for any $E > 0$, we have

$$\sup_{(s,z) \in [0,E] \times [-K_E, K_E]} |f(s,z)| \le \sup_{z \in [-3,3]} \left(z^2 + \sqrt{|z|}\right) = 9 + \sqrt{3},$$

and

$$\begin{aligned} L_E + A_E \left[\sup_{(s,z) \in [0,E] \times [-K_E, K_E]} |f(s,z)| \right] &\le 1 + \frac{1}{2 \cdot 11} \left(9 + \sqrt{3}\right) \\ &< 1 + \frac{1}{2 \cdot 11} (9 + 2) \\ &= \frac{3}{2} = \left(1 - \frac{1}{2}\right) 3 \\ &= (1 - \alpha) K_E, \end{aligned}$$

i.e., condition (8) is satisfied with $K_E = 3$ for any $E > 0$, so Theorem 2.5 is applied. We conclude that for any $\lambda \in (0,1]$, if P_λ is the mapping defined by (3), then $P_\lambda M_3 \subset M_3^o$ for any $E > 0$. In particular, for any $\phi \in M_3$ we have

$$\|P_\lambda \phi\| \le \frac{31 + \sqrt{3}}{11} < 3.$$

To get a more convenient condition than (8), for a fixed (but arbitrary) $z \in \Re$ we set

$$f_E(z) := \sup_{s \in [0,E]} |f(s,z)|, \quad z \in \Re,$$

and consider $\widehat{f}_E : \Re \to \Re^+$ with

$$\widehat{f}_E(z) := \sup_{t \in [-z,z]} f_E(t) \quad z \in \Re.$$

From Theorem 2.5 we have the following corollary.

Corollary 2.1 *If for some $E > 0$ it holds*

$$a_E \liminf_{z\to\infty} \frac{\widehat{f}_E(z)}{z} < 1 - \alpha, \tag{10}$$

then there always exists an $M_E \subset B_E$ such that for the mapping P_λ defined by (3) we have $PM_E \subset M_E)^o$, for any $\lambda \in (0, 1]$.

Proof. Let $E > 0$ be such that (10) holds. Since $\underset{z\to\infty}{lim} \frac{L_E}{z} = 0$, by (10) we may find a $z_0 > 0$ so large that

$$\frac{L_E}{z} + a_E \liminf_{z\to\infty} \frac{\widehat{f}_E(z)}{z} < 1 - \alpha, \quad z > z_0.$$

It follows that we may choose a $K_E > z_0$ so that

$$\frac{L_E}{K_E} + a_E \frac{\widehat{f}_E(z)}{K_E} < 1 - \alpha,$$

i.e.,

$$L_E + a_E \widehat{f}_E(z) < K_E(1 - \alpha), \tag{11}$$

or

$$L_E + a_E \left[\sup_{(s,z)\in[0,E]\times[-K_E,K_E]} |f(s,z)| \right] < K_E(1 - \alpha),$$

which is (8), so Theorem 2.5 is applied.

3 Non-uniqueness and Examples

If M is bounded, then the conclusion that any solution resides in M can be far more important than Krasnoselskii's theorem itself for it can be a suitable substitute for uniqueness, a property that neither Krasnoselskii's nor Schauder's theorem yield. This brings us to another main idea. It was Kneser [12] in 1923 who jolted us with the idea that non-uniqueness is the father of disaster, as the example $x' = x^{1/3}, x(0) = 0$, shows by having both a bounded solution, namely $x(t)$ identically zero, and an infinite collection of unbounded solutions. In such cases what possible good can come from the information that there is a solution in the form of a fixed point of a natural mapping?

That information was one of the great motivating factors in the study of stability theory showing that solutions of differential equations starting near each other will stay near each other.

Our focus here will be on the mapping into M^o and on uniqueness which we believe is a new project.

Example 3.1 We consider the scalar integral equation

$$x(t) = (1/2)x(t)\sin t + \int_0^t a(t-s)x^{1/3}(s)ds$$

in which $a : (0, \infty) \to [0, \infty)$ is continuous and there is an $E > 0$ with

$$\int_0^E a(s)ds \leq 1.$$

The mapping set is

$$M = \{\phi : [0, E] \to \Re\colon |\phi(t)| \le 8\}$$

and our Banach space is the continuous functions on $[0, E]$ with the supremum norm.

