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Solvability Criterion for Integro-Differential Equations with Degenerate Kernel in Banach Spaces

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Abstract: By means of the theory of generalized inversion of operators in Banach spaces, a solvability criterion and a general form of solutions for integro-differential equations with a degenerate kernel in Banach spaces have been established. The obtained results have been illustrated by examples.

Keywords: *integro-differential equation; degenerate kernel; Banach space; generalized invertible operator; general form of solutions.*

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1 Introduction

The investigation of the solvability of integro-differential equations is a problem the specific nature of which lies in the fact that the integro-differential operator has no inverse. Such equations in Euclidean spaces were considered in [1–4] and others.

Sufficient conditions for the existence and uniqueness of piecewise-continuous mild solutions of fractional integro-differential equations in a Banach space with non instantaneous impulses were obtained in [5]. In paper [6] V. Gupta and J. Dabas established the existence and uniqueness of solution for a class of impulsive fractional integro-differential equations with nonlocal boundary conditions.

In this paper, we propose a somewhat different approach to the study of integro-differential equations in Banach spaces. In its realization, the theory of generalized inversion of operators in Banach spaces is effectively used [7, 8].

The proposed approach can be used in the study of the phenomena of energy transfer and diffusion of neutrons, viscoelastic oscillations various systems and structures, in nuclear physics and the mathematical theory of biological populations (see [9–11]).

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2 Formulation of the Problem

Consider the integro-differential equation

$$z(t) - M(t) \int_a^b \big[W(s)z(s) + V(s)\dot{z}(s)\big] ds = f(t), \qquad (1)$$

where the operator-valued function $M(t)$ acts from the Banach space $\mathbf{B}_2$ into the Banach space $\mathbf{B}_1$ and is strongly continuous with the norm $|||M||| = \sup_{t\in\mathcal{I}} \|M(t)\|_{\mathbf{B}_2} = M_0 < \infty$, and the operator-valued functions $W(t)$ and $V(t)$ act from the Banach space $\mathbf{B}_1$ into the Banach space $\mathbf{B}_2$ and are strongly continuous with the norms $|||W||| = \sup_{t\in\mathcal{I}} \|W(t)\|_{\mathbf{B}_1} = W_0 < \infty$ and $|||V||| = \sup_{t\in\mathcal{I}} \|V(t)\|_{\mathbf{B}_1} = V_0 < \infty$, the vector- function $f(t)$ acts from the interval $\mathcal{I}$ into the Banach space $\mathbf{B}_1$: $f(t) \in \mathbf{C}(\mathcal{I}, \mathbf{B}_1) := \big\{f(\cdot) : \mathcal{I} \to \mathbf{B}_1,\ |||f||| = \sup_{t\in\mathcal{I}} \|f(t)\|\big\}$, $\mathbf{C}(\mathcal{I}, \mathbf{B}_1)$ is the Banach space of vector-functions continuous on $\mathcal{I}$ with values in $\mathbf{B}_1$.

By the solution $z(t)$ of the operator equation (1) we mean vector-functions such that $z(t) \in \mathbf{C}(\mathcal{I}, \mathbf{B}_1)$, $\dot{z}(t) \in \mathbf{C}^1(\mathcal{I}, \mathbf{B}_1)$, where $\mathbf{C}^1(\mathcal{I}, \mathbf{B}_1)$ is the Banach space of continuously differentiable vector-functions with the norm $|||z||| = \sum_{k=0}^{1} \sup_{t\in\mathcal{I}} \|z^{(k)}(t)\|\big\}$, where $z^{(k)}(t)$ is the k-th derivative $z(t)$. The derivative $\dot{z}(t)$ is understood in the sense of [12, p. 140].

The problem is to obtain a solvability criterion and to find the structure of solutions for the integro-differential equation (1).

3 Preliminary Information

Consider the linear integral Fredholm equation with a degenerate kernel

$$z(t) - M(t) \int_a^b N(s)z(s)ds = f(t), \qquad (2)$$

where the operator-valued function $N(t)$ acts from the Banach space $\mathbf{B}_1$ into the Banach space $\mathbf{B}_2$ and is strongly continuous with the norm $|||N||| = \sup_{t\in\mathcal{I}} \|N(t)\|_{\mathbf{B}_1} = N_0 < \infty$.

