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Existence and Approximation of Solutions for Systems of First Order Differential Equations

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Abstract: The purpose of this paper is to present some general results concerning the existence of solutions for systems of differential equations. The existence results to be presented will be based on an effective procedure for constructing approximate solution. Namely, a numerical scheme using the Sinc function, in which it is shown that the solution converges exponentially. Furthermore, a numerical example and comparisons are presented to prove the validity of the suggested method.

Keywords: *fixed-point theory; numerical solutions; systems of differential equations; existence of solutions.*

Mathematics Subject Classification (2010): 26A33, 35F25, 35C10.

1 Introduction

In addition to its intrinsic mathematical interest, the theory of ordinary differential equations has extensive applications in many general fields, for instance, physics, chemistry, biology, economics and engineering. The existence and uniqueness of a solution to a first-order differential equation, given a set of initial conditions, is one of the most fundamental results of ordinary differential equations.

In this paper, we shall confine our discussion to systems of first order differential equations of the form

$$\begin{aligned} \tfrac{dx_1}{dt} &= F_1(x_1, x_2, ..., x_n, t), \\ \tfrac{dx_2}{dt} &= F_2(x_1, x_2, ..., x_n, t), \\ &\vdots \\ \tfrac{dx_n}{dt} &= F_n(x_1, x_2, ..., x_n, t), \end{aligned} \tag{1}$$

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that are often encountered in many mathematical models such as the Neutron flow, electrical networks, residential segregation. Also the first order differential equations are shown to be adequate models for various physical phenomena in the areas like damping laws, and diffusion processes. If $\mathbf{x} = (x_1, x_2, ..., x_n)^t$ denotes an element in R^n, the system of equations (1) can be written in a more compact form

$$\mathbf{x}' = F(t)\mathbf{x} + g(t, \mathbf{x}), \tag{2}$$

where $F(t)$ is a continuous $n \times n$ matrix function, its norm is defined as $\| F \| = \sup_{\|x\|=1} \| F\mathbf{x} \|$.

Consider a special case of the system of first order differential equations

$$\begin{aligned} \frac{du_1}{dt} &= b_{11}(t)u_1 + b_{12}(t)u_2 + f_1(t, u_1, u_2), \\ \frac{du_2}{dt} &= b_{21}(t)u_1 + b_{22}(t)u_2 + f_2(t, u_1, u_2) \end{aligned} \tag{3}$$

on the interval $[a, T]$ with the conditions

$$u_1(a) = u_1^0, \;\; u_2(a) = u_2^0. \tag{4}$$

We write equation (3) in a vector form as

$$\frac{d\vec{u}}{dt} = B(t)\vec{u} + \vec{f}(t, \vec{u}), \tag{5}$$

where B is the matrix $[b_{ij}(t)]_{2\times 2},\ i, j = 1, 2$ and $\vec{u} = (u_1, u_2)^t,\ \vec{f} = (f_1, f_2)^t$.

Many other authors have studied, under some conditions, the existence and uniqueness of solutions for systems of first-order differential equations. For example, in [8] Fransis and Miller examined fundamental and general existence theorems along with the Picard iterations. The author in [12] extended the version of Caratheodory's existence theorem for ordinary differential equations. While in [9], representation and approximation of the solutions to linear equations are studied. The authors in [14] used a recent Schauder-type result for discontinuous operators to study the existence of absolutely continuous solutions of first order initial value problems. Existence theorems for iterative differential equations as well as convergence theorems for a fixed point iteration designed to approximate the solutions are proved in [5]. Some applications of the fixed point theory to a nonlinear differential equations are presented in [7]. The fractional derivatives accurately describe natural phenomena that occur in general physical problems, existence and uniqueness of solutions for coupled systems of fractional differential equations are studied in [1]. The authors in [11], studied sufficient conditions for the existence of optimal controls for system of functional-differential equations.

The objectives of this paper are twofold. Firstly, we use the Schauder fixed point theorem to develop an existence theory for a general class of systems of first order differential equations (1), subject to the linear constraint

$$\ell\mathbf{x} = r, \tag{6}$$

where $r \in R^n$. Mainly, we will redo the first section from [2]. Secondly, we aim to implement the Sinc methodology to find approximate solutions for systems of differential equations (5).

