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Generalized Synchronization Between Two Chaotic Fractional Non-Commensurate Order Systems with Different Dimensions

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Abstract: This paper deals with the problem of generalized synchronization between two chaotic and hyperchaotic fractional non-commensurate order systems with different dimensions. By designing an active control technique, the sufficient conditions for achieving generalized synchronization are derived by using the Laplace transform technique and final value theorem. Numerical simulations are also given to illustrate and validate the generalized synchronization results derived in this paper.

Keywords: *chaos; generalized synchronization; fractional non-commensurate order; active control.*

Mathematics Subject Classification (2010): 34A34, 37B25, 35B35, 93C83, 37C25, 37N30.

1 Introduction

Chaos synchronization phenomena have received increasing attention in the study of dynamical systems, because they can be applied in vast areas of engineering and information science, in particular, in secure communication, control processing and cryptology [1–4]. Various methods in chaos synchronization have been proposed [5–7]. Most of the synchronization methods focus on integer order chaotic systems in both continuous and discrete time.

Recently, fractional calculus has attracted a lot of attention and has become an excellent instrument to describe the dynamics of complex systems. Based on the stability

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criterion of linear fractional systems, many fractional-order chaotic systems can be synchronized [8–13].

An interesting aspect is the generalized type of synchronization called $Q-S$ synchronization. It has been investigated for integer order chaotic dynamical systems as well [14–18]. However, to the best of our knowledge, there are few treatments in the literature of the general scheme for generalized $Q-S$ synchronization of fractional non commensurate order systems with different dimensions.

In view of this consideration, this paper investigates an active control technique [15] for generalized synchronization between two different dimensional chaotic fractional non-commensurate order systems, using two suitable real matrices. Based on the Laplace transform technique and final value theorem, the designed control makes the fractional non-commensurate-order chaotic system states asymptotically synchronized. Numerical examples are given to verify the capability of the method.

The rest of the paper is organized as follows. In the following section, we present some basic concepts of fractional calculus fundamentals. In Section 3, we motivate the problem and give the main results. In Section 4, two examples are used to verify the effectiveness of the proposed method. Finally, some concluding remarks are given in Section 5.

2 Fractional Calculus Fundamentals

The three definitions used for the general fractional derivative are the Grunwald–Letnikov (GL) definition, the Riemann– Liouville (RL) and the Caputo definition [19]. The Riemann–Liouville fractional integral of order $\alpha > 0$ is given by

$$J_a^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad t > a, \tag{1}$$

where Γ is the gamma function. The Riemann–Liouville fractional-order derivative ${}_a^{RL}d_t^{\alpha} f$ is defined by

$${}_a^{RL}d_t^{\alpha} f(t) = d^m J_a^{m-\alpha} f(t), \tag{2}$$

where $m = \lceil \alpha \rceil$ is the first integer greater then α.

The Caputo fractional-order derivative ${}_a d_t^{\alpha} f$ is defined by

$${}_a d_t^{\alpha} f(t) = J_a^{m-\alpha} d^m f(t), \ m = \lceil \alpha \rceil. \tag{3}$$

The Grűnwald–Letnikov fractional-order derivative ${}_a^{GL}d_t^{\alpha} f$ is given by

$${}_a^{GL}d_t^{\alpha} f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{k=0}^{\frac{t-a}{h}} (-1)^k \left(\frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1))} \right) f(t-kh). \tag{4}$$

Recall that the Laplace transform of a function $f(t)$ is the function $F(s)$ defined as follows

$$F(s) = L\{f(t), s\} = \int_0^{+\infty} \exp(-st) f(t) dt, \tag{5}$$

$f(t)$ is called original which can be reconstituted from the inverse Laplace transform

$$f(t) = L^{-1}\left\{F(s), t\right\} = \int_{c-i\infty}^{c+i\infty} \exp(st)F(s)ds, \quad c = \Re(s) > 0. \tag{6}$$

Taking into account that the Laplace transform of the convolution is

$$L\left\{f(t) * g(t), s\right\} = F(s).G(s), \tag{7}$$

where $f(t)$ and $g(t)$ are two causal functions for $t < 0$, we see that $F(s)$ and $G(s)$ are their Laplace transforms.

