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Sufficient Conditions for the Existence of Optimal Controls for Some Classes of Functional-Differential Equations

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Abstract: Sufficient conditions for the existence of optimal controls for system of functional-differential equations which is nonlinear by phase variables and linear by control function are given. These conditions are obtained in terms of right-hand sides of the system and the quality criterion function, which makes them convenient for verification. The main differences from the previously obtained results are that the control is in the system as a functional, and the optimal control problem is considered until the exit of the solution from the area of a certain functional space.

Keywords: *optimal control, functional, weakly convergence, convexity, minimizing sequence.*

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1 Introduction

Let $h > 0$ be a value of delay, $|\cdot|$ denote the norm of the vector in the space $\mathbb{R}^d$, $\|\cdot\|$ be the norm of $d \times m$ -dimensional matrix which is consistent with the norm of the vector.

Let us denote by $C = C\left([-h, 0]\,; \mathbb{R}^d\right)$ the Banach space of continuous maps of $[-h, 0]$ into $\mathbb{R}^d$ with the uniform norm $\|\varphi\|_C = \max\limits_{\theta\in[-h;0]} |\varphi(\theta)|$. Also denote by $L_p = L_p\left([-h, 0]\,; \mathbb{R}^m\right),\ p > 1$, the Banach space of p-integrable m-dimensional vector-functions with standard norm $\|\varphi\|_{L_p} = \left(\int_{-h}^{0} |\varphi(\tau)|^p d\tau\right)^{\frac{1}{p}}$.

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Let $x \in C\left([0,T];R^d\right)$, $\varphi \in C$. If $x(0) = \varphi(0)$, then the function

$$x(t,\varphi) = \begin{cases} \varphi(t)\ , t \in [-h,0]\,, \\ x(t)\,, t \geq 0, \end{cases}$$

is continuous on $[-h,T]$.

For each $t \in [0,T]$ in the standart way by $\theta \in [-h,0]$ we put an element $x_t(\varphi) \in C$ as $x_t(\varphi) = x(t+\theta,\varphi)$. In what follows we shall write x_t instead of $x_t(\varphi)$.

Let $t \in [0,T]$, D be some domain in $[0,T] \times C$, ∂D be its boundary and $\overline{D} = D \cup \partial D$.

In this paper we consider the optimal control problems for systems of functional differential equations

$$\dot{x} = f_1(t,x_t) + \int_{-h}^{0} f_2(t,x_t,y)\,u(t,y)\,dy,\ t \in [0,\tau]\,, \tag{1.1}$$

$$x(t) = \varphi_0(t)\,,\ t \in [-h,0]\,,$$

with the quality criterion

$$J[u] = \int_0^{\tau} L(t,x_t,u(t,\cdot))\,dt \to \inf \tag{1.2}$$

on $[0,T]\,,$ where $\varphi_0 \in C$ is a fixed element such that $(0,\varphi_0) \in D$, $x(t)$ is the phase vector in $\mathbb{R}^d$, x_t is the phase vector in C, τ is the moment of the first exit (t,x_t) on the boundary ∂D , $f_1 : D \to \mathbb{R}^d$, $f_2 : D \times [-h,0] \to M^{d\times m}$ are $d \times m$ -dimensional matrices, and for each $(t,\varphi) \in D$, $f_2(t,\varphi,\cdot) \in L_q\left([-h,0]\,; M^{d\times m}\right)$ with the norm $\|f_2(t,\varphi,\cdot)\|_{L_q} = \left(\int_{-h}^{0} \|f_2(t,\varphi,y)\|^q dy\right)^{\frac{1}{q}}$, $\frac{1}{q} + \frac{1}{p} = 1$, $L : D \times L_p \to \mathbb{R}^1$.

The control parameter $u \in L_p([0,T] \times [-h,0])$ is such that $u(t,y) \in U$, and U is a convex and closed set in $\mathbb{R}^m$ for almost all t, y.

Many works are devoted to the optimal control problems for functional-differential equations systems. We note the monograph [1] devoted to the application of the method of dynamic programming and the principle of maximum to such problems. There is also a wide bibliography. Althought these methods, as a rule, give the necessary conditions of optimality, it would be desirable to have suitable sufficient conditions for checking to apply them.

In this regard, we cite the work [2] in which in the case of compactness of the set of admissible controls an analogue of the Filippov theorem on optimal control existence for ordinary differential equations was obtained.

For noncompact set of admissible controls an analogue of the Cessari theorem is obtained in [4]. In the mentioned work the condition of compactness is imposed on a set of constraints and a certain condition of growth is established which connects the right-hand sides of the system and the quality criterion.

In [5] under the condition of compactness of the set of admissible controls values sufficient conditions for optimality on a fixed interval $[t_0,t_1]$ for neutral-type equations are obtained.

In [6] the problem of optimal control of a delayed linear system is rewritten in a form that does not depend on the delay and which is studied by the methods of ordinary differential equations. In the works [7]- [9] the optimal control problem of the system

$$\dot{x}(t) = rx(t) + f_0\left(x(t)\,, \int_{-T}^{0} a(\varsigma)\,x(t+\varsigma)\,d\varsigma\right) - u(t)$$

is considered.

In [7] certain Hamilton-Jacobi-Bellman equations are obtained for certain quality functionals and, in terms of their solutions, sufficient conditions for optimality in the form of a reverse link are obtained.

In [8] similar questions are considered for problems with phase restriction.

In [9] for such problems the authors obtained sufficient conditions for optimality under the condition of nondecreasing function $rx + f_0(x, y)$ in both variables and for the quality criterion

$$J(u) = \int_0^\infty e^{-\varphi t} a^{t\sigma}(t)dt, \quad \sigma \in (0, 1).$$

The main goal of this work is to obtain the theorem on the existence of optimal controls for a wider class of problems under weaker conditions as compared with the above mentioned works [2]- [9].

This paper is organized as follows. In Section 2 we give rigorous formulations of the considered problems and state main results. Section 3 is devoted to the proof of the main results.

In Subection 3.1 we prove the existence theorem, the uniqueness and extension of the solution of the initial problem (1.1) to the boundary ∂D of the domain D.

In Subsection 3.2 the theorem on the existence of optimal control for problem (1.1)-(1.2) is proved.

Examples of the application of the results obtained for ordinary differential equations, equations with delaying argument and equations with maxima are given in Section 4.

2 Statement of the Problems and Main Results

Now we give exact statement of the problem and formulate the main results of this paper. The main conditions for the problem (1.1)-(1.2) are assumed as follows.

Assumption 2.1 Admissible controls are m-dimensional vector functions $u \in L_p([0,T][-h,0], \mathbb{R}^m)$, such that $u(t,y) \in U$ for almost all $t \in [0,T]$ and $y \in [-h,0]$.

The set of admissible controls is denoted by $\mathcal{U}$.