Denote: $D = I_{\mathbf{B}_2} - A$, $\quad A = \int_a^b N(s)M(s)\, ds$, $\quad D : \mathbf{B}_2 \to \mathbf{B}_2$. In [8] it is shown that if D is a bounded generalized invertible operator, then the integral operator L is generalized invertible.

In this case, there exist bounded projections $\mathcal{P}_{N(D)}$, $\mathcal{P}_{Y_D}$ onto the null space $N(D)$ and the subspace $Y_D = I_{\mathbf{B}_2} \ominus R(D)$ of the operator D, respectively [13] and the bounded generalized inverse operator D^- to the operator D [7].

The following theorem holds for the integral equation (2).

Theorem 3.1 *[14] Let $D : \mathbf{B}_2 \to \mathbf{B}_2$. Then the homogeneous (f(t)=0) integral equation (2) has a family of solutions*

$$z(t) = M(t)\mathcal{P}_{N(D)}c,$$

where c is an arbitrary element of the Banach space $\mathbf{B}_2$.

Under and only under the condition

$$\mathcal{P}_{Y_D} \int_a^b N(s)f(s)ds = 0$$

the nonhomogeneous integral equation (2) has a family of solutions

$$z(t) = M(t)\mathcal{P}_{N(D)}c + f(t) + M(t)D^- \int_a^b N(s)f(s)ds.$$

4 The Main Result

1. We obtain the solvability conditions for the general form of solutions of the equation (1).

We make the substitution $\dot{z}(t) = y(t)$ in (1), then

$$z(t) = \int_a^t y(s)ds + c_0, \quad c_0 \in \mathbf{B}_1. \tag{3}$$

Putting (3) in (1), we obtain the integral equation

$$y(t) - M(t) \int_a^b \Big[W(s) \int_a^s y(\tau)d\tau + V(s)y(s)\Big] ds = f(t) + M(t)W_0c_0, \tag{4}$$

where $W_0 = \int_a^b W(s)ds$, $W_0 : \mathbf{B}_1 \to \mathbf{B}_2$.

Changing the order of integration in the integral $\int_a^b W(s) \int_a^s y(\tau)d\tau ds$, we obtain from (4)

$$y(t) - M(t) \int_a^b N(s)y(s)ds = g(t), \tag{5}$$

where

$$N(s) = \int_s^b W(\tau)d\tau + V(s),$$

$$g(t) = f(t) + M(t)W_0c_0. \tag{6}$$

By Theorem 3.1, under and only under the condition

$$\mathcal{P}_{Y_D} \int_a^b N(s)g(s)ds = 0 \tag{7}$$

the integral equation (5) has a family of solutions

$$y(t) = M(t)\mathcal{P}_{N(D)}c + g(t) + M(t)D^{-} \int_a^b N(s)g(s)ds, \tag{8}$$

where c is an arbitrary element of the Banach space $\mathbf{B}_2$.

From the solvability condition (7) we find the value of $c_0 \in \mathbf{B}_1$, for which the integral equation (5) has a solution. We put (6) in (7)

$$\mathcal{P}_{Y_D} \int_a^b N(s)\Big[f(s) + M(s)W_0c_0\Big]ds = 0.$$

After the transformations, we obtain the operator equation

$$Sc_0 = b_0, \tag{9}$$

where

$$b_0 = -\mathcal{P}_{Y_D} \int_a^b N(s)f(s)ds.$$

$$S = \mathcal{P}_{Y_D} \int_a^b N(s)M(s)W_0ds = \mathcal{P}_{Y_D}AW_0 = \mathcal{P}_{Y_D}\big[I - D\big]W_0 = \mathcal{P}_{Y_D}W_0,$$

because $\mathcal{P}_{Y_D}D = 0$.