Using the following property of the Laplace transform of conventional derivative

$$L\left\{f^{m}(t), s\right\} = s^{m}F(s) - \sum_{k=0}^{m-1} s^{k} f^{(m-k-1)}(0), \tag{8}$$

we obtain the Laplace transform of the Riemann-Liouville derivative

$$L\left\{{}^{RL}_{0}d^{\alpha}_{t} f(t), s\right\} = s^{\alpha}F(s) - \sum_{k=0}^{m-1} s^{k}\left[{}^{RL}_{0}d^{\alpha-k-1} f(t)\right]_{t=0} \tag{9}$$

with $m - 1 \leq \alpha < m$. This transform is well known. However its practical application is limited by the absence of the physical interpretation of the function at $t = 0$.

In view of the Laplace transform formula of the Riemann-Liouville integral, the Laplace transform of the Caputo fractional derivative is

$$L\left\{{}^{c}_{0}d^{\alpha} f(t), s\right\} = s^{\alpha}F(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0) \tag{10}$$

with $m - 1 \leq \alpha < m$. Since the initial conditions for the fractional differential equations with the Caputo derivative are of the same form as for the integer-order derivatives, which have clear physical meaning, the Caputo derivative is used in this paper.

Theorem 2.1 *(Final value theorem) Let $F(s)$ be the Laplace transform of function $f(t)$. If the indicated limits exist, then*

$$\lim_{t\to+\infty} f(t) = \lim_{s\to 0} sF(s). \tag{11}$$

Proof. See [20].

3 Problem of Synchronization and Analytical Results

Generally, we consider the following non-commensurate fractional order nonlinear system in the form

$$d^{\alpha}_{t} X = f(X). \tag{12}$$

We take (12) as the drive system. The controlled response system is given by

$$d^{\alpha}_{t} Y = g(Y) + U, \tag{13}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)$ is the vector of rational number between 0 and 1, d_t^{α} is the Caputo fractional derivative of order α, for $i = 1, 2, ..., m$, $X(t) \in \mathbb{R}^n$, $Y(t) \in \mathbb{R}^m$, $(m > n)$ are the state vectors of the drive system (12) and the response system (13), respectively, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^m \to \mathbb{R}^m$ are the non linear vector functions and $U \in \mathbb{R}^m$ is the control input vector.

Our goal is to design an appropriate active control U [15] such that the synchronization between the drive system (12) and the response system (13) is achieved for a given two suitable real matrices $Q = (q_{ij})$, $i = 1, 2, ..., d$, $j = 1, 2, ..., m$ and $S = (s_{kh})$, $k = 1, 2, ..., d$, $h = 1, 2, ..., n$. Particularly, Q and S are chosen such that $q_{ij} = s_{kh} = 0$, for all $i \neq j$ and $k \neq h$.

Hence, the error system is defined as

$$e(t) = QY(t) - SX(t), \tag{14}$$

which means that systems (12) and (13) are globally asymptotically synchronized, i.e.

$$\lim_{t\to+\infty} \|e(t)\| = \lim_{t\to+\infty} \|QY(t) - SX(t)\| = 0.$$

Most existing methods for synchronizing chaos with different dimensions are used only for reduced order or increased order. Motivated by the above idea, in this work, we discuss the two cases: $d = m$ and $d = n$.

3.1 Increased order

In this case assume that $d = m$. By submitting systems (12) and (13) into (14), the error system (14) can be expressed as

$$d_t^{\alpha} e(t) = Q d_t^{\alpha} Y(t) - S d_t^{\alpha} X(t), \tag{15}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)$. Hence

$$\begin{aligned} d_t^{\alpha} e(t) &= Q\left[g(Y(t)) + U\right] - Sf(X(t)) \\ &= A_1 e(t) + K\left(Y(t), X(t)\right) + QU, \end{aligned} \tag{16}$$

where

$$K\left(Y(t), X(t)\right) = -A_1 e(t) + Qg(Y(t)) - Sf(X(t)), \tag{17}$$

and $A_1 \in \mathbb{R}^{m\times m}$ is the linear part of system (13).