Assumption 2.2 The maps $f_1(t,\varphi) : D \to \mathbb{R}^d$ and $f_2(t,\varphi,y) : D \times [-h,0] \to M^{d\times m}$ are defined and measurable with respect to all their arguments in the domains D and $D_1 = \{(t,\varphi) \in D, y \in [-h,0]\}$ respectively, and satisfy the linear growth condition and the Lipschitz condition with respect to φ, i.e. there exists a constant $K > 0$ such that

$$|f_1(t,\varphi)| + \|f_2(t,\varphi,y)\| \leq K(1 + \|\varphi\|_C) \tag{2.1}$$

for any $(t,\varphi) \in D, y \in [-h,0]$,

$$|f_1(t,\varphi_1) - f_1(t,\varphi_2)| + \|f_2(t,\varphi_1,y) - f_2(t,\varphi_2,y)\| \leq K\|\varphi_1 - \varphi_2\|_C \tag{2.2}$$

for all $(t,\varphi_1),(t,\varphi_2) \in D,\ y \in [-h,0]$.

Assumption 2.3 Conditions for the criterion function are:

1) the map $L(t,\varphi,z) : D \times L_p \to \mathbb{R}^1$ is defined and continuous with respect to all its arguments in the domain $D_2 = \{(t,\varphi) \in D, z \in L_p\}$;

2) there exists $a > 0$ such that

$$|L(t, \varphi_1, z) - L(t, \varphi_2, z)\| \leq a\|\varphi_1 - \varphi_2\|_C$$

for all $(t, \varphi_1, z), (t, \varphi_2, z) \in D_2$;

3) the Frechet derivative L_u of the map L is continuous with respect to all its arguments in the domain D_2, and there exist constants $C_1 > 0,\ \alpha > 0$ such that for all $(t, \varphi, z) \in D_2$ the following inequality holds:

$$\|L_u(t, \varphi, z)\|_{L_q} \leq C_1(1 + \|\varphi\|_C^\alpha + \|z\|_{L_p}^{p-1});$$

4) there exists a constant $C > 0$ such that $L(t, \varphi, z) \geq C\|z\|_{L_p}^p$ for all $(t, \varphi, z) \in D_2$;

5) $L(t, \varphi, z)$ is convex with respect to z for any fixed $t,\ \varphi$;

Our first result concerns the existence, uniqueness and extension of the solution of the original problem (1.1) to the boundary ∂D of the domain D. It is some analogue of the Carathéodory theorem for ordinary differential equations.

Definition 2.1 The solution of the initial problem (1.1) on the segment $[-h, A]$, $A > 0$, is called a continuous on the segment $[-h, A]$ function $x(t)$ such that

1) $x(t) = \varphi_0(t),\ t \in [-h, 0]$;

2) $(t, x_t) \in D$ on $t \in [0, A]$;

3) for $t \in [0, A]$ the function $x(t)$ satisfies the integral equation

$$x(t) = \varphi_0(0) + \int_0^t \Big[f_1(s, x_s) + \int_{-h}^0 f_2(s, x_s, y)u(s, y)dy \Big] ds. \tag{2.3}$$

.

Remark 2.1 It is obvious that for $t \in [0, A]$ the solution $x(t)$ is an absolutely continuous function and satisfies the equation (1.1) for almost all t on $[0, A]$.

Theorem 2.1 *Suppose that Assumptions 2.1 and 2.2 are satisfied. Then there exists a solution of the initial problem* (2.3) *on the maximal segment* $[-h, \tau]$*,* $\tau > 0$ *and* $(\tau, x_\tau) \in \partial D$*.*

The following theorem gives for the problem (1.1)-(1.2) the existence conditions of the optimal pair $x^*(t), u^*(t, \theta))$, which provides the minimum of the quality criterion (1.2).

In this case $u^* \in \mathcal{U}$ is called the optimal control and the corresponding trajectory $x^*(t)$ (1.1) is called the optimal trajectory.

Theorem 2.2 *Suppose that Assumptions 2.1-2.3 are satisfied. Then there exists a solution of the optimal control problem (1.1)-(1.2).*

3 Proofs of the Theorems

3.1 Proof of Theorem 2.1

Let us fix an admissible control $u^* \in \mathcal{U}$. First, we shall prove the local existence and uniqueness of the solution of the problem (1.1), on some segment $[-h, \alpha]$, $\alpha > 0$.

To do this, we use the standard principle of contraction mappings.

Obviously, there exist $\alpha_0 > 0$ and $\beta_0 > 0$ such that all (t, φ) for which $0 \le t \le \alpha_0$, and $\|\varphi - \varphi_0\|_C \le \beta_0$ belong to D are equivalent to φ_0 on $[-h, 0]$.

Now we consider the class $B(\alpha, \beta_0)$ of all continuous on $[-h, \alpha]$ functions $x(t)$ that are equivalent to φ_0 on $[-h, 0]$ and $|x(t) - \varphi_0(0)| \le \beta_0$ for $t \in [0, \alpha]$.

Obviously, the set $B(\alpha, \beta_0)$ is closed relatively uniformly metric on $[-h, \alpha]$.

In this case there exists $\alpha_0 \ge \alpha_1 > 0$ such that if $x(t) \in B(\alpha, \beta_0)$ at $0 < \alpha \le \alpha_1$, then the following inequality holds:

$$\|x_t - \varphi_0(0)\|_C \le \beta_0, \quad t \in [0, \alpha]. \tag{3.1}$$

Indeed, under the condition of uniform continuity of φ_0 on $[-h, 0]$ there exists $\alpha_1 > 0$ such that if $|\theta_1 - \theta_2| \le \alpha_1$, then

$$|\varphi_0(\theta_1) - \varphi_0(\theta_2)| \le \frac{\beta_0}{3} \tag{3.2}$$

for all $\theta_1, \theta_2 \in [-h, 0]$.

Hence, for each $t \in [0, \alpha_1]$ at $\alpha \le \alpha_1$ from (3.2) and the properties of the set $B(\alpha, \beta_0)$ we have

$$\|x_t - \varphi_0\|_C \le \sup_{\theta\in[-h,-t]} |x(t+\theta) - \varphi_0(\theta)| + \sup_{\theta\in[-t,0]} |x(t+\theta) - \varphi_0(\theta)| \le$$

$$\le \sup_{\theta\in[-h,-t]} |\varphi_0(t+\theta) - \varphi_0(\theta)| + \sup_{\theta\in[-t,0]} |x(t+\theta) - \varphi_0(\theta)| +$$

$$+ \sup_{\theta\in[-t,0]} |\varphi_0(\theta) - \varphi_0(0)| \le \frac{\beta_0}{3} + \frac{\beta_0}{3} + \frac{\beta_0}{3} = \beta_0.$$

Next we shall prove that $\alpha > 0$ can be choosen so that the operator

$$(Ax)(t) = \begin{cases} \varphi_0(t), \ t \in [-h, 0], \\ \varphi_0(0) + \int_0^t f_1(s, x_s)ds + \int_0^t \int_{-h}^0 f_2(s, x_s, y)u(s, y)dyds, \ t \in [0, \alpha], \end{cases} \tag{3.3}$$

maps the set $B(\alpha, \beta_0)$ into itself and this operator is a contraction.