2 Preliminary Results

Let us review some notations and facts that will be used in this paper. Let $C[\alpha, \beta]$ denote the Banach space of continuous functions such that $x(t)$ is a map that transforms the closed interval $[\alpha, \beta]$ into R^n, where the norm is defined as

$$\| x \| \equiv \max_{[\alpha,\beta]} \| x(t) \| . \tag{7}$$

We also define the domain $\mathbf{D} = \{(t, \mathbf{x}), t \in [\alpha, \beta], \mathbf{x} \in R^n\}$, then we assume the following three hypotheses:

H1. $F(t)$ is a square matrix of size n in which each entry is a continuous function on $[\alpha, \beta]$.

H2. $g(t, \mathbf{x})$ is continuous on the domain $\mathbf{D}$ defined above.

H3. $\mathbf{M} = \mathbf{M}(\alpha, \beta)$ is a bounded linear mapping from $C[\alpha, \beta]$ into R^n with bound $\| \mathbf{M} \| \equiv \sup_{\|\mathbf{x}\|=1} \| \mathbf{Mx} \|$.

A function $\mathbf{x}(t) \in C[\alpha, \beta]$ which has a continuous derivative that satisfies equation (2) on $[\alpha, \beta]$ is called the solution to (2). We consider the following mappings that are defined from the space $C[\alpha, \beta]$ into R^n

$$\begin{aligned} &\mathbf{M_0 x} \equiv \mathbf{x}(\alpha), \\ &\mathbf{M_1 x} \equiv (\mathbf{x}_1(t_1), \mathbf{x}_2(t_2), ..., \mathbf{x}_n(t_n)), \\ &\mathbf{M_2 x} \equiv \textstyle\int_\alpha^\beta \mathbf{x}(\tau) d\tau. \end{aligned} \tag{8}$$

In order to prove our main result in this paper, we do need a sequence of lemmas.

Lemma 2.1 *The above defined mappings* $\mathbf{M_0}, \mathbf{M_1}$, *and* $\mathbf{M_2}$ *satisfy* $\| \mathbf{M_0} \| = 1$, $\| \mathbf{M_1} \| \leq n$ *and* $\| \mathbf{M_2} \| = \beta - \alpha$.

Proof: Linearity of the mappings is obvious. For bounds, we have

$$\| \mathbf{M_0}(\mathbf{x}) \| = \| \mathbf{x}(\alpha) \| \leq \max_{[\alpha,\beta]} \| \mathbf{x}(\mathbf{t}) \| .$$

Using Equation (7), we arrive at $\| \mathbf{M_0} \| \leq \| \mathbf{x} \|$. For $\mathbf{x}(t) \equiv c \in R^n$ we obtain the equality $\| \mathbf{M_0}(\mathbf{x}) \| = \mathbf{1}$. For the mapping $\mathbf{M_1}(\mathbf{x})$, we have

$$\| \mathbf{M_1}(\mathbf{x}) \| = \sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{n}} |\mathbf{x}(\mathbf{t_i})| \leq \mathbf{n} \| \mathbf{x} \|,$$

which implies that $\| \mathbf{M_1} \| \leq n$. Finally,

$$\| \mathbf{M_2}(\mathbf{x}) \| = \| \int_\alpha^\beta \mathbf{x}(\tau) \mathbf{d}\tau \| \leq \int_\alpha^\beta \| \mathbf{x}(\tau) \| \mathbf{d}\tau \leq (\beta - \alpha) \| \mathbf{x} \| .$$

For equality, let $\mathbf{x}(t) \equiv c \in R^n$, that is, $\|\mathbf{M_2}\| = (\beta - \alpha)$.

Lemma 2.2 *(Schauder, see [6]) Let B be a convex compact subset of a normal linear space X, then any continuous mapping L from B into B has a fixed point in B.*

Lemma 2.3 *If B is a closed convex subset of the Banach space X, and L is a continuous mapping of B into B such that $L(B)$ is relatively compact, then L has a fixed point in B.*

Proof: As $L(B)$ is a subset of the closed set B, then $cl(L(B))$ is also a subset of B. Let $\hat{B}$ denote the closed convex hull of $cl(L(B))$. By Lemma 2.2, $\hat{B}$ is compact and is also the smallest closed convex set containing $cl(L(B))$, this would imply that $\hat{B}$ is subset of B. Therefore $L(\hat{B}) \subset L(B) \subset cl(L(B)) \subset \hat{B}$. By Lemma 2.2, L has a fixed point in $\hat{B}$. We would like to impose one more hypothesis in addition to **H1**, **H2** and **H3** mentioned above:

H4 The initial value problem

$$\mathbf{x}' = F(t)\mathbf{x}, \quad \ell\mathbf{x} = r, \tag{9}$$

for any $r \in R^n$, the problem in (9) has a unique solution. Therefore, for the $n-$dimensional space of solutions, call it $\mathcal{F}$, to the problem $\mathbf{x}' = F(t)\mathbf{x}$, $(\ell|\mathcal{F})^{-1}$ exists, i.e., the null space of $\ell|\mathcal{F}$ is $\{0\}$.

