Nonlinear Dynamics and Systems Theory, **18** (3) (2018) 297–306

Solving Two-Dimensional Integral Equations of Fractional Order by Using Operational Matrix of Two-Dimensional Shifted Legendre Polynomials

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Received: November 4, 2017; Revised: June 16, 2018

Abstract: In this paper, we first present a new numerical method for solving two-dimensional integral equations of fractional order. The method is based upon two-dimensional shifted Legendre polynomials. Then we construct an operational matrix for two-dimensional fractional integral. Also, we give the error analysis. Finally, three examples are shown to confirm the theoretical results.

Keywords: *two-dimensional shifted Legendre polynomials; two-dimensional fractional integral equations; operational matrix.*

Mathematics Subject Classification (2010): 45AXX, 26A33.

1 Introduction

In this paper, we present a numerical method for the solution of two-dimensional Volterra integral equations of fractional order in the form

$$f(x,y) - \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} k(x,y,s,t) f(s,t)\, dtds = g(x,y),$$
$$(r_1,r_2) \in (0,\infty)\times(0,\infty), f \in L^1(\Omega), \Omega := [0,l_1]\times[0,l_2]. \tag{1}$$

In [1–3] the authors mentioned that equation (1) is a solution for a class of impulsive partial hyperbolic differential equations involving the Caputo fractional derivative. Therefore, researchers are interested in solving this kind of equations. In recent years, several numerical methods for solving two-dimensional integral equations of fractional

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order have been presented in the literature. Najafalizadeh and Ezzati in [4] used a two-dimensional block pulse operational matrix to solve two-dimensional nonlinear integral equations of fractional order. In [5], two-dimensional orthogonal triangular functions were used for solving two-dimensional integral nonlinear equations of fractional order. In [6], we see that the operational matrix of two-dimensional Bernstein polynomials is used for two-dimensional integral equations of fractional order. Here, we would like to use two-dimensional shifted Legendre polynomials for solving two-dimensional integral equations of fractional order. Firstly, we present some preliminaries in fractional calculus. In Section 3, we review some general concepts concerning one-dimensional and two-dimensional shifted Legendre polynomials, and derive an operational matrix of two-dimensional shifted Legendre polynomials for two-dimensional integration of fractional order. Section 4 is devoted to solving two-dimensional nonlinear fractional integral equations by applying the operational matrix of integration of fractional order. Section 5 represents an error estimation for the presented method. In Section 6, we show accuracy and efficiency of this method through several examples. Finally, a conclusion is given in Section 7.

2 Brief Review of Fractional Calculus

In this section, we present a short introduction of the fractional calculus which will be used in this paper.

Definition 2.1 [7]. The Riemann-Liouville fractional integral operator I^α of order $\alpha \geq 0$ is defined as

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{(\alpha-1)} f(t)\, dt \ , \alpha > 0, \tag{2}$$

where $x > 0$ and $\Gamma(.)$ is the Euler gamma function.

The Riemann-Liouville integral satisfies the following properties:

- $I^\alpha I^\beta f(x) = I^{\alpha+\beta} f(x)$,
- $I^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+1+\beta)} x^{\alpha+\beta}$.

Definition 2.2 [8]. The left-sided mixed Riemann-Liouville integral of order r of f is defined as

$$I_\theta^r f(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t)\, dtds, \tag{3}$$

where $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$ and $\theta = (0,0)$.

Some properties of the left-sided mixed Riemann-Liouville integral are the following:

- $I_\theta^\theta f(x,y) = f(x,y)$,
- if $p, g \in (-1,\infty)$ then, $I_\theta^r x^p y^q = \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+1+r_1)\Gamma(q+1+r_2)} x^{p+r_1} y^{q+r_2}$.

3 Shifted Legendre Polynomials

3.1 One-dimensional shifted Legendre polynomials

Let $L_i(x)$, $x \in [0.l]$, be the shifted Legendre polynomials. Then $L_i(x)$ can be obtained with the aid of the following recurrence formula [9]:

$$L_{i+1}(x) = \frac{2i+1}{i+1}(\frac{2x}{l} - 1)L_i(x) - \frac{i}{i+1}L_{i-1}(x), \quad i = 1, 2, 3, ...,$$

where $L_0(x) = 1, L_1(x) = \frac{2x}{l} - 1$.