Indeed, by Lemma 2.2.1 [10] it follows that x_t is a continuous function with respect to $t \in [0, \alpha]$. Therefore, from Assumption 2.1 $f_1(s, x_s)$ is continuous with respect to $s \in [0, \alpha]$ and the function

$$\int_{-h}^0 f_2(s, x_s, y)u(s, y)dy \tag{3.4}$$

is measurable with respect to s and satisfies the estimate

$$\left| \int_{-h}^0 f_2(s, x_s, y)u(s, y)dy \right| \le C_2 \left(\int_{-h}^0 |u(s, y)|^p \, dy \right)^{\frac{1}{p}}$$

for some constant $C_2 > 0$. From this we have the integrability of (3.4) with respect to s and hence the absolute continuity of the second integral in (3.4).

Now we evaluate the difference $|(Ax)(t) - \varphi_0(0)|$ at $t \in [0, \alpha],\ \alpha \le \alpha_0$.

With (2.1) and (2.2) using Holder's inequality and Fubini's theorem we get

$$|(Ax)(t) - \varphi_0(0)| \le \int_0^t |f_1(s, x_s)|\, ds + \int_0^t \left(\int_{-h}^0 \|f_2(s, x_s, y)\|\, |u(s, y)|\, dy \right) ds \le$$

$$\le \int_0^t K(1 + \|x_s\|_C) ds + \int_0^t \left(\int_{-h}^0 K^q \left(1 + \|x_s\|_C\right)^q dy \right)^{\frac{1}{q}} \cdot \left(\int_{-h}^0 |u(s, y)|^p\, dy \right)^{\frac{1}{p}} ds \le$$

$$\le K(1 + \beta_0 + \|\varphi_0\|_C)\alpha + K h^{\frac{1}{q}} (1 + \beta_0 + \|\varphi_0\|_C) \left(\int_0^\alpha \int_{-h}^0 |u(s, y)|^p\, ds \right)^{\frac{1}{p}} \alpha^{\frac{1}{q}}.$$

Here $q = \frac{p}{p-1}$.

Let us choose now $\alpha_2 \le \alpha_1$ from the condition

$$K(1 + \beta_0 + \|\varphi_0\|_C \left(\alpha + \alpha^{\frac{1}{q}} h^{\frac{1}{q}} \left(\int_0^\alpha \int_{-h}^0 |u(s, y)|^p\, dyds \right)^{\frac{1}{p}} \le \frac{\beta_0}{3}. \tag{3.5}$$

Thus, for all $\alpha \le \alpha_1$ the operator A maps $B(\alpha, \beta_0)$ into itself.

Let us show that there exists $\alpha_3 \in [0, \alpha_2]$ such that the operator A will be a contraction on $B(\alpha_3, \beta_0)$.

Let x and $z \in B(\alpha, \beta_0)$. By (2.2) we have

$$|(Ax)(t) - (Az)(t)| \le \int_0^t K \|x_s - z_s\|_C\, ds + \int_0^t K \|x_s - z_s\|_C \int_{-h}^0 |u(s, y)|\, dyds \le$$

$$\le \left(K\alpha + K\alpha^{\frac{1}{q}} h^{\frac{1}{q}} \left(\int_0^\alpha \int_{-h}^0 |u(s, y)|^p\, dyds \right)^{\frac{1}{p}} \right) \sup_{t \in [-h, \alpha]} |x(t) - z(t)|\,.$$

And now from this we have

$$\sup_{t \in [-h, \alpha]} |(Ax)(t) - (Az)(t)| \le$$

$$\le \left(K\alpha + K\alpha^{\frac{1}{q}} h^{\frac{1}{q}} \left(\int_0^\alpha \int_{-h}^0 |u(s, y)|^p\, dyds \right)^{\frac{1}{p}} \right) \sup_{t \in [-h, \alpha]} |x(t) - z(t)|\,. \tag{3.6}$$

Now choosing $0 < \alpha_3 \le \alpha_2$ from the condition

$$K\alpha + K\alpha^{\frac{1}{q}} h^{\frac{1}{q}} \left(\int_0^\alpha \int_{-h}^0 |u(s, y)|^p\, dyds \right)^{\frac{1}{p}} < 1$$

we get that the operator $A : B(\alpha_3, \beta_0) \to B(\alpha_3, \beta_0)$ is a contraction. Thus, on the segment $[-h, \alpha_3)$ there exists a unique solution to the initial problem (1.1).

To prove the extension of this solution to the boundary ∂D, we use the approach of Theorem 2.3.2 [10]. Note that by (2.1) for $(t, \varphi) \in D$ the following estimates hold:

$$|f_1(t, \varphi)| \le K(1 + \|\varphi\|_C) \tag{3.7}$$

and

$$\left|\int_{-h}^{0} f_2(t,\varphi,y)u(t,y)dy\right| \le K(1+\|\varphi\|_C)h^{\frac{1}{q}}\int_{-h}^{0}|u(s,y)|\,dy. \tag{3.8}$$

Let $[-h,\tau]$ be the maximum interval of existence of the solution $x(t)$. For its extension to the boundary ∂D it is necessary to show that for any closed set $G \in Dt_G$ there exists t_G such that $(t,x_t) \notin G$ for $t \in [t_G,\tau]$. The last statement can be proved by contradiction. Indeed, if this is not the case, then, similar to Theorem 2.3.2 [10], the set $\bar{Q} = \{(t,x_t) : t \in [-h,\tau]\}$ is closed and bounded in D.

Therefore, the estimates (3.7) and (3.8) imply the existence of a constant M such that for $(t,\varphi) \in \bar{Q}$ we have $|f_1(t,\varphi)| \le M$ and $\left|\int_{-h}^{0} f_2(t,\varphi,y)u(t,y)dy\right| \le M\int_{-h}^{0}|u(s,y)|\,dy$.

From (2.3) for each $t_1, t \in [0,\tau]$ we have

$$|x(t_2)-x(t_1)| \le M(t_2-t_1) + Mh(t_2-t_1)^{\frac{1}{q}}\left(\int_Q^T\int_{-h}^{0}|u(s,y)|^p\,dyds\right)^{\frac{1}{p}}. \tag{3.9}$$

This implies that $\{(t,x) : t \in [-h,\tau]\}$ belongs to a compact set in D. The last statement contradicts Corollary 2.3.1 from [10]. The theorem is proved. □

3.2 Proof of Theorem 2.2

First note that controls $u(t,\theta) = u(\theta)$ are admissible.

Let $x(t)$ be a solution that coresponds to $u(t)$ and τ be a moment of the first exit (t,x_t) on the boundary ∂D.

Now we shall prove that $x(t)$ is bounded on $[0,\tau]$.