For the application of Lemma 2.3, we define an appropriate mapping together with the following lemma.

Lemma 2.4 *If $F(t)$ satisfies* **H1**, *then the problem*

$$\mathbf{x}' = \mathbf{F}(t)\mathbf{x} + z(t), \ \mathbf{x}(t_0) = r \tag{10}$$

has a unique solution $\omega(t)$ on $[\alpha, \beta]$ for any $t_0 \in [\alpha, \beta)$, $z \in C[\alpha, \beta]$, and $r \in R^n$. Moreover, for any t, t_0 in $[\alpha, \beta]$,

$$\|\omega(t)\| \leq \|\omega(t_0)\| \exp\Big|\int_{t_0}^{t} \|F(\tau)\| d\tau\Big| + \Big|\int_{t_0}^{t} \exp\Big|\int_{\tau}^{t} \|F(s)\| ds\Big| \|z(\tau) d\tau\|\Big|. \tag{11}$$

The existence and uniqueness of solutions to such initial value problems is well known [10].

Lemma 2.5 *If* **H1**, **H3** *and* **H4** *are satisfied, then the problem*

$$\mathbf{x}' = \mathbf{F}(t)\mathbf{x} + z(t), \ \ell\mathbf{x} = r \tag{12}$$

has a unique solution $\omega(t)$ for any $r \in R^n$ and $z \in C[\alpha, \beta]$.

Proof: Let

$$\omega(t) \equiv \omega_0(t) + (\ell|\mathcal{F})^{-1}(-\ell\omega_0) + (\ell|\mathcal{F})^{-1} r, \tag{13}$$

where ω_0 is the unique solution to

$$\mathbf{x}' = \mathbf{F}(t)\mathbf{x} + z(t), \ \mathbf{x}(\alpha) = 0. \tag{14}$$

It is easily verified by differentiation that $\omega(t)$ is a solution to (12). If $\omega_1(t), \omega_2(t)$ are two solutions to (12), then $\omega_1(t) - \omega_2(t)$ is the unique solution to

$$\mathbf{x}' = \mathbf{F}(t)\mathbf{x}, \quad \ell\mathbf{x} = 0. \tag{15}$$

Hence, by property **H4**, $\omega_1(t) \equiv \omega_2(t)$, that is, $\omega(t)$ is the unique solution to (12).

Lemma 2.6 *Suppose* **H1** *and* **H4** *are satisfied. If* $\hat{\omega}_0(t)$ *is the unique solution to (12) with* $r = 0$*, then*

$$\|\hat{\omega}_0(t)\| \leq \int_\alpha^\beta \Big(\exp \int_\tau^\beta \|F(s)\|ds\Big)\|z(\tau)\|d\tau\Big(1 + \|(\ell|\mathcal{F})^{-1}\ell\|\Big). \tag{16}$$

In particular,

$$\|\hat{\omega}_0\| \leq K_1\|z\|, \tag{17}$$

where $K_1 = (\beta - \alpha)\exp\int_\alpha^\beta \|F(s)\|ds(1 + \|(\ell|\mathcal{F})^{-1}\ell\|)$.

Proof: We have from (2.7) that

$$\hat{\omega}_0(t) = \omega_0(t) + (\ell|\mathcal{F})^{-1}(-\ell\omega_0), \tag{18}$$

where $\omega_0(t)$ is the unique solution to the initial value problem

$$\mathbf{x}' = F(t)\mathbf{x} + z(t), \;\; \mathbf{x}(\alpha) = 0. \tag{19}$$

Thus

$$\|\hat{\omega_0}\| \leq \|\omega_0\| + \|(\ell|\mathcal{F})^{-1}\ell\| \; \|\omega_0\| \leq \|\omega_0\|(1 + \|(\ell|\mathcal{F})^{-1}\ell)\|. \tag{20}$$

From Lemma 2.4 with $t_0 = \alpha$ and $r = 0$, we have

$$\|\omega_0\| \leq \int_\alpha^\beta \Big(\exp\int_\tau^\beta \|F(s)\|ds\Big)\|z(\tau)\|d\tau \tag{21}$$

and so (16) holds.