The shifted Legendre polynomials on $[0, l]$ have the following properties:

- $L_i(x) = \sum_{k=0}^{i} \frac{(-1)^{(i+k)}(i+k)!}{(i-k)!l^k(k!)^2} x^k$,
- $\int_0^l L_i(x)L_j(x)\,dt = \begin{cases} \frac{l}{2i+1}, & for\ i = j, \\ 0, & for\ i \neq j, \end{cases}$
- A function $f(x) \in C[0, l]$ can be expanded by shifted Legendre polynomials in the following form:

$$f(x) \backsimeq \sum_{i=0}^{m} c_i L_i(x) = C^T \Phi(x),$$

where the coefficients c_i are given by

$$c_i = \frac{(2i+1)}{l} \int_0^l L_i(x)y(x)\,dt,$$

and the vectors C, $\Phi(x)$ are given by

$$C^T = [c_0, c_1, ..., c_m], \tag{4}$$
$$\Phi(x) = [L_0(x), L_1(x), ..., L_m(x)]^T. \tag{5}$$

Previously, in [10] the operational matrix of shifted Legendre polynomials for fractional integration in the interval $[0, 1]$ has been presented. Now we present the operational matrix of shifted Legendre polynomials for fractional integration in the interval $[0, l]$ as follows:

$$\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{(\alpha-1)} \Phi(x)\,dt = P^\alpha \Phi(x) \tag{6}$$

and

$$P^\alpha = \begin{pmatrix} \sum_{k=0}^{0} \theta_{0,0,k} & \sum_{k=0}^{0} \theta_{0,1,k} & \cdot & \cdot & \cdot & \sum_{k=0}^{0} \theta_{0,m,k} \\ \sum_{k=0}^{1} \theta_{1,0,k} & \sum_{k=0}^{1} \theta_{1,1,k} & \cdot & \cdot & \cdot & \sum_{k=0}^{1} \theta_{1,m,k} \\ \vdots & \vdots & \cdot & \cdot & \cdot & \vdots \\ \sum_{k=0}^{i} \theta_{i,0,k} & \sum_{k=0}^{i} \theta_{i,1,k} & \cdot & \cdot & \cdot & \sum_{k=0}^{i} \theta_{i,m,k} \\ \vdots & \vdots & \cdot & \cdot & \cdot & \vdots \\ \sum_{k=0}^{m} \theta_{m,0,k} & \sum_{k=0}^{m} \theta_{m,1,k} & \cdot & \cdot & \cdot & \sum_{k=0}^{m} \theta_{m,m,k} \end{pmatrix},$$

where $\theta_{i,j,k}$ is given by

$$\theta_{i,j,k} = (2j+1)\sum_{s=0}^{j} \frac{(-1)^{(s+j+s+k)}(i+k)!(j+s)!l^{\alpha+s-k}}{(i-k)!k!\Gamma(k+\alpha+1)(j-s)!(s!)^2(k+s+\alpha+1)},$$

and P^{α} is called the shifted Legendre polynomials operational matrix for fractional integration.

The proof is similar to the proof of Theorem 3 in [10].

3.2 Two-dimensional shifted Legendre polynomials

The two-dimensional shifted Legendre polynomials are defined on $\Omega = [0, l_1] \times [0, l_2]$ as follows [11]:

$$\psi_{m,n}(x, y) = L_m(x)L_n(y), \quad m, n = 0, 1, 2, \cdots,$$

where $L_m(x)$ and $L_n(y)$ are shifted Legendre polynomials which are defined in the same way as on the intervals $[0, l_1]$ and $[0, l_2]$, respectively. In the following, we study the important properties of the two-dimensional shifted Legendre polynomials.

The two-dimensional shifted Legendre polynomials are orthogonal with each other

$$\int_0^{l_1}\int_0^{l_2} \psi_{m,n}(x, y)\psi_{i,j}(x, y)\, dydx = \begin{cases} (\frac{l_1 l_2}{(2m+1)(2n+1)}), & i = m, j = n, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that $\Theta = L^2(\Omega)$, the inner product in this space is defined by

$\langle (f(x, y), g(x, y)\rangle = \int_0^{l_1}\int_0^{l_2} f(x, y)g(x, y)\, dydx,$

and the norm is as follows:

