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Different Schemes of Coexistence of Full State Hybrid Function Projective Synchronization and Inverse Full State Hybrid Function Projective Synchronization

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Abstract: This paper presents new synchronization schemes, which assure the co-existence of the full-state hybrid function projective synchronization (FSHFPS) and the inverse full-state hybrid function projective synchronization (IFSHFPS) between wide classes of three-dimensional master systems and four-dimensional slave systems. In order to show the capability of co-existence approaches, numerical examples are reported, which illustrate the co-existence of FSHFPS and IFSHFPS between 3D chaotic system and 4D hyperchaotic system in different dimension.

Keywords: *chaos; full-state hybrid function projective synchronization; inverse full-state hybrid function projective synchronization; co-existence; Lyapunov stability.*

Mathematics Subject Classification (2010): 37B25, 37B55, 93C55, 93D05.

1 Introduction

Synchronization refers to a process wherein two dynamical systems (master and slave systems, respectively) adjust their motion to achieve a common behavior, mainly due to a control input [1]. The issue of synchronization of chaotic dynamical systems was first studied by Pecora and Carroll [2]. By considering the historical timeline of the topic, it can be observed that a large variety of synchronization types has been proposed such as matrix projective synchronization [3], generalized synchronization [4], inverse generalized synchronization [5], $\Lambda - \phi$ generalized synchronization [6,7] and $\Phi - \Theta$ synchronization [8,9] and so on. Among the different types, *full state hybrid projective synchronization* (FSHPS) has been introduced, wherein each slave system variable synchronizes with a linear combination of master system variables [10]. Different types

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of synchronization such as complete synchronization, anti-synchronization, projective synchronization and hybrid synchronization can be achieved from the FSHPS scheme depending on the choice of scaling functions. On the other hand, when the inverted scheme is implemented, i.e., each master system state synchronizes with a linear combination of slave system states, the *inverse full-state hybrid projective synchronization* (IFSHPS) is obtained [11]. Moreover, when the scaling factors are replaced by scaling functions, function-based hybrid synchronization schemes are obtained, i.e., the *full-state hybrid function projective synchronization* (FSHFPS) [12] and the *inverse full-state hybrid function projective synchronization* (IFSHFPS) [13], respectively.

Recently, the topic of coexistence of several synchronization types between chaotic systems has recently started to attract increasing attention. In fact, very recent papers have investigated the co-existence of different types of synchronization when synchronizing two chaotic systems. For example, the approach developed in [14,15] has illustrated a rigorous study to prove the co-existence of some synchronization types between discrete-time chaotic (hyperchaotic) systems. Referring to integer-order chaotic systems, in [16] two synchronization schemes of co-existence have been proposed. The problem of coexistence of some types of synchronization between different dimensional fractional order chaotic systems has been studied [17, 18]. New approaches to study the co-existence of some types of synchronization between integer order and fractional order chaotic systems with different dimensions have been introduced in [19]. Meanwhile, to the best of our knowledge, the investigation of coexistence of FSHFPS and IFSHFPS for integer-order differential dynamical systems with different dimensions is not yet explored. The present research work focuses on coexistence of FSHFPS and IFSHFPS between chaotic and hyperchaotic systems.

Based on these considerations, this paper aims to give a further contribution to the topic by considering the co-existence of FSHFPS and IFSHFPS between non-identical and different dimensions chaotic and hyperchaotic systems. Specifically, the paper illustrates new schemes, which prove the co-existence of the full-state hybrid function projective synchronization (FSHFPS) and the inverse full-state hybrid function projective synchronization (IFSHFPS) between a three-dimensional master system and a four-dimensional slave system in 4D and 3D, respectively. These master-slave systems belong to general classes, which include several chaotic (hyperchaotic) systems characterized by different dimensions. The conceived schemes are general approches and the only restriction on the scaling functions is that they must be differentiable and bouned functions.

The paper is organized as follow: Section 2 gives some definitions related to FSHFPS and IFSHFPS. Sections 3 and 4 give the basic mathematical background of the coexistence of FSHFPS and IFSHFPS in 4D and 3D respectively. Section 5 presents some numerical examples of co-existence of synchronization types with the aim to show the effectiveness of the approach developed herein. Section 6 concludes the paper.

2 Definition of FSHFPS and IFSHFPS

We consider the following master and slave systems

$$\dot{X}(t) = F(X(t)), \tag{1}$$

$$\dot{Y}(t) = G(Y(t)) + U, \tag{2}$$

where $X(t) = (x_i(t))_{1\leq i\leq n}$, $Y(t) = (y_i(t))_{1\leq i\leq m}$ are the states of the master system and the slave system, respectively, $F : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^m \to \mathbb{R}^m$ and $U = (u_i)_{1\leq i\leq m}$ is a vector controller.

Definition 2.1 The master systems (1) and the slave system (2) are said to be full state hybrid function projective synchronized (FSHFPS), if there exist a controller $U = (u_i)_{1 \leq i \leq m}$ and differentiable functions $\alpha_{ij}(t) : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, ..., m$; $j = 1, 2, ..., n$, such that the synchronization errors

$$e_i(t) = y_i(t) - \sum_{j=1}^{n} \alpha_{ij}(t) x_j(t), \quad i = 1, 2, ..., m, \tag{3}$$

satisfy $\lim_{t\to\infty} e_i(t) = 0$.