The following notation will be used in this paper. Let H be a positive number, $\vec{H} = (H_1, H_2, ..., H_n)$, where each H_i is positive, and $H(t)$ be a continuous positive function for $\alpha \leq t \leq \beta$. Let

$$\begin{aligned}
&\psi(r) = \psi(t, r, \alpha, \beta) \equiv (\ell(\alpha, \beta)|\mathcal{F})^{-1}r,\\
&C(H) = C(H, r, \alpha, \beta) \equiv \{y \in C[\alpha, \beta] : \|y - \psi(r)\| \leq H\},\\
&D(H) = D(H, r, \alpha, \beta) \equiv \{(t, y) \in D(\alpha, \beta) : \|y - \psi(t; r)\| \leq H\},\\
&D(H, t) = D(H, r, t) \equiv \{y \in R^n : \|y - \psi(t; r)\| \leq H\},\\
&C(H(t)) = C(H(t), r, \alpha, \beta) \equiv \{y \in C[\alpha, \beta] : \|y(t) - \psi(t; r)\| \leq H(t), \forall t \in [\alpha, \beta]\},\\
&C(\vec{H}) = C(\vec{H}(t), r, \alpha, \beta) \equiv \{y \in C[\alpha, \beta] : \|y_i(t) - \psi_i(t; r)\| \leq H_i, \; i = 1, ..., n; \; t \in [\alpha, \beta]\}.
\end{aligned} \tag{22}$$

Note that $C(H), C(H(t))$, and $C(\vec{H})$ are closed, convex subsets of $C[\alpha, \beta]$. Note also that if $y \in C(H)$, then $(t, y(t)) \in D(H)$ and $y(t) \in C(H, t)$ for $t \in [\alpha, \beta]$. If **H2** is satisfied and $y \in C[\alpha, \beta]$, then $g(t, y(t))$ is continuous on $[\alpha, \beta]$. By Lemma 11, the problem

$$\mathbf{x}' = F(t)\mathbf{x} + g(t, y(t)), \; \ell\mathbf{x} = r, \tag{23}$$

has a unique solution. We denote this solution by $u(r, y) = u(t; r, y)$. Note that

$$u(r, y) = \psi(r) + u(0, y). \tag{24}$$

We now define a mapping L on $C[\alpha, \beta]$ by

$$L(y) \equiv u(r, y). \tag{25}$$

Note that if $Lx = x$, then x is a solution to the problem (2), (6).

Lemma 2.7 *If* **H1** *to* **H4** *are satisfied, then $L(C(H))$ is relatively compact in $C[\alpha,\beta]$.*

Proof: By Lemma 12 and equation (24), for $y \in C(H)$, we have

$$\begin{aligned} \|Ly\| &= \|u(r,y)\| \leq \|\psi(r)\| + \|u(0,y)\| \leq \|(\ell|\mathcal{F})^{-1}\|\|r\| + K_1 \max_{[\alpha,\beta]} \|g(t,y(t))\| \\ &\leq \|(\ell|\mathcal{F})^{-1}\|\|r\| + K_1 \max_{D(H)} \|g(t,z)\| \equiv B_1, \end{aligned}$$

that is, $L(C(H))$ is bounded by B_1. By Ascoli's theorem, it is sufficient to show that $L(C(H))$ is equicontinuous. For $y \in C(H), Ly = u(r,y)$ and

$$\|u'(t;r,y)\| = \|F(t)u(t;r,y) + g(t,y(t))\| \leq \|F\|B_1 + \max_{D(H)} \|g(t,z)\| \equiv B_2. \tag{26}$$

By the mean value theorem, for $t_1, t_2 \in [\alpha,\beta]$,

$$\|u(t_2;r,y) - u(t_1;r,y)\| \leq B_2|t_2 - t_1|. \tag{27}$$

Thus $L(C(H))$ is equicontinuous.

Lemma 2.8 *If* **H1** *to* **H4** *are satisfied, then L is continuous on $C(H)$.*

Proof: Let $\epsilon > 0$ be given. Since $D(H)$ is compact, g is uniformly continuous on $D(H)$. There exists $\delta > 0$ such that if (t_1,x_1) and (t_2,x_2) are in $D(H)$ and $|t_1 - t_2| + \|x_1 - x_2\| < \delta$, then $\|g(t_1,x_1) - g(t_2,x_2)\| \leq \epsilon/K_1$, where K_1 is defined in Lemma 12. If $y_1, y_2 \in C(H)$, then $Ly_1 - Ly_2$ is the solution to

$$\mathbf{x}' = F(t)\mathbf{x} + g(t,y_1(t)) - g(t,y_2(t)),\ \ell\mathbf{x} = 0. \tag{28}$$