The natural mapping $P : M \to M$ is defined by $\phi \in M$ implies

$$(P\phi)(t) = (1/2)\phi(t)\sin t + \int_0^t a(t-s)\phi^{1/3}(s)ds$$

so that by inserting λ as in (3) and then taking norms we obtain

$$\|P\phi\| \le (1/2)\|\phi\| + \|\phi\|^{1/3}\int_0^E a(s)ds \le (1/2)8 + 8^{1/3} = 6 < 8$$

and so $P : M \to M^o$ for every $\lambda \in (0, 1]$. A result in [10] shows the continuity and the compactness of the integral maps. The conditions of Theorem 2.1 are satisfied and there is a fixed point. Moreover, every fixed point resides in M. More general results on compact mappings by integrals are found in [7] and [6].

Our next example will show that the mapping into M but not into M^o results in a fixed point in M and one which is not in M. This problem starts from a differential equation about which we know a great deal. This enables us to see what the fixed point theorem can do and cannot do.

Example 3.2 Consider the initial value problem

$$x' = x^{1/3}, \quad x(0) = 0$$

which has one solution $x(t) \equiv 0$. Separation of variables yields

$$x^{-1/3}dx = dt \implies (3/2)x^{2/3} = (3/2)x(0)^{2/3} + t$$

and

$$x^{2/3} = (2/3)t \implies x = \pm\left(\frac{2}{3}t\right)^{3/2}$$

as second and third solutions which are unbounded and hence are of a very different type than the first solution.

This problem occurs in elementary text books, but it is enormously complicated. Kneser's theorem tells us that there is a continuum of solutions between

$$x = 0 \textit{ and } x = (2t/3)^{3/2}$$

as well as between

$$x = 0 \textit{ and } x = -(2t/3)^{3/2}.$$

If we applied Schauder's fixed point theorem it would tell us that there is a solution, but would not suggest which of the three types it might be. But things can get worse. Do we really know that these are the only kinds which might arise? We are going to find out.

It is a routine matter to convert our initial value problem into an integral equation which is

$$x(t) = \int_0^t e^{-(t-s)}(x(s) + x^{1/3}(s))ds, \tag{12}$$

still retaining the solution $x(t) \equiv 0$.

Let $E > 0$ be given and find

$$\int_0^E e^{-s} ds = 1 - e^{-E}.$$

Next, find $a > 0$ so that

$$(a + a^{1/3})\Big(1 - e^{-E}\Big) < a. \tag{13}$$

This is possible since

$$\frac{a + a^{1/3}}{a} \to 1$$

as $a \to \infty$.

Let

$$M = \{\phi : [0, E] \to \Re \colon 0 \le \phi(t) \le a\}$$

and define $P : M \to M$ by $\phi \in M$ implies

$$0 \le (P\phi)(t) = \int_0^t e^{-(t-s)} \Big[\phi(s) + \phi^{1/3}(s)\Big] ds \le [a + a^{1/3}] \left(1 - e^{-E}\right) < a.$$

Notice that PM is not in M^o since $\phi(t) \ge 0$. It is shown in [10] that P is a compact map, so by Schauder's theorem P has a fixed point. As this is not an open map, we cannot be sure that all fixed points are in M, as we already know.

First, there is a parallel mapping with

$$M = \{\phi : [0, E] \to \Re \colon 0 \ge \phi(t) \ge -a\}$$

which is mapped into itself, but not into its interior as it still contains the zero function.

We now combine the two and take

$$M = \{\phi : [0, E] \to \Re \colon \|\phi\| \le a\}$$

with a and E generated as before. This time when we take the natural mapping we find that M is mapped into M^o and we now know that all solutions reside in this set.

Note that Theorem 2.5 can be applied here with $g \equiv 0, a(t, s) = e^{-(t-s)}, f(s, x) = x + x^{1/3}$. We find

$$L_E = 0, \quad a_E = 1 - e^{-E}, \quad \sup_{(s,z)\in[0,E]\times[-x,x]} |f(s, z)| = x + x^{1/3}.$$

Since $g \equiv 0$ may be considered as a contraction with contraction constant any $\alpha \in (0, 1)$, condition (8) reduces to asking for a constant K_E with

$$a_E \Big[K_E + (K_E)^{1/3}\Big] < (1 - \alpha)K_E \tag{14}$$

and α being any convenient number in $(0, 1)$. This is equivalent to asking for a $K_E > 0$ with

$$a_E \Big[K_E + (K_E)^{1/3}\Big] < K_E. \tag{15}$$

Indeed, if a $K_E > 0$ satisfying (15) exists, then (14) is satisfied by taking $0 < \alpha < 1 - \frac{a_E\left[K_E + (K_E)^{1/3}\right]}{K_E}$. Clearly, (15) is (13) with $a = K_E$.