From (2.3) for $\in [0,\tau]$ we have

$$|x(t)| \le |\varphi_0(0)| + \int_0^t K(1+\|x_s\|_C)ds + Kh^{\frac{1}{q}}\int_0^t(1+\|x_s\|_C)ds\,\|u\|_{L_p} \le$$

$$\le |\varphi_0(0)| + KT + Kh^{\frac{1}{q}}\|u\|_{L_p}T + (K + h^{\frac{1}{q}}u\|u\|_{L_p})\int_0^t\|x_s\|_C\,ds =$$

$$= C_3 + C_4\int_0^t\|x_s\|\,ds \le C_3 + C_4\int_0^t\max_{s_1\in[-h,s]}|x(s_1)|\,ds \tag{3.10}$$

Since

$$\max_{s\in[-h,t]}|x(s)| \le \max_{s\in[-h,0]}|\varphi_0(s)| + \max_{s\in[0,t]}|x_1(s_1)|,$$

and from (3.10) we have

$$\max_{s\in[-h,t]}|x(s)| \le C_5 + C_4\int_0^t\max_{s\in[-h,s]}|x(s_1)|\,ds$$

for some constant $C_5 > 0$. Using Gronwall's inequality we have $\max_{s\in[-h,t]}|x(s)| \le C_5$, $t \in [0,\tau]$ for some constant $C_6 > 0$ which does not depend on t. From this it follows that

$$\max_{s\in[-h,\tau]}|x(s)| \le C_6$$

and $\max_{s\in[-h,\tau]}\|x_t\| \le C_6$.

Since from Lemma 2.2.1 [10] x_t is continuous with respect to $t \in [0, \tau]$, under the first condition of Assumption 2.3 we have $L(t, x_t, u(\theta))$ (where $(u(t, \theta)) = u(\theta)$ is continuous with respect to t and hence

$$\int_0^\tau L(t, x_t, u(\theta) dt \tag{3.11}$$

is bounded. Therefore $\inf_{u \in U} J(u) < \infty$. Since $J(u) \geq 0$, there exists a nonnegative lower limit m of the values $J(u)$. Let $u^{(n)}(t, \theta)$ be a minimizing sequence such that $J(u^n) \to m, \quad n \to \infty$ monotonously.

Let $x^{(n)}$ be a sequence of coresponding to $u^{(n)}$ solutions of equation (2.3), $[-h, \tau_n]$ be a maximal interval of its existense. From Theorem 2.1 it follows that $[\tau_n, x^{(n)}_{\tau_n}] \in \partial D$.

We have

$$m + 1 \geq \int_0^{\tau_n} L(t, x^{(n)}_t, u^{(n)}) dt \geq C \int_0^T \int_{-h}^0 \left| u^{(n)}(t, y) \right|^p dy dt \tag{3.12}$$

for sufficiently large n. Consequently $u^{(n)}(t, y)$ is weakly compact in $L_p([0, T] \times [-h, 0])$.

Therefore one can choose a sequence (also denoted by $u^{(n)}(t, y)$) which is weakly converging to $u^k(f) \in L_p([0, T] \times [-h, 0])$ in $L_p([0, T] \times [-h, 0])$.

By Mazur's lemma ([11], Ch. 5) there exists a convex combination $b_k(t, y) = \sum_{\tau=1}^{n(k)} \alpha_i \cdot (K) u^{(i)} \cdot (t, u)$ of elements $u^{(i)}(t, y)$ such that $b_k \to u^{(*)}$ strongly converges in $L_p([0, T] \times [-h, 0])$.

Therefore there exists a subsequence $b_{k_j}(t, y)$ of sequence $b_k(t, y)$ such that for almost all (t, y) on $[0, T] \times [-h, 0]$ it converges to $u^x(t, y)$.

Since U is convex, we have $b_{k_j}(t, y) \in U$, and from the closedness of U it follows that $u^*(t, y) \in U$ for almost all (t, y). So, the control function $u^*(t, y)$ is admissible.

Let us prove uniform boundedness of solutions $x^{(n)}$ on $[-h, \tau_n]$. From (2.3) under Assumptions 2.1 and 2.2 we have for $t \in [0, \tau_n]$

$$\left| x^{(n)}(t) \right|^q \leq 3^{q-1} \left| \varphi(0) \right|^q + K^q T^{\frac{q}{p}} 2^{q-1} \int_0^t (1 + \left\| x^{(n)}_s \right\|^q_C) ds +$$

$$+ h \left(\int_0^T \int_{-h}^0 \left| u^{(n)}(t, y) \right|^p \right)^{\frac{1}{p}} K^q h \int_0^t 2^{q-1} (1 + \left\| x^{(n)}_s \right\|^q_C) ds.$$

With (3.12), from the last inequality for some positive constants C_7 and C_8 which do not depend on t, y and n, we have

$$\left| x^{(n)}(t) \right|^q \leq C_7 + C_8 \int_0^t \left\| x_s \right\|^q ds$$

for $t \in [0, \tau_n]$.

Thus we have the estimate

$$\max_{s \in [-h, t]} \left| x(s) \right| \leq C_9 + C_8 \int_0^t \max_{s_1 \in [-h, s]} \left| x(s_1) \right| ds$$

for some constant C_9.

Using Gronwall's inequality we have

$$\max_{t\in[-h,\tau_n]} \left|x^{(a)}(t)\right| \le C_{10}, \tag{3.13}$$

where C_{10} is a positive constant which does not depend on n.

So $x^{(n)}(t)$ are uniform bounded. Let us extend the functions $x^{(n)}(t)$ to the whole segment $[0,T]$ as follows

$$y^{(n)}(t) = \begin{cases} x^{(n)}(t), t \in [0,\tau_n], \\ x^{(n)}(\tau_n), t \in [\tau_n, T]. \end{cases} \tag{3.14}$$

If $s_1 \le s_2 \le \tau_n$, then from (3.9) it follows the estimate

$$\left|y^{(n)}(s_1) - y^n(s_2)\right| \le C_{11}(s_2 - s_1) + C_{12}(s_2 - s_1)^{\frac{1}{q}}. \tag{3.15}$$

If $s_1 \le \tau_n \le s_2$, then similarly to (3.15) we have

$$\left|y^{(n)}(s_1) - y^n(s_2)\right| = \left|x^{(n)}(s_1) - x^{(n)}(\tau_n)\right| \le C_{11}|\tau_n - s_1| + C_{12}|\tau_n - s_1| \le$$

$$\le C_{11}\ (s_2\ -\ s_1)\ +\ C_{12}(s_2\ -\ s_1)^{\frac{1}{q}}.$$

This implies the equicontinuity of the function set $\{y^{(n)}(t)\}$ on $[0,T]$ and from (3.14) and (3.13) it follows uniform boundedness of this set. Hence the set $\left\{y^{(n)}(t)\right\}$ includes a subsequence which converges uniformly on $[0,T]$ and which we denote as $\left\{y^{(n)}(t)\right\}$. Let $y^{(x)}(t)$ be its uniform limit on $[0,T]$.