By Lemma 12, we have

$$\|Ly_1 - Ly_2\| \leq K_1 \max_{[\alpha,\beta]} \|g(t,y_1(t)) - g(t,y_2(t))\|. \tag{29}$$

If $\|y_1 - y_2\| < \delta$, that is, $\|y_1(t) - y_2(t)\| < \delta$ for $t \in [\alpha,\beta]$, then

$$\max_{[\alpha,\beta]} \|g(t,y_1(t)) - g(t,y_2(t))\| < \frac{\epsilon}{K_1},\ \ \|Ly_1 - Ly_2\| < K_1\frac{\epsilon}{K_1} = \epsilon. \tag{30}$$

Hence, L is continuous on $C(H)$.

3 Existence Results

With the aid of the preceding lemmas we can prove our main results.

Theorem 3.1 *Suppose* **H1** *to* **H4** *are satisfied. If there exists $H > 0$ such that*

$$M(H) = M(H,r,\alpha,\beta) \equiv \sup_{y\in C(H)} \|u(0,y)\| \leq H, \tag{31}$$

then problem (2), (6) has a solution $\mathbf{x}(\mathbf{t}) \in C(H)$.

Proof: From (24) we have $Ly - \psi(r) = u(r, y) - \psi(r) = u(0, y)$. Thus, for $y \in C(H)$, (31) yields

$$\|Ly - \psi(r)\| = \|u(0, y)\| \leq H, \tag{32}$$

that is, $Ly \in C(H)$, and $L(C(H)) \subset C(H)$. Since $C(H)$ is closed and convex, we can conclude from Lemmas 2.7 and 2.8 and the Schauder theorem in the form of Lemma 2.3 that L has a fixed point $x \in C(H)$; that is, problem (2),(6) has a solution $\mathbf{x} \in C(H)$. □

We may easily obtain some natural generalization of Theorem 31.

Theorem 3.2 *Suppose* **H1** *to* **H4** *are satisfied. If there exists a positive, continuous function $H(t)$ on $[\alpha, \beta]$ such that*

$$M(t, H(t)) \equiv \sup_{y \in C(H(t))} \|u(t; 0, y)\| \leq H(t), \tag{33}$$

for $t \in [\alpha, \beta]$, then the problem (2), (6) has a solution $\mathbf{x} \in C(H(t))$.

Proof: We have $Ly(t) - \psi(t; r) = u(t; r, y) - \psi(t; r) = u(t; 0, y)$. The condition (33) implies that, for $y \in C(H(t))$, (31) yields

$$\|Ly(t) - \psi(t; r)\| = \|u(t; 0, y)\| \leq H(t), \tag{34}$$

for $t \in [\alpha, \beta]$. Thus $Ly \in C(H(t))$, that is, $L(C(H(T))) \subset C(H(t))$. If $H \equiv \max_{[\alpha,\beta]} H(t)$, then $C(H(t)) \subset C(H)$. Moreover, $L(C(H(t))) \subset L(C(H))$. By Lemma 2.7, $L(C(H))$ is a relatively compact subset of $C[\alpha, \beta]$; hence, $L(C(H(t)))$ is relatively compact. By Lemma 2.8, L is continuous on $C(H)$; hence L is continuous on $C(H(t))$. Since $C(H(t))$ is a closed, convex subset of $C[\alpha, \beta]$, we may conclude from Lemma 2.3 that L has a fixed point $\mathbf{x}$ in $C(H(t))$, that is, problem (2),(6) has a solution $\mathbf{x} \in C(H(t))$.

In what follows, we use the Sinc methodology to find a numerical solution for equation (5).

4 Description of the Sinc Approximation

The goal of this section is to recall notations and definitions of the Sinc function that will be used in this paper. These are discussed in [3,15]. The Sinc function is defined on the whole real line R by

$$sinc(x) = \frac{\sin(\pi x)}{\pi x}, \quad x \in R. \tag{35}$$

Recall that a radial basis function is a function whose value depends only on the distance of its input to a central point. For a series of nodes equally spaced h apart, the Sinc function can be written as a radial basis function:

$$S(j, h)(kh) = \delta_{jk}^{(0)} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases} \tag{36}$$

Let

$$\delta_{kj}^{(-1)} = \frac{1}{2} + \delta_{kj} = \int_0^{k-j} \frac{\sin(\pi t)}{\pi t} dt.$$