4 The Transformation: Large Kernels

The reader has noticed that some of our examples involve small kernels, quite unlike the vast majority of real-world problems. That can be remedied in an interesting and useful way provided that the kernel $a(t)$ satisfies a set of conditions ensuring the existence of a resolvent, $R(t)$, having a small integral. The conditions are as follows and they include kernels typified by $(t-s)^{q-1}$, $0 < q < 1$, occurring in heat problems and fractional differential equations of both Riemann-Liouville and Caputo type, as well as many others. That class is discussed in depth by Miller [13, p. 209] with consequences on pp. 212–213 and Gripenberg [11]. They are defined as follows:

(A1) $a(t) \in C(0,\infty) \cap L^1(0,1)$.

(A2) $a(t)$ is positive and non-increasing for $t > 0$.

(A3) For each $T > 0$ the function $a(t)/a(t+T)$ is non-increasing in t for $0 < t < \infty$.

In those references it is shown that when a has an infinite integral then the resolvent equation is

$$R(t) = a(t) - \int_0^t a(t-s)R(s)ds \tag{16}$$

and that

$$0 < R(t) \le a(t), \quad \int_0^\infty R(t)dt = 1. \tag{17}$$

When

$$\int_0^\infty a(t)dt = \alpha < \infty$$

then

$$\int_0^\infty R(t)dt = \frac{\alpha}{1+\alpha}$$

and that can simplify some of the calculations we previously saw with E.

In the work to follow, notice that if J is a positive constant, then $Ja(t)$ still satisfies (A1)–(A3).

In a sequence of papers we showed the advantages of transforming an integral equation

$$x(t) = b(t) - \int_0^t a(t-s)f(s,x(s))ds \tag{18}$$

using a variation of parameters formula of Miller [13, pp. 191-192] into

$$x(t) = z(t) + \int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right]ds, \tag{19}$$

with

$$z(t) = b(t) - \int_0^t R(t-s)b(s)ds. \tag{20}$$

Here are the steps. Starting with (18) and $b(t)$ continuous on $[0,\infty)$ while a satisfies (A1)–(A3) we have

$$\begin{aligned} x(t) &= b(t) - \int_0^t a(t-s)[Jx(s) - Jx(s) + f(s,x(s))ds] \\ &= b(t) - \int_0^t Ja(t-s)x(s)ds + \int_0^t Ja(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right]ds. \end{aligned}$$

The linear part is

$$z(t) = b(t) - \int_0^t Ja(t-s)z(s)ds \tag{21}$$

and the resolvent equation is

$$R(t) = Ja(t) - \int_0^t Ja(t-s)R(s)ds \tag{22}$$

so that by the linear variation-of-parameters formula we have

$$z(t) = b(t) - \int_0^t R(t-s)b(s)ds \tag{23}$$

and the non-linear variation of parameters formula then yields

$$x(t) = z(t) + \int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right]ds. \tag{24}$$

Miller [13, p. 192] points out that all steps are reversible and that will be important for our next step. This transformation was first given in [3] for integral equations of Caputo type and has since been worked out in a variety of situations. Many fine details are found in [1].

5 Extending the Transformation

Our transformation involves

$$x(t) = b(t) - \int_0^t a(t-s)f(s,x(s))ds \tag{25}$$

and that does not cover the sum of two operators as in

$$x(t) = g(t,x(t)) - \int_0^t a(t-s)f(s,x(s))ds \tag{26}$$

with g a contraction and the last term a compact mapping. But we are reminded of an old trick that can be found in Bellman [2, p. 35]. The idea is to say: if there is a solution $x(t)$, then we can identify $g(t,x(t))$ as the forcing function $b(t)$ and perform the transformation. Recall that Miller [13, p. 192] points out that all steps are reversible and that will be important for our next step. According to our transformation, this equation is transformed into

$$x(t) = b(t) - \int_0^t R(t-s)b(s)ds + \int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right]ds. \tag{27}$$

If we can show that this equation has a solution, then reversing the steps we have $b(t) = g(t,x(t))$ so that this last equation is now

$$x(t) = g(t,x(t)) + \int_0^t R(t-s)\left[x(s) - g(s,x(s)) - \frac{f(s,x(s))}{J}\right]ds. \tag{28}$$

Once again we are looking at the sum of two operators, one is the same original function, while the integral is the compact map.