We redefine the control function $U = (u_1, u_2, ...u_m)^T$ to eliminate all terms which cannot be shown in the form e such that

$$QU = -K\left(y(t), x(t)\right) + Be(t), \tag{18}$$

and $B \in \mathbb{R}^{m\times m}$ is a feedback gain matrix to be determined. We find the error system as

$$d_t^{\alpha} e(t) = (A_1 + B)e(t). \tag{19}$$

Applying the Laplace transform for the previous system, letting

$$F_i(s) = L(e_i(t)), \quad i = 1, 2, ..., m, \tag{20}$$

and using the formula

$$L\left\{d_t^{\alpha_i} e_i(t)\right\} = s^{\alpha_i} F_i(s) - s^{\alpha_i - 1} e_i(0), \quad i = 1, 2, ..., m, \tag{21}$$

we find a new system

$$s^{\alpha} F(s) = s^{\alpha-1} e(0) + (A_1 + B) F(s), \tag{22}$$

where

$$F = (F_1, F_2, ..., F_m)^T, \quad s^{\alpha} = (s^{\alpha_1}, s^{\alpha_2}, ..., s^{\alpha_m}).$$

Hence, we have the following result.

Theorem 3.1 *If the matrix B is chosen such that all poles of $sF_i(s)$ lie in the open left half plane, then the drive system (12) and response system (13) are globally generally synchronized.*

Proof. Suppose that the matrix B is chosen such that all poles of $sF_i(s)$ lie in the open left half plane. Using Theorem 2.1, we have

$$\lim_{t \to +\infty} e_i(t) = \lim_{s \to 0^+} sF_i(s) = 0, \quad \text{for all } i = 1, 2, ..., m.$$

This means that the drive system (12)and the response system (13) achieve the synchronization. □

3.2 Reduced order

In this case assume that $d = n$. Using the notation (14), the error system can be derived as

$$d_t^{\alpha} e(t) = A_2 e(t) + H\left(Y(t), X(t)\right) + QU, \tag{23}$$

where

$$H\left(Y(t), X(t)\right) = -A_2 e(t) + Qg(Y(t)) - Sf(X(t)), \tag{24}$$

and $A_2 \in \mathbb{R}^{n \times n}$ is the linear part of system (12).

We redefine the control function $U = (u_1, u_2, ...u_n, 0, 0..., 0)^T$ to eliminate all terms which cannot be shown in the form $e = (e_1, e_2, ...e_n)^T$ such that

$$Q^0 U^0 = -H\left(Y(t), X(t)\right) + Ce(t), \tag{25}$$

where $U^0 = (u_1, u_2, ...u_n)^T$, $C \in \mathbb{R}^{n \times n}$ is a feedback gain matrix to be determined and $Q^0 = diag(Q_{11}, Q_{22}, ..., Q_{nn})$. Then the error system is changed to

$$d_t^{\alpha} e(t) = (A_2 + C) e(t). \tag{26}$$

Applying the Laplace transform for the previous system, letting

$$F_i(s) = L(e_i(t)), \quad i = 1, 2, ..., n, \tag{27}$$

and using the formula

$$L\left\{d_t^{\alpha_i} e_i(t)\right\} = s^{\alpha_i} F_i(s) - s^{\alpha_i - 1} e_i(0), \quad i = 1, 2, ..., n, \tag{28}$$

we find a new system

$$s^{\alpha} F(s) = s^{\alpha-1} e(0) + (A_1 + C) F(s), \tag{29}$$

where

$$F = (F_1, F_2, ..., F_n)^T, \quad s^{\alpha} = (s^{\alpha_1}, s^{\alpha_2}, ..., s^{\alpha_n}).$$

Hence, we have the following result.