We define a matrix $I^{(-1)}$ whose $(k, j)th$ entry is given by $\delta_{kj}^{(-1)}$. If a function $f(x)$ is defined on the real line, then for $h > 0$, the series

$$C(f, h)(x) = \sum_{j=-\infty}^{\infty} f(jh) sinc\Big(\frac{x - jh}{h}\Big), \quad j = 0, \mp 1, ...$$

is called the Whittaker cardinal expansion, which has been extensively studied in [13,15]. In practice, we need to use a finite number of terms in the above series, say $j = -N, ..., N$, where N is the number of Sinc grid points. For a restricted class of functions known as the Paly-Weiner class, which are entire functions, the Sinc interpolation and quadrature formulae are exact [15]. A less restricted class of functions that are analytic only on an infinite strip containing the real line, and that allow specific growth restriction has exponentially decaying absolute errors in the Sinc approximation.

Definition 4.1 Let $\mathcal{D}_d$ denote the infinite strip domain of width $2d,\ d > 0$, given by

$$\mathcal{D}_d = \{w = u + iv : |v| < d \leq \pi/2\}.$$

To construct an approximation on the interval $\Gamma = (a, T)$, which is our space interval in this paper, we consider the conformal map $\phi(x) = \ln(\frac{x-a}{T-x})$, the map ϕ carries the eye-shaped region

$$\mathcal{D} = \Big\{ z = x + iy : \Big| \arg\Big(\frac{z-a}{T-z}\Big)\Big| < d \leq \pi/2 \Big\}$$

onto the infinite strip $\mathcal{D}_d$. For the Sinc method, the basis functions on the interval Γ at $z \in \mathcal{D}$ are derived from the composite translated Sinc functions

$$S_j(z) = S(j,h) \circ \phi(z) = sinc\Big(\frac{\phi(z) - jh}{h}\Big).$$

The function $z = \phi^{-1}(w) = \frac{a+T\exp(w)}{1+\exp(w)}$ is an inverse mapping of the $w = \phi$. We define the range of ϕ^{-1} on the real line as

$$\Gamma = \{\phi^{-1}(y) \in \mathcal{D} : -\infty < y < \infty\} = (0, T).$$

The Sinc grid points $z_k \in \Gamma$ in $\mathcal{D}$ will be denoted by x_k, because they are real, and are given by

$$x_k = \phi^{-1}(kh) = \frac{a + T\exp(kh)}{1 + \exp(kh)},\quad k = 0, \mp 1, \mp 2, ...$$

To further explain the Sinc method, an important class of functions is denoted by $\mathbf{L}_\alpha(\mathcal{D})$. The properties of the functions in $\mathbf{L}_\alpha(\mathcal{D})$ and detailed discussion are given in [15]. We recall the following definition followed by two theorems for our purpose.

Definition 4.2 Let $\mathbf{L}_\alpha(\mathcal{D})$ be the class of all analytic functions f in $\mathcal{D}$, for which there is a number C_0 such that, for $\rho(z) = \exp(\phi(z))$, we have

$$|f(z)| \leq C_0 \frac{|\rho(z)|^\alpha}{[1 + |\rho(z)|]^{2\alpha}},\ \forall z \in \mathcal{D}.$$

The class $\mathbf{L}_\alpha(\mathcal{D})$ is important in Sinc methodology since it guarantees the rapid convergence of Sinc approximations. In the next theorem, we shall give a general formula for approximating the integral $\int_a^\nu F(u)du, \nu \in \Gamma$. To this end, we state the following result, which we will use to approximate the obtained integral equation.

Theorem 4.1 *Let* $\frac{F(t)}{\phi'(t)} \in \mathbf{L}_\alpha(\mathcal{D})$*, with* $0 < \alpha \leq 1$*,* $\delta_{jk}^{(-1)}$ *be defined as above,* N *be a positive integer, and* h *be selected as* $h = \sqrt{\pi d/(\alpha N)}$*, then there exists a positive constant* K *independent of* N*, such that*

$$\Big| \int_a^{t_k} F(t)dt - h \sum_{j=-N}^{N} \delta_{jk}^{(-1)} \frac{F(t_k)}{\phi'(t_k)} \Big| \leq K \exp(-\sqrt{\pi d \alpha N}).$$