Let the operator $S : \mathbf{B}_1 \to \mathbf{B}_2$ be generalized invertible. Then there exist bounded projectors $\mathcal{P}_{N(S)} : \mathbf{B}_1 \to \mathbf{B}_1$ and $\mathcal{P}_{Y_S} : \mathbf{B}_2 \to \mathbf{B}_2$ and a bounded generalized inverse operator $S^{-} : \mathbf{B}_2 \to \mathbf{B}_1$ to the operator S. The operator equation (9) is solvable under and only under the condition [7]

$$\mathcal{P}_{Y_S}b_0 = \mathcal{P}_{Y_S}\mathcal{P}_{Y_D} \int_a^b N(s)f(s)ds = 0, \tag{10}$$

and, under this condition, the equation (9) has a family of solutions

$$c_0 = \mathcal{P}_{N(S)}\tilde{c} + S^{-}b_0,$$

where $\tilde{c}$ is an arbitrary element of the Banach space $\mathbf{B}_1$.

Then $g(t)$ takes the form

$$g(t) = f(t) + M(t)W_0\Big[\mathcal{P}_{N(S)}\tilde{c} + S^{-}b_0\Big].$$

We put $g(s)$ in the solution (8) of the integral equation (5)

$$\begin{aligned} y(t) = {} & M(t)\mathcal{P}_{N(D)}c + f(t) + M(t)W_0\Big[\mathcal{P}_{N(S)}\tilde{c} + S^{-}b_0\Big] + \\ & + M(t)D^{-} \int_a^b N(s)\Big\{f(s) + M(s)W_0\Big[\mathcal{P}_{N(S)}\tilde{c} + S^{-}b_0\Big]\Big\}ds. \end{aligned} \tag{11}$$

Denoting $\widetilde{D} = (I_{\mathbf{B}_1} + D^{-}A)W_0$, after the transformations we obtain the general solution of the equation (5)

$$y(t) = M(t)\Big[\mathcal{P}_{N(D)},\ \ \widetilde{D}\mathcal{P}_{N(S)}\Big]\left[\begin{array}{c} c \\ \tilde{c} \end{array}\right] + f(t) +$$

$$+M(t)D^{-}\int_a^b N(s)f(s)ds - M(t)\widetilde{D}S^{-}\mathcal{P}_{Y_D}\int_a^b N(s)f(s)ds =$$

$$= M(t)\Big[\mathcal{P}_{N(D)},\ \ \widetilde{D}\mathcal{P}_{N(S)}\Big]\left[\begin{array}{c} c \\ \tilde{c} \end{array}\right] + f(t) +$$

$$+M(t)\Big[D^{-} - \widetilde{D}S^{-}\mathcal{P}_{Y_D}\Big]\int_a^b N(s)f(s)ds,$$

where $c \in \mathbf{B}_2$, $\tilde{c} \in \mathbf{B}_1$ are arbitrary constants.

Putting the obtained $y(t)$ in (2), we obtain the general solution of the integro-differential equation (1)

$$z(t) = \Big[\widetilde{M}(t)\mathcal{P}_{N(D)},\ \ \widetilde{M}(t)\big(\widetilde{D}\mathcal{P}_{N(S)} + \mathcal{P}_{N(S)}\big)\Big]\left[\begin{array}{c} c \\ \tilde{c} \end{array}\right] + \tilde{f}(t) + F(t),$$

where

$$\begin{gathered} \widetilde{M}(t) = \int_a^t M(s)ds, \ \ \tilde{f}(t) = \int_a^t f(s)ds, \\ F(t) = \Big\{\widetilde{M}(t)\Big[D^{-} - \widetilde{D}S^{-}\mathcal{P}_{Y_D}\Big] - S^{-}\mathcal{P}_{Y_D}\Big\}\int_a^b N(s)f(s)ds. \end{gathered} \tag{12}$$

Thus, the following theorem holds for the integro-differential equation (1).