Function $y^*(t)$ is defined and continuous on $[0,T]$. Therefore, we also have $y^*_t = y^*(t+\theta)$ for all $t \in [0,T]$. Let τ^* be a moment of the first exit (t, y^*_t) on the boundary ∂D, i.e.

$$\tau^* = \begin{cases} \inf\{t \in [O,T] : (t, y^*_t) \in \partial D\}, \\ T, \quad if \quad (t, y^*_t) \in D, \forall t \in [O,T]. \end{cases}$$

Note that if

$$y^{(n)}_{\tau_n} = y^{(n)}(\tau_n + \theta) = x^{(n)}(\tau_n + \theta) = x^{(n)}_{\tau_n},$$

then τ_n is the moment of the first exit (t, y^n_t) on ∂D.

Let us prove that

$$\tau^* \le \lim_{n\to\infty} \inf \tau_n. \tag{3.16}$$

Suppose that it is not true. Then

$$\tau^* > \lim_{n\to\infty} \inf \tau_n = \tau. \tag{3.17}$$

Obviously, there exists a subsequence τ_{n_k} such that $\tau_{n_k} \to \tau$ for $n_k \to \infty$. Therefore for sufficiently large n_k we have $\tau < \tau^*$ and

$$(\tau, y^*_\tau) \in D. \tag{3.18}$$

But $(\tau_{n_k}, y^{(n_k)}_{\tau_{n_k}}) \in \partial D$.

On the other hand, taking into account the uniform convergence of the sequence $y^n(t)$ to $y^*(t)$ on $[-h,T]$ and uniform on $[-h,T]$ continuity of $y^*(t)$ it is not difficult to see that $y^{(n_k)}_{\tau_{n_k}} \to y^*_\tau$ in C for $n_k \to \infty$.

Since the set ∂D is closed, we have $(\tau, y^*_\tau) \in D$. The latter contradicts (3.18). Therefore

$$\tau^* \leq \tau = \liminf_{n\to\infty} \tau_n.$$

Let $x^*(t) = y^*(t)$ for $t \in [0, \tau^*]$. Show that $x^*(t)$ is a solution of the equation (1.1) which corresponds to the equation $u^*(t)$.

We consider two cases.

1. Let $\tau^* < \tau$. Then by the theorem of the characterization of the lower bound, the set $\{n \in N : \tau_n \leq \tau^*\}$ is finite. Consequently, there exists a subsequence $\{\tau_{n_k}\}$ of the sequence τ_n, such that $\tau_{n_k} > \tau^*$. Then $y^{(n_k)}(t) = x^{(n_k)}(t)$ for $t \in [0, \tau^*]$ and $x^{(n_k)}(t)$ converges uniformly to $x^*(t)$ for $n_k \to \infty$. We have

$$x^{(n_k)}(t) = \varphi_0(0) + \int_0^t f_1(s, x_s^{n_k})ds + \int_0^t \int_{-h}^0 f_2(s, x_s^{n_k} y)u^{n_k}(s, y)dyds \tag{3.19}$$

for $t \in [0, \tau^*]$.

Then we get

$$x^{(u_k)}(t) = \varphi_0(0) + \int_0^t f_1(s, x_s^{n_k})ds + \int_0^t \int_{-h}^0 f_2(s, x_s^{n_k} y)u^*(s, y)dyds+$$

$$+ \int_0^t \int_{-h}^0 (f_2(s, x_s^{(n_k)}, y) - f_2(s, x_s^*, y))(u^{(u_k)}(s, y) - u^*(s, y)dyds+$$

$$+ \int_0^t \int_{-h}^0 f_2(s, x_s^*, y)(u^{(u_k)}(s, y) - u^*(s, y)dyds. \tag{3.20}$$

It is obvious that $x_t^{(n_k)} \to x_t^*$ on C for all $t \in [0, \tau^*]$.

From (2.2) we have

$$\int_0^t f_1(s, x_s^{(n_k)})ds \to \int_0^t f_1(s, x_s^*)ds \tag{3.21}$$

and in view of the Lebesgue theorem on dominanted convergence, we also obtain that

$$\int_0^t \int_{-h}^0 f_2(s, x_s^{(n_k)}, y)u^*(s, y)dyds \to \int_0^t \int_{-h}^0 f_2(s, x_s^*, y)u^*(s, y)dyds. \tag{3.22}$$

Similarly, we establish that the third integral in (3.20) tends to zero for $n_k \to \infty$.

Taking into account Assumption 2.2 with respect to f_2, it is easy to see that the expression $\int_0^t \int_{-h}^0 f_2(s, x_s^*, y)u(s, y)dyds$ defines a linear continuous functional on $L_2([0, t] \times [-h, 0])$.

Therefore the last integral in (3.20) tends to zero because of the weak convergence of $u^{(n_k)}(s, y)$ to $u^*(s, y)$. Using the limiting transition in (3.20) we obtain that $x^*(t)$ is the solution of the initial problem (1.1) on $[0, \tau^*]$ which corresponds to the control $u^*(t, y)$.

2. Let $\tau^* = \tau$. Take an arbitrary $t_1 \in [0, \tau]$ such that $t_1 < \tau^*$. Then the set $\{n \in N : \tau_n \leq t_1\}$ is finite.

In the case of the finiteness of the set $Z = \{n \in N : t_1 < \tau_n \leq t^*\}$ the proof reduces to the preceding case. Let Z be infinite and δ_{n_k} be a subsequence of the sequence τ_n such that $\tau_{n_k} \in Z$. Then for each $t \in [0, t]$ we have $y^{(n_k)}(t) = x^{(n_k)}(t)$ $y^*(t) = x^*(t)$.

Similarly to the previous case then $x^*(t)$ is the solution of the initial problem (1.1) on $[0,t_1]$ corresponding to the control $u^*(t,y)$, that is

$$x^*(t)=\varphi_0(0)+\int_0^t f_1(s,x_s^*)ds+\int_0^t\int_{-h}^0 f_2(s,x_s^*,y)u^*(s,y)dyds \tag{3.23}$$

for $t\in[0,t_1]$. Since $t_1<\tau^*$ is arbitrary, the equality (3.23) holds on the interval $[0,\tau^*]$.

Let us show it holds also for $t=\tau^*$. Let $t_n\in[0,\tau^*]$ and $t_n\to\tau^*$, then $x^*(t_n)\to x^*(\tau^*)$.

Similarly to the inequality (3.9) we get for $n\to\infty$

$$\left|\int_0^{\tau^*}[f_1(s,x_s^*)+\int_{-h}^0 f_2(s,x_s^*,y)u^*(s,y)dy]ds-\int_0^{t_n}[f_1(s,x_s^*)+\int_{-h}^0 f_2(s,x_s^*,y)u^*(s,y)dy]ds\right|\to 0.$$

Therefore $x^*(t)$ satisfies (3.23) for $t=\tau^*$ too.

It remains to show that the control $u^*(s,y)$ is optimal. We have two cases.