$\|f(x, y)\|_2 = \langle f(x, y), f(x, y)\rangle^{\frac{1}{2}} = (\int_0^{l_1}\int_0^{l_2} |f(x, y)|^2\, dydx)^{\frac{1}{2}}.$

For every $f(x, y) \in \Theta$, we have

$$f(x, y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f_{ij}\phi_{ij}(x, y). \tag{7}$$

If the infinite series in (7) is truncated, then we will have

$$f(x, y) \simeq \sum_{i=0}^{M}\sum_{j=0}^{N} f_{ij}\phi_{ij}(x, y) = F^T\phi(x, y) = \phi^T(x, y)F, \tag{8}$$

where $\phi(x, y)$ and F are $(M+1)(N+1) \times 1$ vectors of the following form

$$F = [f_{00}, ..., f_{0N}, ..., f_{M0}, ..., f_{MN}]^T, \tag{9}$$

$$\phi(x, y) = [\phi_{00}(x, y), ..., \phi_{0N}(x, y), ..., \phi_{M0}(x, y), ..., \phi_{MN}(x, y)]^T \tag{10}$$

and $\phi_{i,j}(x, y) = \phi_i(x).\phi_j(y)$.

The two-dimensional shifted Legendre polynomials coefficients $f_{i,j}$ are obtained by

$$f_{i,j} = \frac{\langle f(x, y), \phi_{i,j}(x, y)\rangle}{\|\phi_{i,j}(x, y)\|_2^2}.$$

By using the Kronecker product of $\phi(x)$ and $\phi(y)$ we can show $\phi(x, y)$ as

$$\phi(x, y) = \phi(x) \otimes \phi(y), \tag{11}$$

where $\otimes$ denotes the Kronecker product defined for two arbitrary matrices A and B as

$$A \otimes B = (a_{i,j} B),$$

also it has the following two basic properties [12]:

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD), \qquad (A + B) \otimes C = A \otimes C + B \otimes C. \tag{12}$$

Similarly, the function $k(x, y, s, t)$ in $L^2(\Omega \times \Omega)$ can be expanded in terms of two shifted Legendre polynomials as

$$k(x, y, s, t) \simeq \phi^T(x, y) K \phi(s, t), \tag{13}$$

where K is a block matrix of the form

$$K = [K^{(i,m)}]_{i,m=0}^{M}$$

in which

$$K^{(i,m)} = [k_{ijmn}]_{j,n=0}^{N}, \quad i, m = 0, 1, ..., M$$

and the two-shifted Legendre polynomials coefficient k_{ijmn} is given by

$$k_{ijmn} = \frac{\langle\langle k(x, y, s, t)\phi_{m,n}(s, t)\rangle, \phi_{i,j}(x, y)\rangle}{\|\phi_{i,j}(x, y)\|_2^2 \|\phi_{m,n}(s, t)\|_2^2}, \quad i, m = 0, 1, ..., M. \quad j, n = 0, 1, ..., N.$$

The product of two vectors $\phi(x, y)$ and $\phi^T(x, y)$ with the vector F is given by

$$\phi(x, y)\phi^T(x, y) F \simeq \tilde{F}\phi(x, y), \tag{14}$$

where F is defined by (9) and $\tilde{F}$ is an $(M + 1)(N + 1) \times (M + 1)(N + 1)$ matrix

$$\tilde{F} = [F^{(i,j)}]_{i,j=0,1,...M}, \tag{15}$$

where $F^{(i,j)}$, $i, j = 0, 1, ..., M$, are given by

$$F^{(i,j)} = \frac{2j + 1}{l_2} = \sum_{m=0}^{M} W_{i,j,m} \Lambda_m,$$

in which $W_{i,j,m}$ is defined as

$$W_{i,j,m} = \int_0^{l_1} L_i(\frac{2}{l_1}x - 1) L_j(\frac{2}{l_1}x - 1) L_m(\frac{2}{l_1}x - 1)\, dx.$$

and Λ_m, $m = 0, 1, ..., M$, are $(N + 1) \times (N + 1)$ matrices

$$[\Lambda_m]_{kh} = \frac{2h + 1}{l_1} = \sum_{n=0}^{N} \acute{W}_{k,h,n} f_{mn}, \quad k, h = 0., 1, ..., N,$$

where

$$\acute{W}_{k,h,n} = \int_0^{l_2} L_k(\frac{2}{l_2}y - 1) L_h(\frac{2}{l_2}y - 1) L_n(\frac{2}{l_2}y - 1)\, dy.$$

3.3 Operational matrix of fractional order

Now, we construct an operational matrix of two-dimensional shifted Legendre polynomials for the fractional integration.