Theorem 4.1 *Let the operators $D : \mathbf{B}_2 \to \mathbf{B}_2$ and $S : \mathbf{B}_1 \to \mathbf{B}_2$ be generalized invertible. Then the integro-differential equation (1) is solvable for those and only those $f(t) \in \mathbf{C}([a,b],\mathbf{B}_1)$, that satisfy the condition*

$$\mathcal{P}_{Y_S}\mathcal{P}_{Y_D}\int_a^b N(s)f(s)ds = 0$$

and has a family of solutions

$$z(t) = \Big[\widetilde{M}(t)\mathcal{P}_{N(D)},\ \ \big(\widetilde{M}(t)\widetilde{D}\mathcal{P}_{N(S)} + \mathcal{P}_{N(S)}\big)\Big]\left[\begin{array}{c} c \\ \tilde{c} \end{array}\right] + \tilde{f}(t) + F(t).$$

Remark 4.1 As shown in [3] the integro-differential equation

$$(Lz)(t) := \dot{z}(t) + H(t)z(t) - M(t)\int_a^b \big[W(s)z(s) + V(s)\dot{z}(s)\big]ds = f(t),$$

where the operator-valued function $H(t)$ acts from the Banach space $\mathbf{B}_1$ to the Banach space $\mathbf{B}_1$ and is strongly continuous with the norm $|||H||| = \sup_{t\in\mathcal{I}}\|H(t)\|_{\mathbf{B}_1} = H_0 < \infty$, with the help of substitution $z(t) = X(t)y(t)$, where $X(t)$ is the fundamental operator [12, p. 148] $\dot{z}(t) = -H(t)z(t)$, is reduced to an equation of the form (1).

Remark 4.2 The integro-differential equation

$$(Lz)(t) := \dot{z}(t) - \sum_{i=1}^{q} M_i(t) \int_a^b \big[W_i(s)z(s) + V_i(s)\dot{z}(s)\big] ds = f(t)$$

is reduced to an equation of the form (1), if we denote the operator matrices $M(t) = \big[M_1(t), M_2(t), \dots, M_q(t)\big]$, $W(t) = \mathrm{col}\big[W_1(t), W_2(t), \dots, W_q(t)\big]$, $V(t) = \mathrm{col}\big[V_1(t), V_2(t), \dots, V_q(t)\big]$.

2. In the case when the integro-differential equation is considered in Euclidean spaces, the proposed method of investigation can be refined and concretized.

Consider the equation (1) under the assumption that $M(t)$ is an $(n \times m)$-dimensional matrix, $W(t)$ and $V(t)$ are $(m \times n)$-dimensional matrices, $f(t)$ is an $(n \times 1)$-dimensional matrix whose elements belong to the space $\mathbf{L}_2[a, b]$. The solution will be sought in the class of functions $z(t) \in \mathbf{D}_2^n[a, b]$, $\dot{z}(t) \in \mathbf{L}_2^n[a, b]$.

In this case, the operator $D = I_m - A$, $\quad A = \int_a^b N(s)M(s)\, ds$ and orthoprojectors $P_{N(D)}$, $P_{N(D^*)}$ [15,16] are $(m \times m)$-dimensional matrices.

Let rank$D = n_1$. Denote an $(m \times r)$-dimensional matrix by $P_{N_r(D)}$, which is composed of $r = m - n_1$ linearly independent columns of the orthoprojector matrix $P_{N(D)}$, and an $(r \times m)$-dimensional matrix by $P_{N_r(D^*)}$, which is composed of r linearly independent rows of the orthoprojector matrix $P_{N(D^*)}$.

Then by Theorem 3.1, under and only under r linearly independent conditions

$$P_{N_r(D^*)} \int_a^b N(s)g(s)ds = 0 \tag{13}$$

the integral equation (5) has r linearly independent solutions

$$y(t) = M(t)P_{N_r(D)}c_r + g(t) + M(t)D^+ \int_a^b N(s)g(s)ds, \tag{14}$$

where c_r is an arbitrary element of the Euclidean space $\mathbf{R}^r$, D^+ is the Moor-Penrose pseudoinverse matrix to the matrix D [15,16].

From the condition (13) we obtain an algebraic system with respect to the vector $c_0 \in \mathbf{R}^n$

$$Sc_0 = b_0, \tag{15}$$

where $S = P_{N_r(D^*)}W_0$ is an $(r \times n)$-dimensional matrix, $b_0 = -P_{N_r(D^*)} \int_a^b N(s)f(s)ds$.