1. Let $\tau^*<T$.

a) Let $\tau^*<\lim_{n\to\infty}\inf\tau_n=\tau$. Then, similarly to the above, there exists a subsequence τ_{n_k} of the sequence τ_n such that $\tau_{n_k}>\tau^*$ and for $t\in[0,\tau^*]$ $y^{(n_k)}(t)=x^{(n_k)}(t)$ and $y^*(t)=x^*(t)$.

We show the integrability of the function $L(t,x_t^*,u^{(n_k)}(t,\cdot))$ on $[0,\tau^*]$

Using inequality

$$\left|L(t,x_t^*,u^{(n_k)}(t,\cdot))-L(t,x_t^*,u_0)\right|\le$$

$$\le\sup_{\lambda\in[0,1]}\left\|L_u(t,x_t^*,u_0+\lambda(u^{(n_k)}(t,\cdot)-u_0)\right\|_{L_q}\left\|u^{(n_k)}(t,\cdot)-u_0\right\|_{L_p},$$

where $u_0=const$, $u_0\in\mathcal{U}$, we have

$$L(t,x_t^*u^{(n_k)}(t,\cdot))\le L(t,x_t^*,u_0)+$$

$$+\sup_{\lambda\in[0,1]}\left\|L_u(t,x_t^*,u_0+\lambda(u^{(n_k)}(t,\cdot)-u_0)\right\|_{\lambda_q}\left\|u^{(n_k)}(t,\cdot)-u_0\right\|_{L_p}.$$

Using condition 3) of Assumption 2.3 we have

$$L(t,x_t^*,u^{(n_k)}(t,\cdot))\le L(t,x_t^*,u_0)+C_1\left\|u^{(n_k)}(t,\cdot)-u_0\right\|_{L_p}+$$

$$+C_1\left\|x_t^*\right\|_C^\alpha\left\|u^{(n_k)}(t,\cdot)-u_0\right\|_{L_p}+$$

$$+C_1\sup_{\lambda\in[0,1]}\left\|u_0+\lambda(u^{(n_k)}(t,\cdot)-u_0)\right\|_{L_p}^{p-1}\left\|u^{(n_k)}(t,\cdot)-u_0\right\|_{L_p}. \tag{3.24}$$

The first term in (3.24) is integrable in accordance with (3.11). The second and third terms are also integrable on $[0,\tau^*]$ due to (3.12), (3.13) and the uniform convergence of $x^{(n_k)}(t)$ to $x^*(t)$ on $[0,\tau^*]$.

The integrability of the last term in (3.24) follows from the estimate

$$\int_0^{\tau^*}(\|u_0\|_{L_p}+\|u^{n_k}(t,\cdot)-u_0\|)^{p-1}\left\|u^{(n_k)}(t,\cdot)-u_0\right\|_{L_p}dt\le$$

$$\leq 2^{\frac{(p-1)^2}{p}} \left(\int_0^{\tau^*} (\|u_0\|^p_{L_p} + \|u^{n_k}(t,\cdot) - u_0\|^p_{L_p}) dt \right)^{\frac{p-1}{p}} \left(\int_0^{\tau^*} (\|u^{n_k}(t,\cdot) - u_0\|^p_{L_p}) dt \right)^{\frac{1}{p}}.$$

Therefore the function $\alpha(t, x^*_t, u^{(n_k)}(t,\cdot)$ is integrable on $[0, \tau^*]$.

Let $\chi_R(t)$ be a characteristic function of the set $\left\{ t \in [0,t] : \|u^*(t,\cdot)\|_{L_p} < R \right\}$ for some $R > 0$.

Since $L(t,y,z)$ is convex on z (condition 5) from Assumption 2.3), the following inequality holds

$$L(t, x^*_t, \upsilon(t,\cdot)\chi_R(t) \geq L(t, x^*_t, u^*(t,\cdot))\chi_R(t) + \left\langle L'_u(t, x^*_t, u^*(t,\cdot), \upsilon(t,\cdot) - u^*(t,\cdot) \right\rangle \chi_R(t) \tag{3.25}$$

for any admissible control $\upsilon(t,y) \in U_p \;\; t \in [0,\tau^*]$. Here $\left\langle L'_u, \upsilon - u^* \right\rangle$ is the action of the linear continuous functional L_u on the element $\upsilon(t,\cdot) - u^*(t,\cdot) \in L_p$. Putting in (3.25) $\upsilon(t,\cdot) = u^{(n_k)}(t,\cdot)$ we have

$$\int_0^{\tau^*} L(t, x^*_t, u^{(n_k)}(t,\cdot))\chi_R(t)dt \geq \int_0^{\tau^*} L(t, x^*_t, u^*(t,\cdot))\chi_R(t)dt + \\ + \int_0^{\tau^*} \left\langle L'_u(t, x^*_t, u^*(t,\cdot), u^{(n_k)}(t,\cdot) - u^*(t,\cdot) \right\rangle \chi_R(t)dt. \tag{3.26}$$

Under condition 3) of Assumption 2.3 we have

$$\|L_u(t, x^*_t, u^*(t,\cdot)\|_{L_q} \chi_R(t) \leq K(1 + \|x^*_t\|^\alpha_C + R)^{p-1},$$

therefore, the second term defines a linear continuous functional in $L_p([0,\tau^*] \times [-h,0])$. So, the second integral in (3.26) tends to zero, because of the weak convergence of $u^{n_k}(t,s)$ to $u^*(t,s)$.

Therefore

$$\lim_{n\to\infty} \inf \int_0^{\tau^*} L(t, x^*_t, u^{(u_k)}(t,\cdot))\chi_R(t)dt \geq \int_0^{\tau^*} L(t, x^*_t, u^*(t,\cdot))\chi_R(t)dt.$$

Since $L(t,y,z) \geq 0$, $\chi_R(t) \leq 1$ and $\chi_R(t) \to 1$ for$R \to \infty$, we get from the last inequality that

$$\int_0^{\tau^*} L(t, x^*_t, u^*(t,\cdot)dt \leq \lim_{n\to\infty} \inf \int_0^{\tau^*} \alpha(t, x^*_t, u^{(u_k)}(t,\cdot)dt. \tag{3.27}$$

The integrability of $L(t, x^*_t, u^{(n_k)}(t,\cdot)$ on $[0,\tau^*]$ is taken into account.

Let us also consider the difference

$$\int_0^{\tau^*} \left| L(t, x^{(n_k)}_t, u^{(u_k)}(t,\cdot)) - L(t, x^*_t, u^{(u_k)}(t,\cdot)) \right| dt. \tag{3.28}$$

Using condition 2) of Assumption 2.3 we have

$$\int_0^{\tau^*} \left| L(t, x^{(n_k)}_t, u^{(u_k)}(t,\cdot)) - L(t, x^*_t, u^{(u_k)}(t,\cdot)) \right| dt \leq$$

$$\leq \alpha \int_0^{\tau^*} \left\| x_t^{(n_k)} - x_t^* \right\| dt \to 0, n_k \to \infty. \tag{3.29}$$

The limit transition in (3.29) is possible by the Lebesgue theorem on the majorization of convergence (3.13) and the uniform convergence of $x^{(n_k)}(t)$ to $x^*(t)$ on $[0,\tau^*]$. From (3.29) we find that the expression (3.28) tends to zero for $n_k \to \infty$.