By using equations (10), (11) we have

$$\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\phi(s,t)\,dtds =$$
$$\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\phi(s)\otimes\phi(t)\,dtds =$$
$$\frac{1}{\Gamma(r_1)}\int_0^x (x-s)^{r_1-1}\phi(s)\,ds\otimes\frac{1}{\Gamma(r_2)}\int_0^y (y-t)^{r_2-1}\phi(t)\,dt = *.$$

From equation (6) we get

$$\begin{aligned} * =& p^{r_1}\phi(x)\otimes p^{r_2}\phi(y)\\ =&(p^{r_1}\otimes p^{r_2})(\phi(x)\otimes\phi(y))\\ =&(p^{r_1}\otimes p^{r_2})\phi(x,y). \end{aligned}$$

Hence,

$$\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\phi(s,t)\,dtds = p^{r_1,r_2}\phi(x,y), \tag{16}$$

where

$$p^{r_1,r_2} = (p^{r_1}\otimes p^{r_2}).$$

4 Numerical Solution of Two-Dimensional Volterra Integral Equations of Fractional Order

In this section, we present an effective method to solve equation (1). For this purpose, by using the method mentioned in Section 3, the functions $f(x,y)$, $g(x,y)$ and $k(x,y,s,t)$ can be approximated by

$$\begin{aligned} f(x,y) &= \phi(x,y)^T F,\\ g(x,y) &= \phi(x,y)^T G,\\ k(x,y,s,t) &= \phi(x,y)^T K\phi(s,t), \end{aligned} \tag{17}$$

where $\phi(x,y)$ is defined in equation (10) and the vectors F,G and matrix K are two-dimensional shifted Legendre polynomials coefficients of $f(x,y), g(x,y)$ and $k(x,y,s,t)$, respectively. Now, substituting equation (17) in equation (1), we have

$$\begin{aligned} \phi^T(x,y)F - \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\phi^T(x,y)K\phi(s,t)\phi^T(s,t)F\,dtds\\ \simeq \phi^T(x,y)G. \end{aligned} \tag{18}$$

By using equations (14) and (16) we conclude that

$$\phi^T(x,y)F - \frac{\phi^T(x,y)K\tilde{F}}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\phi(s,t)\,dtds \simeq \phi^T(x,y)G, \tag{19}$$

$$\phi^T(x,y)F - \phi^T(x,y)K\tilde{F}P^{r_1,r_2}\phi(x,y) \simeq \phi^T(x,y)G. \tag{20}$$

If in the above equation we substitute $\simeq$ with $=$, we get the following equation

$$F - K\tilde{F}P^{r_1,r_2}\phi(x,y) = G. \tag{21}$$

Now we collocate equation (21) in $(M+1)(N+1)$ Newton-Cotes nodes as

$$x_m = \frac{2m+1}{2(M+1)}, y_n = \frac{2n+1}{2(N+1)}, m = 0,1,...,M, n = 0,1,...,N.$$

We will have a linear system of algebraic equations

$$F - K\tilde{F}P^{r_1,r_2}\phi(x_m,y_n) = G, m = 0,1,...,M, n = 0,1,...,N. \tag{22}$$

It is clear that, by solving this system, we can obtain the approximate solution of equation (1) according to equation (8).

5 Error Analysis

Theorem 5.1 . *[11] Let $\tilde{f}(x,y) = \sum_{i=0}^{M}\sum_{j=0}^{N} f_{ij}\phi_{ij}(x,y)$ be the two-dimensional shifted Legendre polynomials expansion of the real sufficiently smooth function $f(x,t)$ in Ω, then there exist real numbers C_1 , C_2 and C_3 such that*

$$\|f(x,y) - \tilde{f}(x,y)\|_2 \le C_1\frac{(\frac{l_1}{2})^{M+1}}{(M+1)!2^M} + C_2\frac{(\frac{l_2}{2})^{N+1}}{(N+1)!2^N} + C_3\frac{(\frac{l_1}{2})^{M+1}(\frac{l_2}{2})^{N+1}}{(M+1)!(N+1)!2^{M+N}}.$$