Theorem 3.2 *If the matrix C is chosen such that all poles of $sF_i(s)$ lie in the open left half plane, then the drive system (12) and response system (13) are globally generally synchronized.*

Proof. The proof is similar to that of Theorem 3.1.

4 Numerical Examples

In this section, we present some simulation examples to illustrate our proposed general method.

4.1 Simulation results (Increased order)

In this case , we assume that the new memristor-based simplest chaotic circuit system of non-commensurate fractional order (MBSCCS) [21] is the drive system. The dynamic of the circuit is described by the mathematical model

$$\begin{cases} d_t^{\alpha_1} x_1(t) = a_1 x_2, \\ d_t^{\alpha_2} x_2(t) = -b_1(x_1 + M(x_3)x_2), \\ d_t^{\alpha_3} x_3(t) = -x_2 - c_1 x_3 + x_2^2 x_3. \end{cases} \tag{30}$$

In (30), x_1, x_2, x_3 are the states, $a_1, b_1, c_1, \beta, \gamma$ are the positive parameters, M is the memristor function defined by

$$M(x_3(t)) = \gamma x_3^2(t) - \beta, \tag{31}$$

and $\alpha_i, i = 1, 2, 3$ are rational numbers between 0 and 1.

For all numerical simulation, we take the initial states of system (30) as

$$x_1(0) = 0.1, x_2(0) = -0.5, x_3(0) = 1. \tag{32}$$

The parameters values are taken as

$$(a_1, b_1, c_1, \beta, \gamma) = (1, \frac{1}{3}, 0.9, 3, 0.4). \tag{33}$$

The proposed fractional orders are taken as

$$(\alpha_1, \alpha_2, \alpha_3) = (0.97, 0.98, 0.99). \tag{34}$$

The system (30) exhibits chaotic behaviour as shown in Figure 1.

The linear part A_2 of system (30) is given by

$$A_2 = \begin{pmatrix} 0 & a_1 & 0 \\ -b_1 & b_1\beta & 0 \\ 0 & -1 & -c_1 \end{pmatrix}.$$

Assume that the fractional-order hyperchaotic Lorenz system [22] is the response system. The controlled hyperchaotic Lorenz system is expressed by the mathematical model

$$\begin{cases} d_t^{\alpha_1} y_1(t) = a_2(y_2 - y_1) + y_4 + u_1, \\ d_t^{\alpha_2} y_2(t) = c_2 y_1 - y_2 - y_1 y_3 + u_2, \\ d_t^{\alpha_3} y_3(t) = y_1 y_2 - b_2 y_3 + u_3, \\ d_t^{\alpha_4} y_4(t) = -y_2 y_3 + r y_4 + u_4. \end{cases} \tag{35}$$

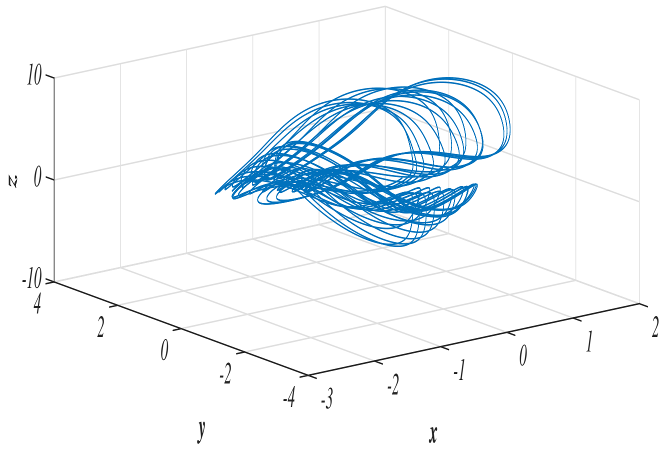

Figure 1: Chaotic Attractor of the Fractional Order MBSCCS System with $(\alpha_1, \alpha_2, \alpha_3) = (0.97, 098, 0.99)$.