Let rank$S = n_2$. Denote an $(n \times k)$-dimensional matrix by $P_{N_k(S)}$, which is composed of $k = n - n_2$ linearly independent columns of the orthoprojector matrix $P_{N(S)}$, and an $(d \times r)$-dimensional matrix by $P_{N_d(S^*)}$, which is composed of $d = r - n_2$ linearly independent rows of the orthoprojector matrix $P_{N(S^*)}$.

The system (15) is solvable if and only if the vector b_0 satisfies the condition

$$P_{N_d(S^*)}b_0 = P_{N_d(S^*)}P_{N_r(D^*)} \int_a^b N(s)f(s)ds = 0, \tag{16}$$

under which the equation (15) has a family of solutions

$$c_0 = P_{N_k(S)}\tilde{c}_k + S^+ b_0,$$

where $\tilde{c}_k$ is an arbitrary element of the Euclidean space $\mathbf{R}^k$, S^+ is the Moor-Penrose pseudoinverse matrix to the matrix S [15,16].

The condition (16) consists of d linearly independent conditions. Indeed, since the matrices $P_{N_d(S^*)}$, $P_{N_r(D^*)}$ are of the full rank: $\text{rank}P_{N_d(S^*)} = d$, $\text{rank}P_{N_r(D^*)} = r$ and $d \le r$, we have from the Sylvester inequality [17, p. 31] that

$$\text{rank}P_{N_d(S^*)} + \text{rank}P_{N_r(D^*)} - r \le \text{rank}\big(P_{N_d(S^*)}P_{N_r(D^*)}\big) \le$$

$$\le \min\big(\text{rank}P_{N_d(S^*)}, \text{rank}P_{N_r(D^*)}\big)$$

or

$$d + r - r \le \text{rank}\big(P_{N_d(S^*)}P_{N_r(D^*)}\big) \le d.$$

It follows that $\text{rank}\big(P_{N_d(S^*)}P_{N_r(D^*)}\big) = d$.

Then the following theorem holds for the integro-differential equation (1).

Theorem 4.2 *Let* $\text{rank}D = n_1$, *and* $\text{rank}S = n_2$.

Then the integro-differential equation (1) is solvable for those and only those $f(t) \in \mathbf{R}^n$, *that satisfy* $d = r - n_2$ *linearly independent conditions*

$$P_{N_d(S^*)}P_{N_r(D^*)}\int_a^b N(s)f(s)ds = 0,$$

and at the same time it has an $(r+k)$*-parametric family of linearly independent solutions*

$$z(t) = \Big[\widetilde{M}(t)P_{N_r(D)},\ \ \big(\widetilde{M}(t)\widetilde{D}P_{N_k(S)} + P_{N_k(S)}\big)\Big]\begin{bmatrix} c_r \\ \tilde{c}_k \end{bmatrix} + \tilde{f}(t) + F(t),$$

where $c_r \in \mathbf{R}_r$, $\tilde{c}_k \in \mathbf{R}_k$ *are arbitrary constants;* $\widetilde{D} = (I_m + D^+A)W_0$*;* $\widetilde{M}(t), \tilde{f}(t)$ *have the form (12);*

$$F(t) = \Big\{\widetilde{M}(t)\Big[D^+ - \widetilde{D}S^+P_{N_r(D^*)}\Big] - S^+P_{N_r(D^*)}\Big\}\int_a^b N(s)f(s)ds.$$

Example 4.1 Consider the integro-differential equation

$$(Lz)(t) := \dot{z}(t) - M(t)\int_0^2 \left[W(s)z(s) + V(s)\dot{z}(s)\right]ds = f(t), \tag{17}$$

where

$$M(t) = \text{diag}\left\{\begin{bmatrix} 0 & t-1 & 0 \\ 1 & 0 & 3t \end{bmatrix}, \begin{bmatrix} 0 & t-1 & 0 \\ 1 & 0 & 3t \end{bmatrix}, \dots\right\},$$

$$W(s) = \text{diag}\left\{\begin{bmatrix} 0 & s-\frac{3}{2} \\ -\frac{3}{2} & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & s-\frac{3}{2} \\ -\frac{3}{2} & 0 \\ 1 & 0 \end{bmatrix}, \dots\right\},$$

$$V(s)=\operatorname{diag}\left\{\begin{bmatrix}1&0\\-1&0\\s-1&\frac{s-1}{2}\end{bmatrix},\begin{bmatrix}1&0\\-1&0\\s-1&\frac{s-1}{2}\end{bmatrix},\dots\right\}.$$