Further, we have

$$\lim_{n\to\infty} \inf \int_0^{\tau^*} L(t, x_t^{(n_k)}, u^{(n_k)}(t,\cdot))dt \geq$$

$$\geq \lim_{n\to\infty} \inf \int_0^{\tau^*} [L(t, x_t^{(n_k)}, u^{(n_k)}(t,\cdot) - L(t, x^*, u^{(n_k)})]dt+$$

$$+ \lim_{n\to\infty} \inf \int_0^{\tau^*} \left| L(t, x_t^*, u^{(u_k)}(t,\cdot)) - L(t, x_t^*, u^*(t,\cdot)) \right| dt + \int_0^{\tau^*} L(t, x_t^*, u^*(t,\cdot))dt. \tag{3.30}$$

As is shown above, the first limit on the right-hand side (3.30) is zero, and the second limit is non-negative with (3.27).

Then

$$m = \lim_{n_k\to\infty} \inf \int_0^{\tau_{n_k}} L(t, x_t^{(n_k)}, u^{(n_k)}(t,\cdot))dt \geq \lim_{n_k\to\infty} \inf \int_0^{\tau^*} L(t, x_t^{(n_k)}, u^{(n_k)}(t,\cdot))dt \geq$$

$$\geq \int_0^{\tau^*} L(t, x_t^*, u^*(t,\cdot))dt.$$

Thus $J(u^*) = m$, so the pair $(x^*(t), u^*(t,s))$ is optimal.

b) Let $\tau^* = \tau = \lim\limits_{n\to\infty} \inf \tau_n$.

Let us consider the set $Z = \{n \in N : t_1 < \tau_n \leq \tau^*\}$, where we again take an arbitrary $t_1 \in [0,T]$ such that $t_1 < \tau^*$. It is enough to consider the case when this set is infinite. Then, in the same way as in a), we can show that

$$\int_0^{t_1} L(t, x_t^*, u^*(t,\cdot))dt \leq m.$$

Thus, by the limit transition for $t_1 \to \tau^*$ we establish that

$$\int_0^{\tau^*} L(t, x_t^*, u^*(t,\cdot))dt \leq m.$$

Hence $J(u^*) = m$.

2. Let $\tau^* = T$. Then from (3.16) we have $\tau = \lim\limits_{n\to\infty} \inf \tau_n = \tau^*$, and the proof reduces to case 1, b). The theorem is proved. □

Remark 3.1 The method of proving the existence of optimal control and optimal trajectory is constructive if we take into account the fact that the approach of works [16, Chapter 7], or [17, Chapter 4] can be used to construct a minimizing control sequence.

4 Applications

As an application of the obtained results, we consider some particular cases of problem (1.1)-(1.2).

Example 4.1 If $u = u(t)$ and does not depend on the value y, then the problem (1.1)-(1.2) reduces to the "ordinary" optimal control problem for functional-differential equations

$$\dot{x}(t) = f_1(t, x_t) + g(t, x_t)u(t),\ t \in [0, \tau],$$

$$x(t) = \varphi_0(t),\ t \in [-h, 0],$$

$$J[u] = \int_0^\tau L(t, x_t, u(t))dt \ \to \inf,$$

where $g(t, x_t) \in M^{d\times m}$ and $g(t, x_t) = \int_{-h}^0 f_2(t, x_t, y)dy$, $u(t) \in L_p([0, T])$, $u(t) \in \mathcal{U}$.

Example 4.2 Equations with maximum.

A particular case of the problem (1.1)-(1.2) is the optimal control problem with maximum on the interval $[-h, T], h > 0$.

$$\dot{x}(t) = f_1(t, x_t, \max_{s\in I(t)} x(s)) + f_2(t, x_t, \max_{s\in I(t)} x(s))u(t), \tag{4.1}$$

$$x(t) = \varphi(t),\ t \in [-h, 0],$$

$$J[u] = \int_0^\tau L(t, x(t), u(t))dt \ \to \inf, \tag{4.2}$$

where $I(t) = [\beta(t), \alpha(t)]$, $\max x(s) = (\max x_1(s), \dots, \max x_d(s))$, $\beta(t), \alpha(t)$ are continuous on $[0, T]$ functions such that $\beta(t) \le \alpha(t) \le t$ and $\min\limits_{t\in[0,T]}(\beta(t)-t) = -h$, G is a domain in $\mathbb{R}^d$, $f(t, x, y) : [0, T] \times G \times G \to M^{d\times m}$, $u \in U \subset \mathbb{R}^m$, $L(t, x, u) : [0, T] \times G \times U \to \mathbb{R}^1$.

The general theory of equations with maxima is presented in the monograph [12].

The problem (4.1)-(4.2) reduces to problem (1.1)-(1.2) if we put

$$u(t, y) = u(t) \in L_p([0, T]),$$

$$\tilde{f}_1(t, \varphi) = f_1(t, \varphi(0), \max_{\theta\in[\beta(t)-t,\alpha(t)-t]} \varphi(\theta)),$$

$$\tilde{f}_2(t, \varphi) = \int_{-h}^0 f_2(t, \varphi(0), \max_{\theta\in[\beta(t)-t,\alpha(t)-t]} \varphi(\theta), s)ds.$$

Let the following conditions be satisfied:

4.A. Functions $f_1(t, x, y)$ and $f_2(t, x, y, s)$ are defined and measurable with respect to all its arguments in domains $Q = \{t \in [0, T], x \in G, y \in G\}$, and $Q_1 = \{t \in [0, T], x \in G, y \in G, s \in [-h, 0]\}$ and satisfies with respect to x, y the linear growth and the Lipschitz condition with constant $K > 0$ in these domains, i. e.

$$|f_1(t, x, y)| + \|f_2(t, x, y, s)\| \le K(1 + |x| + |y|) \tag{4.3}$$

$$|f_1(t, x, y) - f_1(t, x_1, y_1)| + \|f_2(t, x, y, s) - f_2(t, x_1, y_1, s)\| \le$$
$$\le K(|x_0 - x_1| + |y - y_1|) \tag{4.4}$$

for all $t \in [0,T]$, $x, y, x_1, y_1 \in Q$ $s \in [-h, 0]$.