In (35), y_1, y_2, y_3,y_3 are the states, $a_2, c_2,\ b_2, r$ are the positive parameters and $\alpha_i, i = 1, 2, 3, 4$ are rational numbers between 0 and 1.

For all numerical simulation, we take the initial states of system (35) as

$$y_1(0) = 1, y_2(0) = 1, y_3(0) = 0, y_4(0) = -1. \tag{36}$$

The parameters values are taken as

$$(a_2, c_2,\ b_2, r\ ) = (10, 28, \frac{8}{3}, 1.3). \tag{37}$$

The proposed fractional orders are taken as

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.97, 0.98, 0.99, 0.999). \tag{38}$$

The system (35) (with $u_1 = u_2 = u_3 = u_4 = 0$) exhibits chaotic behaviour as shown in Figure 2.

The linear part A_1 of system (35) is given by

$$A_1 = \begin{pmatrix} -a_2 & a_2 & 0 & 1 \\ c_2 & -1 & 0 & 0 \\ 0 & 0 & -b_2 & 0 \\ 0 & 0 & 0 & r \end{pmatrix}.$$

Here, we choose

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \text{ and } S = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0. \end{pmatrix}.$$

Let us define the error variables between the slave system (35) to be controlled and the master system (30) as

$$e(t) = QY(t) - SX(t),$$

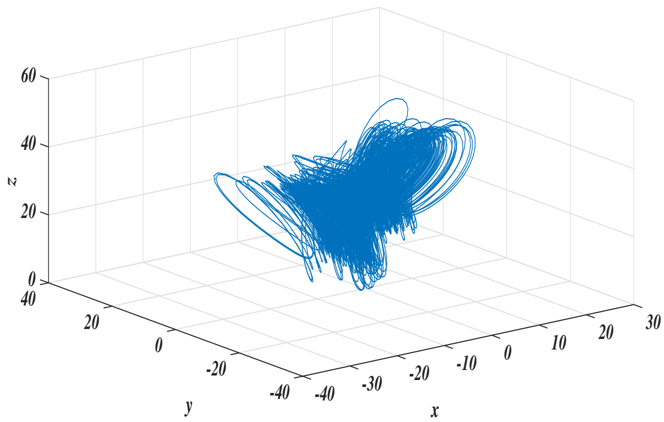

Figure 2: Chaotic Attractor of the Fractional Order Lorenz System with $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.97, 0.98, 0.99, 0.999)$.

i.e.

$$\begin{cases} e_1 = -2x_1 + y_1, \\ e_2 = -2(x_2 - y_2) - x_3, \\ e_3 = -3x_3 + y_3, \\ e_4 = 2y_4. \end{cases} \tag{39}$$

For simplicity, choose the suitable feedback gain matrix B such that

$$A_1 + B = \begin{pmatrix} -a_2 & 0 & 0 & 0 \\ 0 & -b_2 & 0 & 0 \\ 0 & 0 & -c_2 & 0 \\ 0 & 0 & 0 & -r \end{pmatrix}. \tag{40}$$

Hence

$$\begin{cases} u_1 = -e_1 a_2 + 2a_1 x_2 - y_4 + a_2 y_1 - a_2 y_2, \\ u_2 = -\frac{1}{2} b_2 e_1 - M b_1 x_2 x_3 + \frac{1}{2} x_2^2 x_3 - \frac{1}{2} x_2 - \frac{1}{2} c_1 x_3 - b_1 x_1 + y_2 - c_2 y_1 + y_1 y_3, \\ u_3 = -c_2 e_1 + 3x_2^2 x_3 - 3x_2 - 3c_1 x_3 + b_2 y_3 - y_1 y_2, \\ u_4 = -\frac{1}{2} r e_1 - r y_4 + y_2 y_3. \end{cases} \tag{41}$$

The error system can be rewritten as

$$d_t^{\alpha_i} e_i(t) = (A_1 + B) e_i, \text{ for all } i = 1, 2, 3, 4. \tag{42}$$