In the special case when $M = N$ and $l_1 = l_2 = 1$ we get

$$\|f(x,y) - \tilde{f}(x,y)\|_2 \le (C_1 + C_2 + C_3\frac{1}{(M+1)!2^{2M+1}})\frac{1}{(M+1)!2^{2M+1}},$$

hence

$$\|f(x,y) - \tilde{f}(x,y)\|_2 = O(\frac{1}{(M+1)!2^{2M+1}}).$$

Theorem 5.2 *Suppose $M = N$, $l_1 = l_2 = 1$ and $f(x,y)$ is an exact solution of the fractional integral equation (1) and $\tilde{f}(x,y)$ shows the approximate solution by the two-dimensional shifted Legendre polynomials. If $|(x-s)^{r_1-1}(y-t)^{r_2-1}k(x,y,s,t)| < C$, $f(x,y)$ and $k(x,y,s,t)$ are sufficiently smooth functions, then*

$$\|f(x,y) - \tilde{f}(x,y)\|_2^2 \le \frac{C^2}{(\Gamma(r_1)\Gamma(r_2)(M+1)!2^{2M+1})^2}(C_1 + C_2 + C_3\frac{1}{(M+1)!2^{2M+1}})^2.$$

Proof.

$$\|f(x,y)-\tilde{f}(x,y)\|_2^2 =$$

$$\frac{1}{(\Gamma(r_1)\Gamma(r_2))^2}\|\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}k(x,y,s,t)(f(s,t)-\tilde{f}(s,t))\,dtds\|_2^2$$

$$\leq \frac{1}{(\Gamma(r_1)\Gamma(r_2))^2}\int_0^x\int_0^y \|(x-s)^{r_1-1}(y-t)^{r_2-1}k(x,y,s,t)(f(s,t)-\tilde{f}(s,t))\|_2^2\,dtds$$

$$\leq \frac{C^2}{(\Gamma(r_1)\Gamma(r_2))^2}\int_0^x\int_0^y \|(f(s,t)-\tilde{f}(s,t))\|_2^2\,dtds$$

$$\leq \frac{C^2xy}{(\Gamma(r_1)\Gamma(r_2)(M+1)!2^{2M+1})^2}(C_1+C_2+C_3\frac{1}{(M+1)!2^{2M+1}})^2$$

$$\leq \frac{C^2}{(\Gamma(r_1)\Gamma(r_2)(M+1)!2^{2M+1})^2}(C_1+C_2+C_3\frac{1}{(M+1)!2^{2M+1}})^2. \ \square$$

6 Illustrative Examples

In this section we will implement our method by three examples. For justifying our method, we compare our computed results and those by other authors. Outcomes show the accuracy and the validity of the presented method. In these examples we let $l_1 = l_2 = 1$, $M = N$ and denote the following error function

$$e(x,y) = |f(x,y)-\tilde{f}_{M,N}(x,y)|,$$

where $f(x,y)$ and $\tilde{f}_{M,N}(x,y)$ are the exact and approximate solutions of the two-dimensional fractional integral equation, respectively.

Example 6.1 Consider the two-dimensional fractional integral equation given in [5]

$$f(x,y)-\frac{1}{\Gamma(3.5)\Gamma(3.5)}\int_0^x\int_0^y (x-s)^{2.5}(y-t)^{2.5}xyt^{\frac{1}{2}}f(s,t)\,dtds = \frac{1}{2}xy-\frac{x^{5.5}y^6}{9450}.$$

The exact solution of this equation is $f(x,y)=\frac{1}{2}xy$. Table 1 shows the absolute error obtained by using the present method and by using the 2D-Tf method [5].

Example 6.2 Consider the two-dimensional fractional integral equation given in [6]

$$f(x,y)-\frac{1}{\Gamma(3.5)\Gamma(2.5)}\int_0^x\int_0^y (x-s)^{2.5}(y-t)^{1.5}e^{-t}(y^2+s)f(s,t)\,dtds =$$

$$x^2e^y-\frac{1024x^{5.5}y^{2.5}(6x+13y^2)}{2027025\pi}$$

and the exact solution of the above equation is $f(x,y)=e^yx^2$. Table 2 shows the absolute error obtained by using the present method and by using the two-dimensional Bernstein polynomials method [6].