4.B. 1) The function $L(t,x,y) : [0,T] \times G \times U \to \mathbb{R}^1$ is defined and continuous with respect to all its arguments and satisfies the Lipschitz condition with respect to x;

2) the partial derivative L_u is continuous in the domain of definition and satisfies for some $C_0 > 0,\ \alpha > 0$ the following estimate:

$$\|L_u(t,x,u)\| \le C_0(1 + |x|^\alpha + |u|^{p-1});$$

3) there exists a constant $C_1 > 0$ such that

$$L(t,x,u) \ge C_1 \left|u\right|^p, p > 1;$$

4) the function $L(t,x,u)$ is convex with respect to u for each fixed $t \in [0,T],\ x \in G$.

The optimal control problem (4.1)-(4.2) can be written as follows:

$$\dot{x}(\theta) = f_1(t, x_t), \max_{\theta \in I(t)} x(t)) + \int_{-h}^{0} f_2(t, x(t)) \max_{\theta \in I(t)} x_t, s) ds\, u(t) \tag{4.5}$$

$$x(t) = \varphi(t), t \in [-h, 0]$$

$I(t) = [\beta(t) - t, \alpha(t) - t]$

$$I(u) = \int_0^\tau L(t, x(t), u(t)) dt \to \inf. \tag{4.6}$$

Then all of the conditions of Assumptions 2.1-2.3 hold.

Moreover, the domain $D \subset [-h,T] \times C$ is a set $\{(t,\varphi) : t \in [-h,T]\,,\ \varphi \in \Omega$, where Ω is a set of functions $\varphi \in C$ such that $\varphi(\theta) \in G$ for $\theta \in [-h,0]$, $\partial\Omega$ is a set of functions $\varphi \in C$ such that $\varphi(\theta) \in \bar{G}$, and for each of these functions there exists a point $\theta \in [-h,0]$ such that $\varphi(\theta) \in \partial G$. It is obvious that the set $[0,T] \times \Omega = D$ is open, and $\partial D = ([0,T] \times \partial\Omega) \cup (\{T\} \times \bar{\Omega})$ is closed.

Let us check the conditions of Assumptions 2.1-2.3. Indeed, by 4.A we have

$$\left|\tilde{f}_1(t,\varphi)\right| = \left|f_1(t, \varphi(0), \max_{\theta \in [\beta(t)-t, \alpha(t)-t]} \varphi(\theta))\right| \le$$

$$\le K\left(1 + |\varphi(0)| + \left|\max_{\theta \in [\beta(t)-t, \alpha(t)-t]} \varphi(\theta)\right|\right) \le K\left(1 + \|\varphi\|_C + \|\varphi\|_C\right)$$

and for $\tilde{f}_1(t,\varphi)$ the condition (2.1) holds. For $\tilde{f}_2(t,\varphi)$ the situation is similar. Further

$$\left|\tilde{f}_1(t,\varphi) - \tilde{f}_1(t,\varphi_1)\right| \le$$

$$\le K\left(|\varphi(0) - \varphi_1(0)| + \left|\max_{\theta \in [\beta(t)]-t, \alpha(t)-t]} \varphi(\theta) - \max_{\theta \in [\beta(t)-t, \alpha(t)-t]} \varphi_1(\theta)\right|\right) \le$$

$$\le K\left(\|(\varphi - \varphi_1\|_C + \left|\max_{\theta \in [\beta(t)-t, \alpha(t)-t]} |\varphi(\theta) - \varphi_1(\theta)|\right|\right) \le 2K \|\varphi - \varphi_1\|_C\,,$$

that is the condition (2.2) holds.

Further, since the mapping L is finite-dimensional with respect to u, the Frechet derivative L_u is the Jacobi matrix $\frac{\partial L}{\partial u}$, and the norm $\|L_u\|_{L_q} = \|L_u\|$. Therefore, condition 3) from Assumption 2.3 is trivially satisfied. It is obvious that other conditions from Assumption 2.3 are also satisfied. Therefore, for problems (4.1) and (4.2), when conditions 4.1 and 4.2 are satisfied, Theorems 2.1 and 2.2 hold.

References

[1] Kolmanovskii, V.B. and Shaikhet, L.E. *Control of Systems with Aftereffect.* Amer. Math. Society (Book 157), Providence, 1996.

[2] Angell, T.S. Existence theorems for optimal control problems involving functional differential equations. *Journ. of Optimiz. Theory and Applic.* **7** (3) (1971) 149–169.

[3] Filippov, A.F. On certain questions in the theory of optimal control. *SIAM. Journal on Control* **1** (1) (1962) 76–84.

[4] Cesari, L. Existence theorems for weak and usual optimal solutions in Lagrange problems with unilateral constraints I and II. *Transactions of the American Math. Society* **124** (3) (1966) 369–430.

[5] Banks, H.T. and Keut, G.A. Control of functional differential equations of retarded and neutral type to target sets in functional space *SIAM. Journ. on Control* **10** (4) (1972) 567–593.

[6] Guo-Ping, C., Jin-Zhi, H. and Simon, X. Y. An optimal control method for linear systems with time delay. *Computers and Structures* **81** (15) (2003) 1539 – 1546.

[7] Carlier, G. and Taharoni, R. On some optimal control problems governed by a state equation with memory. *ESIAM Control, Optimization and Calculus of Variations* **14** (4) (2008) 725–743.

[8] Federico, S., Goldys, B. and Gozzi, F. HJB equations for the optimal control of differential equations with delays and state constraints I: regularity of viscosity solutions. *SIAM. Journ. on Control and Optimiz.* **48** (8) (2010) 4910–4937.

[9] Federico, S., Goldys, B. and Gozzi,F. HJB equations for the optimal control of differential equations with delays and state constraints II: verification and optimal feedbacks. *SIAM Journ. on Control and Optimiz.* **49** (6) (2011) 2378-2414.

[10] Hale, G. *Theory of Functional-Differential Equations.* Springer-Verlag, Berlin, New York, 1977.

[11] Yosida, K. *Functional Analysis.* Springer-Verlag, Berlin, New York, 1980.

[12] Bainov, D.D. and Hristova, S.G. *Differential Equations with Maxima.* CRS Press Tylor and Francis Group, 2011.

[13] Lavrova, O., Mogylova, V., Stanzhytskyi, O. and Misiats O. Approximation of the Optimal Control Problem on an Interval with a Family of Optimization Problems on Time Scales. *Nonlinear Dynamics and Systems Theory* **17** (3) (2017) 303-314.

[14] Martynyuk, A.A. Analysis of a Set of Trajectories of Generalized Standard Systems: Averaging Technique. *Nonlinear Dynamics and Systems Theory* **17** (1) (2017) 29-41.

[15] Kamaljeet and Bahuguna, D. Extremal Mild Solutions for Nonlocal Semilinear Differential Equations with Finite Delay in an Ordered Banach Space. *Nonlinear Dynamics and Systems Theory* **16** (3) (2016) 300-311.

[16] Bryson, A.E., Ho, Y.C. *Applied Optimal control: Optimization, Estimation and Control.* Waltham, MA: Blaisdell, 1969.

[17] Beltrami, E.J. *An Algorithmic Approach to Nonlinear Analysis and Optimization.* Math. in Sci. and Engr., **63**, Academic Press, 1970.