Strategy In this form, the equation is more algebraic than it is differential. With the integral of R being 1, when we take the norm of both sides the integrand slips out as the sup of g and f, multiplied by the integral of R which is bounded by 1. Now in the derivation of the transformation the idea is to work inside a bounded mapping set in which we can take J so large that $x(s)$ dominates $f(s,x(s))/J$. But now we have $g(s,x(s))$ in the integrand but it is a contraction, so $x(s)$ can still dominate it AND $f(s,x(s))/J$. In many problems from applied mathematics f satisfies the "spring condition", f has the sign of x.

What this means is that we can very often get a mapping set as simply

$$M = \{\phi\colon \|\phi\| \leq \text{constant}\}$$

and that is going to be a set where $P : M \to M^o$. The problems become almost entirely algebraic.

Example 5.1 We will go through the details for an integral equation of the type of (1), namely

$$x(t) = g(t,x(t)) - \int_0^t C(t-s)f(s,x(s))ds. \tag{1}$$

Using the transformation and the nonlinear variation of parameters formula, we write (1) as

$$x(t) = g(t,x(t)) + \int_0^t R(t-s)\left[x(s) - g(s,x(s)) - \frac{f(s,x(s))}{J}\right]ds \tag{29}$$

where J is an arbitrary positive constant and

$$0 < R(t) \leq C(t), \quad \int_0^\infty R(s)ds = 1. \tag{30}$$

Compactness: There are many known conditions under which $\int_0^t R(t-s)h(s,\phi(s))ds$ maps bounded sets into compact sets. The prime example is $C(t) = t^{q-1}, 0 < q < 1$, which includes heat transfer problems as well as fractional differential equations of both Caputo and Riemann-Liouville type. See, for example, [13, pp. 207-213] and [9].

To get a bound on solutions we ask the "spring conditions"

$$x \neq 0 \implies xg(t,x) > 0, \quad xf(t,x) \geq 0. \tag{31}$$

In view of Krasnoselskii's theorem we ask that $g(t,0) = 0$ and that there exist $0 < \alpha < 1$ with

$$|g(t,x) - g(t,y)| \leq \alpha|x-y|, \tag{32}$$

for all $x, y \in \Re$, $t \geq 0$. Concerning f we ask that there exist a $K > 0$ and a $J > 0$ such that $|x| \leq K$ and $t \geq 0$ imply that

$$1 - \alpha \leq \frac{f(t,x)}{Jx} \leq 1, \quad x \neq 0. \tag{33}$$

Proposition 5.1 *For h continuous and bounded for ϕ bounded, let $\int_0^t R(t-s)h(s,\phi(s))ds$ map bounded sets in $(\mathcal{B}, \|\cdot\|_{[0,E]})$ into compact sets. Under conditions (30)–(33) with $\alpha \le 1/2$, $K > 0$ fixed and*

$$M = \{\phi \in \mathcal{B} \colon \|\phi\| \le K\},$$

there is a solution of (29) in M and if there is any other solution of (29), then it also resides in M. As M is bounded, all possible solutions of (29) share that bound.

Proof. We verify the conditions of Theorem 2.1. For a fixed $K > 0$ let $J > 0$ be such that (32) and (33) hold. Following (2) we change (29) to

$$x(t) = \lambda g\left(t, \frac{x(t)}{\lambda}\right) + \lambda \int_0^t R(t-s)\left[x(s) - g(s, x(s)) - \frac{f(s, x(s))}{J}\right]ds, \tag{34}$$

for $0 < \lambda \le 1$, and define $P_\lambda : M \to \mathcal{B}$ by $\phi \in M$ implies

$$(P_\lambda \phi)(t) = \lambda g(t, \phi(t)/\lambda) + \lambda \int_0^t R(t-s)\left[\phi(s) - g(s, \phi(s)) - \frac{f(s, \phi(s))}{J}\right]ds, t \ge 0.$$

Then, in view of (32) we have $|g(s,x)| \le a|x|$ and

$$1 - \alpha \le \frac{g(s,x)}{x} + \frac{f(s,x)}{Jx} \le 1 + \alpha$$

or

$$-\alpha \le \left[\frac{g(s,x)}{x} + \frac{f(s,x)}{Jx}\right] - 1 \le \alpha,$$

thus

$$\left|1 - \left(\frac{g(s,x)}{x} + \frac{f(s,x)}{Jx}\right)\right| \le \alpha.$$