Let the vector-function $f(t)$ act from the interval $[0,2]$ into the Banach space $\mathbf{c}$ of all convergent numerical sequences: $f(t)\in\mathbf{C}([0,2],\mathbf{c}):=\{f(\cdot):[0,2]\to\mathbf{c}\}$, the operator-valued functions $M(t)$, $W(t)$ and $V(t)$ act from the Banach space $\mathbf{C}([0,2],\mathbf{c})$ to itself with the norms $|||M|||_{\mathbf{C}([0,2],\mathbf{c})}=\sup_{t\in[0,2]}\|M(t)\|_{\mathbf{c}}$, $|||W|||_{\mathbf{C}([0,2],\mathbf{c})}=\sup_{t\in[0,2]}\|W(t)\|_{\mathbf{c}}$, $|||V|||_{\mathbf{C}([0,2],\mathbf{c})}=\sup_{t\in[0,2]}\|V(t)\|_{\mathbf{c}}$.

It is obvious that the operator L is a linear bounded operator acting from the Banach space of continuously differentiable functions $\mathbf{C}^1([0,2],\mathbf{c})$ on the interval $[0,2]$ into the Banach space of continuous functions $\mathbf{C}([0,2],\mathbf{c})$.

For this equation we have:

$$N(s)=\int_s^2 W(s)ds+V(s)=$$

$$=\operatorname{diag}\left\{\begin{bmatrix}1&-1-\frac{s^2}{2}+\frac{3s}{2}\\-4+\frac{3s}{2}&0\\1&\frac{s-1}{2}\end{bmatrix},\begin{bmatrix}1&-1-\frac{s^2}{2}+\frac{3s}{2}\\-4+\frac{3s}{2}&0\\1&\frac{s-1}{2}\end{bmatrix},\dots\right\},$$

$$W_0=\int_0^2 W(s)ds=\operatorname{diag}\left\{\begin{bmatrix}0&-1\\-3&0\\2&0\end{bmatrix},\begin{bmatrix}0&-1\\-3&0\\2&0\end{bmatrix},\dots\right\}.$$

Then

$$D=I-A=I-\int_0^2 N(s)M(s)ds=\operatorname{diag}\left\{\begin{bmatrix}\frac{4}{3}&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}\frac{4}{3}&0&0\\0&0&0\\0&0&0\end{bmatrix},\dots\right\},$$

$$P_{N(D)}=\mathcal{P}_{Y_D}=\operatorname{diag}\left\{\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix},\dots\right\},$$

$$D^-=\operatorname{diag}\left\{\begin{bmatrix}\frac{3}{4}&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}\frac{3}{4}&0&0\\0&0&0\\0&0&0\end{bmatrix},\dots\right\}.$$

To find the solvability condition, we compute the operator

$$S=\mathcal{P}_{Y_D}W_0=\operatorname{diag}\left\{\begin{bmatrix}0&0\\-3&0\\2&0\end{bmatrix},\begin{bmatrix}0&0\\-3&0\\2&0\end{bmatrix},\dots\right\},$$

$$\mathcal{P}_{N(S)}=\operatorname{diag}\left\{\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix},\dots\right\},\mathcal{P}_{Y_S}=\operatorname{diag}\left\{\begin{bmatrix}1&0&0\\0&1&\frac{3}{2}\\0&0&0\end{bmatrix},\begin{bmatrix}1&0&0\\0&1&\frac{3}{2}\\0&0&0\end{bmatrix},\dots\right\}.$$