It is convenient, for deriving an approximate solution for the system (5) by the Sinc-Galerkin method, to start with the scalar first order differential equation

$$\frac{du}{dt} = B(t)u(t) + f(t),\ t \in (a,T) \tag{37}$$

subject to the initial condition

$$u(a) = u^0. \tag{38}$$

Integrating with respect to t, and using the initial condition we arrive at the integral equation

$$u(t) = \int_a^t \Big[B(\tau)u(\tau) + f(\tau)\Big] d\tau + u^0. \tag{39}$$

To approximate over $\Gamma = (a,T)$, we make use of the conformal mapping $\phi(x)$ mentioned above. We also assume that both B/ϕ' and f belong to the class of functions $\mathbf{L}_\alpha(\mathcal{D})$. If B is a matrix, this shall imply that all the components of B/ϕ' (or f as a vector) are in the class $\mathbf{L}_\alpha(\mathcal{D})$. Now in equation (39) we collocate via the use of the indefinite integration formula (as in Theorem 4.1). We use the notation $\mathcal{D}(1/\phi'(t_i)) = diag[1/\phi'(t_{-N}), ..., 1/\phi'(t_N)]$, then equation (39) can be written as a system of $m = 2N + 1$ linear equations

$$U = hI_m^{(-1)}\mathcal{D}(B/\phi'(t_i))U + hI_m^{(-1)}\mathcal{D}(1/\phi'(t_i))F + U^0, \tag{40}$$

where $U = [u_{-N}, ..., u_N]^t$, $F = [f_{-N}, ..., f_N]^t$ with the nodes $t_i = \phi^{-1}(ih)$ for $i = -N, ..., N$ where $h = \sqrt{\pi d/\alpha N}$, and U^0 denotes the vector of $2N + 1$ constant values $U^0 = [u^0_{z_{-N}}, ..., u^0_{z_N}]^t$. Define the matrices A and E by $A = hI_m^{(-1)}\mathcal{D}(B/\phi'(t_i))$, $E = hI_m^{(-1)}\mathcal{D}(1/\phi'(t_i))$. Then equation (40) can be written as

$$U = AU + EF + U^0. \tag{41}$$

To prove convergence of the method, we evaluate the integral in (39) at the nodes t_i, where $i = -N, ..., N$, to get

$$u(t_i) = \int_a^{t_i} \Big[B(\tau)u(\tau) + f(\tau)\Big] d\tau + u^0$$

with the same matrices A, E and U^0 as mentioned above, and using the approximation in Theorem 4.1 we get, in matrix form, the approximation

$$U + AU + EF + U^0 + \tilde{K}\exp(-\sqrt{\pi d\alpha N}),$$

where the constant $\tilde{K}$ is a vector such that each entry is bounded by the constant K in Theorem 4.1. So, the error ERR can be bounded as

$$\|ERR\| \le \|U - (AU + EF + U^0)\| \le \tilde{K}\exp(-\sqrt{\pi d\alpha N}),$$

i.e., the discretization error that arises when a differential equation is replaced by a discrete system of algebraic equations is exponentially small. With the notation as above, we just proved the following theorem.

Theorem 4.2 *Let $B/\phi', f \in \mathbf{L}_\alpha(\mathcal{D})$, let the function $u(t)$ be defined as in (39), and let the matrix U be defined as in (41). Then for $h = \sqrt{\pi d/(\alpha N)}$ there exists a constant K independent of N such that*

$$\sup \|[u(t_i)] - U\| \leq K \exp(-\sqrt{\pi d \alpha N}).$$

Now, we may attempt to solve the linear systems of equations (41) by successive approximations, that is, by means of the iterative scheme:

$$U^{(n+1)} = AU^{(n)} + EF + U^0. \tag{42}$$

It is easy to show that the convergence of the scheme depends on the ℓ^∞ norm of B, as noted in the following theorem.

Theorem 4.3 *The sequence $U^{(n)}$ defined in (42) converges, for all N being sufficiently large, to the exact solution provided that $(T-a) < 11/(10\|B\|_\infty)$.*

Proof: Recall that by definition of $\delta^{(-1)}$ as defined in Section 2, it satisfies the inequality [15, p. 172] $\delta^{(-1)} \leq 11/10$, we have

$$\begin{aligned}
\|B\|_\infty &= \|hI_m^{(-1)}\mathcal{D}(1/\phi'(t_i))\| = |\max_i \sum_{j=-N}^{N} h\delta_{i-j}^{(-1)}(B(z_j)/\phi'(z_j))| \\
&\leq \frac{11}{10} h \sum_{j=-N}^{N} (B(z_j)/\phi'(z_j)) \approx \frac{11}{10} \int_a^T |B(t)| dt \leq \frac{11}{10}(T-a) \sup_{t\in(a,T)} |B(t)| \\
&\leq \frac{11}{10}(T-a)\|B\|_\infty,
\end{aligned}$$

where in the third inequality we used Theorem 4.1, with the fact that $B/\phi' \in \mathbf{L}_\alpha(\mathcal{D})$. For the iteration scheme to converge we require that $\|B\|_\infty < 1$. Therefore we can achieve convergence of the scheme (42) by choosing $(T-a) < \frac{11}{10\|B\|_\infty}$.