It follows that for any $\phi \in M$ we take for $t \in [0, E]$

$$\begin{aligned} |(P_\lambda \phi)(t)| \le& \lambda |g(t, \phi(t)/\lambda)| + \int_0^t R(t-s)|\phi(s)| \left|1 - \left(\frac{g(s,\phi)}{\phi} + \frac{f(s,\phi)}{J\phi}\right)\right| ds \\ &\le \lambda\alpha|\phi(t)|/\lambda + \int_0^t R(t-s)\|\phi\|\alpha ds \\ &\le \|\phi\| \left[\alpha + \alpha \int_0^t R(t-s)ds\right] \le \alpha\left(1 + \int_0^E R(t-s)ds\right)\|\phi\| \end{aligned}$$

so, by $\alpha \le 1/2$ and in view of $\int_0^E R(t-s)ds < 1$ we conclude that $P_\lambda M \subset M^o$.

The conditions of Theorem 2.1 hold, so there is a solution in M which also contains any other solution of (29). This proves Proposition 5.1.

We may easily see that if there exist positive numbers m_1, m_2 with $1 - \alpha \le \frac{m_1}{m_2}$ and such that

$$m_1|x| \le |f(t,x)| \le m_2|x|, \quad t \in [0, E], |x| \le K,$$

then condition (33) is satisfied by taking $J = m_2$.

We note that the requirement that the contraction constant satisfies $\alpha \leq 1/2$ may be relaxed to $\alpha \leq 2/3$ by replacing (33) in Proposition 5.1 by the assumption that there exists a $J > 0$ such that f satisfies

$$\alpha \leq \frac{f(t,x)}{Jx} \leq 2(1-\alpha), \quad t \in [0,E], 0 \neq |x| \leq K. \tag{35}$$

Indeed, by (32) and (35) we take

$$\alpha - 1 \leq \frac{g(s,x)}{x} + \frac{f(s,x)}{Jx} - 1 \leq 1 - \alpha,$$

so

$$\left|\frac{g(s,x)}{x} + \frac{f(s,x)}{Jx} - 1\right| \leq 1 - \alpha,$$

and the result is obtained following the arguments in the proof of Proposition 5.1.

References

[1] Becker, L.C., Burton, T.A., and Purnaras, I.K. An inversion of a fractional differential equation and fixed points. *Nonlinear Dynamics and Systems Theory* **15** (4) (2015) 242–271.

[2] Bellman, Richard. *Stability Theory of Differential Equations*, McGraw-Hill, New York, 1953.

[3] Burton, T.A. Fractional differential equations and Lyapunov functionals. *Nonlinear Anal.: TMA* **74** (2011) 5648–5662.

[4] Burton, T.A. and Kirk, Colleen. A fixed point theorem of Krasnoselskii-Schaefer type. *Math. Nachr.* **189** (1998) 23–31.

[5] Burton, T. A. and Purnaras, I.K. Equivalence of differential, fractional differential, and integral equations: Fixed points by open mappings *Mathematics in Engineering, Science and Aerospace* **8**(3) (2017) 293–305.

[6] Burton, T.A. and Zhang, Bo. Fixed points and fractional differential equations: Examples. *Fixed Point Theory* **14** (2) (2013) 313–326.

[7] Burton, T.A. and Zhang, Bo. A NASC for equicontinuous maps for integral equations. *Nonlinear Dynamics and Systems Theory* **17** (3) (2017) 247–265.

[8] Corduneanu, C. *Integral Equations and Applications*, Cambridge, 1991.

[9] Diethelm, Kai. *The Analysis of Fractional Differential Equations*. Springer, Berlin Heidelberg, 2010

[10] Dwiggins, D.P. Fixed point theory and integral equations. *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis* **23** (2016) 47–57.

[11] Gripenberg, G. On positive, nonincreasing resolvent of Volterra equations. *J. Differential Equations* **30** (1978) 380–390.

[12] Kneser, H. Uber die Lösungen eines Systems gewöhnlicher Differential gleichungen das der Lipschitzschen Bedingung nicht genğt. *S. B. Preuss, Akad. Wiss. Phys.-Math. Kl.* (II4) (1923) 171–174.

[13] Miller, R.K. *Nonlinear Volterra Integral Equations*. Benjamin, Menlo Park, CA, 1971.

[14] Park, Sehie. Generalizations of Krasnosleskii fixed point theorem. *Nonlinear Analysis* **67** (2007) 3401–3410.

[15] Smart, D.R. *Fixed Point Theorems*. Cambridge, 1980.