Then the solvability condition for the equation (17) takes the form

$$\mathcal{P}_{Y_S}\mathcal{P}_{Y_D}\int_0^2 N(s)f(s)ds=\operatorname{diag}\left\{\begin{bmatrix}0&0&0\\0&1&\frac{3}{2}\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&\frac{3}{2}\\0&0&0\end{bmatrix},\dots\right\}\times$$

$$\times \int_0^2 \mathrm{diag}\left\{\begin{bmatrix} 1 & -1-\frac{s^2}{2}+\frac{3s}{2} \\ -4+\frac{3s}{2} & 0 \\ 1 & \frac{s-1}{2} \end{bmatrix}, \begin{bmatrix} 1 & -1-\frac{s^2}{2}+\frac{3s}{2} \\ -4+\frac{3s}{2} & 0 \\ 1 & \frac{s-1}{2} \end{bmatrix}, \dots\right\} f(s)ds =$$

$$= \int_0^2 \mathrm{diag}\left\{\begin{bmatrix} 0 & 0 \\ \frac{3s-5}{2} & \frac{3(s-1)}{4} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ \frac{3s-5}{2} & \frac{3(s-1)}{4} \\ 0 & 0 \end{bmatrix}, \dots\right\} f(s)ds = 0.$$

After the transformations, we obtain the results which show that the components of the vector-function $f(t) = \mathrm{col}\big(f_1(t),\ f_2(t), \dots\big)$ must satisfy the conditions

$$\int_0^2 \Big[2(3s-5)f_{2k-1}(s) + 3(s-1)f_{2k}(s)\Big] ds = 0, \quad k = 1, 2, 3, \dots.$$

These conditions are satisfied, for example, by the vector $f(t) = \mathrm{col}(0, 1, 0, 1, 0, 1, \dots)$.

For this vector, the solution of the equation will have the form

$$z(t) = \mathrm{col}\left[\begin{bmatrix} \frac{t^2}{2} - t)c_2 \\ \frac{4-3t}{4}\tilde{c}_2 + \frac{3t^2}{2}c_3 \end{bmatrix}, \begin{bmatrix} \frac{t^2}{2} - t)c_4 \\ \frac{4-3t}{4}\tilde{c}_4 + \frac{3t^2}{2}c_5 \end{bmatrix}, \dots, \right].$$

Example 4.2 We find the conditions for the solvability of the integro-differential equation, which is considered in the finite-dimensional Euclidean space [1, 3]

$$z(t) - M(t)\int_0^{2\pi} W(s)z(s)ds = g(t),$$

where

$$M(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}, \quad W(s) = \frac{1}{2\pi}\begin{bmatrix} 0 & 0 \\ \cos s & \sin s \end{bmatrix},$$

$$V(s) = 0, \quad g(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} f(t).$$

For this equation we have

$$W(s) = \int_s^{2\pi} W(s)ds = \frac{1}{2\pi}\begin{bmatrix} 0 & 0 \\ \sin s & \cos s - 1 \end{bmatrix}, \quad W_0 = \int_0^{2\pi} W(s)ds = \frac{1}{2\pi}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$D = I_2 - A = I_2 - \int_0^{2\pi} W(s)M(s)ds = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_{N(D)} = \mathcal{P}_{Y_D} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrix $S = \mathcal{P}_{Y_D}W_0$ is zero, so $\mathcal{P}_{Y_S} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and the solvability condition (10) will have the form

$$\mathcal{P}_{Y_S}\mathcal{P}_{Y_D}\int_0^{2\pi} W(s)g(s)ds =$$

$$= \frac{1}{2\pi}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\int\limits_0^{2\pi}\begin{bmatrix} 0 & 0 \\ \sin s & \cos s - 1 \end{bmatrix}\begin{bmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{bmatrix} f(s).$$

After the transformations, we obtain the condition

$$\int\limits_0^{2\pi}\big[f_1(s)\sin s + f_2(s)(1-\cos s)\big]ds,$$

which completely coincides with the conditions from [1, 3], obtained by other methods.

The proposed research method can be used to study the solvability conditions for integro-differential systems of the Volterra type equations [18] in the case when the system is not everywhere solvable.

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