It remains to show that the approximate solution U^* of node values of equation (41) converges to the node values of the exact solution U (see, [4]). For that end, choose a constant R so that U and $U*$ belong to the ball $\mathcal{B} = \{X : \|X\|_\infty < \frac{11}{10}\|B\|_\infty < R/2\}$. It is enough to show that $|U - U^*|$ is small. If U^* is the approximate solution and satisfies equation (41), then $(U - U^*) - A(U - U^*) = Error$, or

$$\|(U - U^*)\| \leq \|A(U - U^*)\| + \|Error\|. \tag{43}$$

Now we can find a small constant r, that is $0 < r < 1$ such that the Jacobian of the matrix A is less than r, so by the mean-value theorem, we obtain $\|U - AU\| \leq r\|U - U^*\|$, so equation (43) reduces to

$$\|(U - U^*)\| \leq \frac{1}{1-r}\|Error\|. \tag{44}$$

This shows that the approximate solution is sufficiently close to the exact solution. With the above notations, we have proved the following theorem.

Theorem 4.4 *For a constant $R > 0$, with $\|B\|_\infty < 1$, the solution in equation (5) with the iteration scheme (41) converges to the unique solution.*

t	Errors $x(t)$	Errors $y(t)$
0.1	1.048 E -07	2.098 E -08
0.3	3.549 E -07	1.560 E -08
0.6	2.963 E -06	3.905 E -08
0.9	4.998 E -06	4.848 E -08
1.2	1.009 E -05	3.555 E -08
1.5	2.286 E -05	9.098 E -08
0.8	5.201 E -05	8.948 E -08
1.8	7.579 E -05	8.011 E -08

Table 1: Numerical results for the example given in 45. Comparison between the Sinc solution and the exact solution.

5 Test Example

Consider the initial value problem in equation (2) of the form

$$\mathbf{X}'(\mathbf{t}) = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} \mathbf{X}(\mathbf{t}) + \begin{pmatrix} e^{2t} \\ 1 \end{pmatrix}, \quad \mathbf{X}(\mathbf{0}) = \begin{pmatrix} -1 \\ 0, \end{pmatrix}, \tag{45}$$

that is, we consider equation (2) with $n = 2$, $F(t) = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}$, $g(t, X) = \begin{pmatrix} e^{2t} \\ 1 \end{pmatrix}$ and $r = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. Since $F(t)$ and $g(t, \mathbf{X})$ are continuous, **H1** and **H2** are satisfied. Assumption **H3** can be established by Lemma (8). While assumption **H4** is an immediate consequence of Lemma (34). To prove the existence of the solution for (45), with the fact that in our case $u(t; r; y)$ is given by $u(t; r, y) = re^t + \int_0^t e^{t-s} \begin{pmatrix} e^{2s} \\ 1 \end{pmatrix} ds$, it is easy to manipulate the steps of Example 4.2 in [2]. To show the efficiency of the Sinc method in comparison with the exact solution of the given equation, which is known to be

$$\mathbf{X}(\mathbf{t}) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -\frac{9}{2}e^t - \frac{5}{6}e^{-t} + \frac{4}{3}e^{2t} - 3 \\ \\ -\frac{3}{2}e^t - \frac{5}{6}e^{-t} + \frac{1}{3}e^{2t} + 2 \end{pmatrix},$$

we use the Sinc method to solve the problem in (45) with the parameters $d = \frac{1}{2}, \alpha = 1$ and $N = 32$. In Table 1, the comparison of the numerical results demonstrates the accuracy of this approach.

Conclusions

The study of systems of ODEs is still a very active area of research due to its application in modeling various physical, chemical, biological, engineering and social systems. This paper mainly focused on the application of the Schauder fixed point theorem to study the existence of solutions for systems of ODE. On the other hand, a numerical scheme using Sinc functions is developed to approximate the solution of a 2×2 system of first order differential equations. The numerical results demonstrate the reliability and efficiency of using the Sinc method to solve such problems. The error in the numerical solution is shown to converge exponentially.

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