Example 6.3 As the last example, we have the two-dimensional fractional integral equation

$$f(x,y)-\frac{1}{\Gamma(3.5)\Gamma(3.5)}\int_0^x\int_0^y (x-s)^{3.5}(y-t)^{3.5}52\sqrt{tx}f(s,t)\,dtds = xy^2-\frac{x^5y^5}{5670}$$

	Present method	Present method	Method [5]
x=y	$m = 1$	$m = 2$	m=8
0.1	2.2349×10^{-6}	$7.84825\ \times10^{-8}$	1.126×10^{-4}
0.2	$2.13487\ \times10^{-6}$	$3.48089\ \times10^{-8}$	$1.363\ \times10^{-4}$
0.3	2.03717×10^{-6}	1.47598×10^{-7}	6.22×10^{-5}
0.4	1.94179×10^{-6}	2.60141×10^{-7}	1.27×10^{-5}
0.5	1.84874×10^{-6}	3.7269×10^{-7}	1.983×10^{-4}
0.6	1.758×10^{-6}	4.8549×10^{-7}	4.6×10^{-5}
0.7	1.66959×10^{-6}	5.9879×10^{-7}	5.2×10^{-5}
0.8	1.58351×10^{-6}	7.1281×10^{-7}	6.8×10^{-4}
0.9	1.49975×10^{-6}	8.2781×10^{-7}	6.8×10^{-4}

Table 1: Absolute error for Example 1.

	Present method	Present method	Method [6]
x=y	$m = 1$	$m = 2$	m=4
0.0	1.1458×10^{-2}	2.4215×10^{-5}	4.086×10^{-4}
0.1	1.1130×10^{-2}	$2.1511\ \times10^{-5}$	4.181×10^{-4}
0.2	$1.0799\ \times10^{-2}$	$1.9207\ \times10^{-5}$	$4.471\ \times10^{-4}$
0.3	1.0466×10^{-2}	1.7355×10^{-5}	4.970×10^{-4}
0.4	1.0131×10^{-2}	1.6000×10^{-5}	5.656×10^{-4}
0.5	9.7937×10^{-3}	1.5188×10^{-5}	6.474×10^{-4}
0.6	9.4538×10^{-3}	1.4957×10^{-5}	7.316×10^{-4}
0.7	9.1117×10^{-3}	1.5342×10^{-5}	7.817×10^{-4}
0.8	8.7676×10^{-3}	1.6374×10^{-5}	6.788×10^{-4}
0.9	8.4215×10^{-3}	1.8082×10^{-5}	1.004×10^{-4}

Table 2: Absolute error for Example 2.

and the exact solution of the above equation is $f(x, y) = xy^2$. Table 3 illustrates the numerical results for this example.

7 Conclusion

In this paper a general formulation for the two-dimensional shifted Legendre polynomials operational matrix of two-dimensional fractional integral equations has been derived. This matrix is used to approximate numerical solution of the two-dimensional nonlinear fractional integral equations. The properties of two-dimensional shifted Legendre polynomials and the operational matrices are used to reduce the two-dimensional fractional integral equations to a system of algebraic equations that can be solved easily. Finally, illustrative examples are presented to show the validity and the accuracy of the proposed method.

x=y	$m = 1$	$m = 2$	m=3
0.0	5.9600×10^{-3}	1.3080×10^{-6}	2.7534×10^{-7}
0.1	6.3096×10^{-3}	1.4595×10^{-6}	2.8692×10^{-7}
0.2	6.6175×10^{-3}	1.6017×10^{-6}	2.9819×10^{-7}
0.3	6.8844×10^{-3}	1.7349×10^{-6}	3.0917×10^{-7}
0.4	7.1110×10^{-3}	1.8596×10^{-6}	3.1989×10^{-7}
0.5	7.2982×10^{-3}	1.9763×10^{-6}	3.3038×10^{-7}
0.6	7.4466×10^{-3}	2.0852×10^{-6}	3.4065×10^{-7}
0.7	7.5570×10^{-3}	2.1868×10^{-6}	3.5073×10^{-7}
0.8	7.6301×10^{-3}	2.2815×10^{-6}	3.6063×10^{-7}
0.9	7.6668×10^{-3}	2.3696×10^{-6}	3.7036×10^{-7}

Table 3: Absolute error for Example 3.

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