To prove that the error system converges to 0, we apply the formulas (20) and (21), we obtain

$$\begin{cases} s^{\alpha_1} F_1(s) = s^{\alpha_1 - 1} e_1(0) - a_2 F_1(s), \\ s^{\alpha_2} F_2(s) = s^{\alpha_2 - 1} e_2(0) - b_2 F_2(s), \\ s^{\alpha_3} F_3(s) = s^{\alpha_3 - 1} e_3(0) - c_2 F_3(s), \\ s^{\alpha_4} F_4(s) = s^{\alpha_4 - 1} e_4(0) - r F_4(s). \end{cases} \tag{43}$$

It follows from the equations of the system(43) that

$$\begin{cases} F_1(s) = \dfrac{s^{\alpha_1-1}e_1(0)}{s^{\alpha_1}+a_2}, \\ F_2(s) = \dfrac{s^{\alpha_2-1}e_2(0)}{s^{\alpha_2}+b_2}, \\ F_3(s) = \dfrac{s^{\alpha_3-1}e_3(0)}{s^{\alpha_3}+c_2}, \\ F_4(s) = \dfrac{s^{\alpha_3-1}e_4(0)}{s^{\alpha_4}+r}. \end{cases} \tag{44}$$

Since a_2, b_2, c_2, r are positive parameters, we can conclude that all poles of $sF_i(s)$, $i = 1, 2, 3, 4$ lie in the open left half plane. Thus, by using Theorem 3.1, we get

$$\lim_{t\to+\infty} e_i(t) = \lim_{s\to 0^+} sF_i(s) = 0, \quad \text{for all } i = 1, 2, 3, 4. \tag{45}$$

This means that the drive system (30) and the response system (35) achieve the synchronization. The error functions evolution, in this case, is shown in Figure 3.

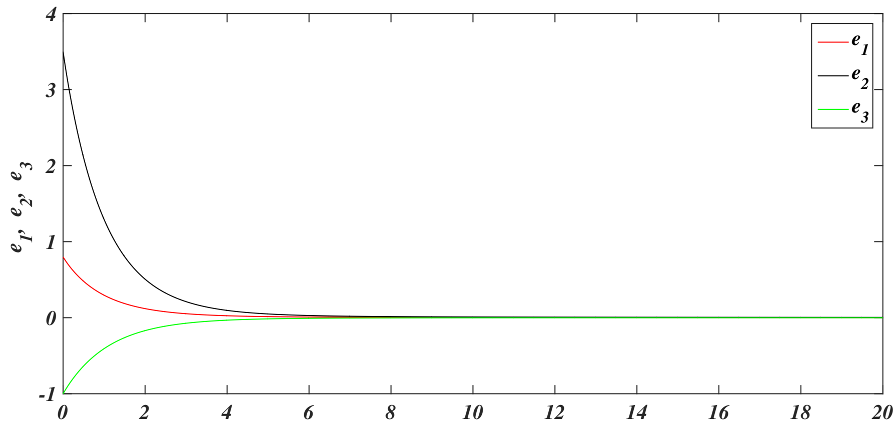

Figure 3: Error Functions Evolution of System (42).

4.2 Simulation results (Reduced order)

Let us take the same previous systems. Here, we choose

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \end{pmatrix}, \quad \left(Q^0\right)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ and } S = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1. \end{pmatrix}.$$

To investigate the generalized synchronization of the systems (30) and (35), we define the error states as

$$e(t) = QY(t) - SX(t), \tag{46}$$

i.e.

$$\begin{cases} e_1 = -2x_1 + y_1, \\ e_2 = -3x_2 + 2y_2, \\ e_3 = -x_3 + \frac{1}{2}y_3. \end{cases} \tag{47}$$

For simplicity, choose the suitable feedback gain matrix C such that

$$(A_2 + C) = \begin{pmatrix} -a_1 & 0 & 0 \\ 0 & -b_1\beta & 0 \\ 0 & 0 & -c_1. \end{pmatrix}. \tag{48}$$

Hence

$$\begin{cases} u_1 = -a_1e_1 + 2a_1x_2 - y_4 + a_2y_1 - a_2y_2, \\ u_2 = -\frac{1}{2}b_1\beta e_1 - \frac{3}{2}Mb_1x_2x_3 - \frac{3}{2}b_1x_1 + y_2 - c_2y_1 + y_1y_3, \\ u_3 = -2e_1 - 2x_2 - 2c_1x_3 + 2x_2^2x_3 + b_2y_3 - y_1y_2 - 2c_1, \\ u_4 = 0. \end{cases} \tag{49}$$

The error system can be rewritten as

$$d_t^{\alpha_i}e_i(t) = (A_2 + C)e_i, \text{ for all } i = 1, 2, 3. \tag{50}$$

To prove that the error system converges to 0, we apply the formulas (20) and (21), we

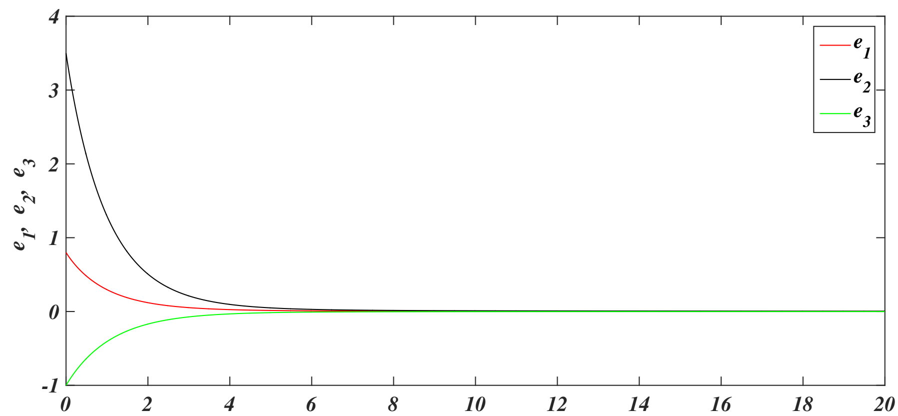

Figure 4: Error Functions Evolution of System (50).

obtain

$$\begin{cases} s^{\alpha_1}F_1(s) = s^{\alpha_1-1}e_1(0) - a_1F_1(s), \\ s^{\alpha_2}F_2(s) = s^{\alpha_2-1}e_2(0) - b_1\beta F_2(s), \\ s^{\alpha_3}F_3(s) = s^{\alpha_3-1}e_3(0) - c_1F_3(s)). \end{cases} \tag{51}$$

It follows from the equations of the system (51) that

$$\begin{cases} F_1(s) = \dfrac{s^{\alpha_1-1}e_1(0)}{s^{\alpha_1} + a_1}, \\ F_2(s) = \dfrac{s^{\alpha_2-1}e_2(0)}{s^{\alpha_2} + b_1\beta}, \\ F_3(s) = \dfrac{s^{\alpha_3-1}e_3(0)}{s^{\alpha_3} + c_1}. \end{cases} \tag{52}$$

Since a_1, b_1, c_1 are positive parameters, we can conclude that all poles of $sF_i(s),\ i = 1, 2, 3$ lie in the open left half plane. Thus, by using Theorem 3.2, we get

$$\lim_{t\to+\infty} e_i(t) = \lim_{s\to 0^+} sF_i(s) = 0, \quad \text{for all } i = 1, 2, 3, \tag{53}$$

which clearly demonstrates that the drive system (30) and the response system (35) achieve the generalized synchronization. The error functions evolution, in this case, is shown in Figure 4.

5 Conclusion

In this paper, we have investigated the generalized synchronization between two different dimensional chaotic fractional non-commensurate order systems. The analytical conditions for the synchronization between these chaotic systems are derived by using the Laplace transform technique and final value theorem. Numerical simulations of chaotic and hyperchaotic systems have been given to illustrate and validate the effectiveness of the proposed generalized synchronization.

Our future work is to develop some type of synchronization and we suggest some potential applications in secure communication.

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