Definition 2.2 The master systems (1) and the slave system (2) are said to be inverse full state hybrid function projective synchronized (IFSHFPS), if there exist a controller $U = (u_i)_{1 \leq i \leq m}$ and differentiable functions $\beta_{ij}(t) : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, ..., n$; $j = 1, 2, ..., m$, such that the synchronization errors

$$e_i(t) = x_i(t) - \sum_{j=1}^{m} \beta_{ij}(t) y_j(t), \quad i = 1, 2, ..., n, \tag{4}$$

satisfy $\lim_{t\to\infty} e_i(t) = 0$.

3 Scheme 1

Here, we assume that the master system can be considered as

$$\dot{x}_i(t) = f_i(X(t)), \quad i = 1, 2, 3, \tag{5}$$

where $X(t) = (x_i(t))_{1 \leq i \leq 3}$ is the state vector of the master system (5), $f_i : \mathbb{R}^3 \to \mathbb{R}$, $i = 1, 2, 3$. Also, consider the slave system as

$$\dot{y}_i(t) = \sum_{j=1}^{4} b_{ij} y_j(t) + g_i(Y(t)) + u_i, \quad i = 1, 2, 3, 4, \tag{6}$$

where $Y(t) = (y_i)_{1 \leq i \leq 4}$ is the state vector of the slave system (6), $(b_{ij}) \in \mathbb{R}^{4\times 4}$, $g_i : \mathbb{R}^4 \to \mathbb{R}$ are nonlinear functions and u_i, $i = 1, 2, 3, 4$, are controllers to be designed.

Definition 3.1 Let $(\alpha_j(t))_{1 \leq j \leq 4}$, $(\beta_j(t))_{1 \leq j \leq 3}$, $(\gamma_j(t))_{1 \leq j \leq 4}$ and $(\theta_j(t))_{1 \leq j \leq 3}$ be continuously differentiable and boundary functions, it is said that IFSHFPS and FSHFPS coexist in the synchronization of the master system (5) and the slave system (6), if there exist controllers u_i, $= 1, 2, 3, 4$, such that the synchronization errors

$$\begin{aligned} e_1(t) &= x_1(t) - \sum_{j=1}^{4} \alpha_j(t) y_j(t), \\ e_2(t) &= y_2(t) - \sum_{j=1}^{3} \beta_j(t) x_j(t), \\ e_3(t) &= x_3(t) - \sum_{j=1}^{4} \gamma_j(t) y_j(t), \\ e_4(t) &= y_4(t) - \sum_{j=1}^{3} \theta_j(t) x_j(t), \end{aligned} \tag{7}$$

satisfy $\lim_{t \longrightarrow +\infty} e_i(t) = 0, \;\; i = 1, 2, 3, 4.$

Sufficient conditions for co-existence of IFSHFPS and FSHFPS between systems (5) and (6) are given by the following theorem.

Theorem 3.1 *The coexistence of IFSHFPS and FSHFPS between the master system (5) and the slave system (6) will occur if* $\alpha_3(t)\gamma_1(t) - \alpha_1(t)\gamma_3(t) \neq 0$ *and the control law is designed as follows:*

$$\begin{aligned} u_1 &= \sum_{i=1}^{4} P_i \left(\sum_{j=1}^{4} (b_{ij} - c_{ij}) e_j(t) - R_i \right), \\ u_2 &= \sum_{j=1}^{4} (b_{2j} - c_{2j}) e_j(t) - R_2, \\ u_3 &= \sum_{i=1}^{4} Q_i \left(\sum_{j=1}^{4} (b_{ij} - c_{ij}) e_j(t) - R_i \right), \\ u_4 &= \sum_{j=1}^{4} (b_{4j} - c_{4j}) e_j(t) - R_4, \end{aligned} \tag{8}$$

where $(c_{ij})_{4\times 4}$ *are control constants to be selected and*

$$\begin{aligned} P_1 &= \frac{\gamma_3(t)}{\alpha_3(t)\gamma_1(t) - \alpha_1(t)\gamma_3(t)}, \\ P_2 &= \frac{\gamma_3(t)\alpha_2(t) - \alpha_3(t)\gamma_2(t)}{\alpha_3(t)\gamma_1(t) - \alpha_1(t)\gamma_3(t)}, \\ P_3 &= \frac{-\alpha_3(t)}{\alpha_3(t)\gamma_1(t) - \alpha_1(t)\gamma_3(t)}, \\ P_4 &= \frac{\gamma_3(t)\alpha_4(t) - \alpha_3(t)\gamma_4(t)}{\alpha_3(t)\gamma_1(t) - \alpha_1(t)\gamma_3(t)}, \\ Q_1 &= \frac{-\gamma_1(t)}{\alpha_3(t)\gamma_1(t) - \alpha_1(t)\gamma_3(t)}, \\ Q_2 &= \frac{\alpha_1(t)\gamma_2(t) - \alpha_2(t)\gamma_1(t)}{\alpha_3(t)\gamma_1(t) - \alpha_1(t)\gamma_3(t)}, \\ Q_3 &= \frac{\alpha_1(t)}{\alpha_3(t)\gamma_1(t) - \alpha_1(t)\gamma_3(t)}, \\ Q_4 &= \frac{\alpha_1(t)\gamma_4(t) - \alpha_4(t)\gamma_1(t)}{\alpha_3(t)\gamma_1(t) - \alpha_1(t)\gamma_3(t)}, \end{aligned} \tag{9}$$

and

$$R_1 = f_1(X(t)) - \sum_{j=1}^{4} \dot{\alpha}_j(t) y_j(t) - \sum_{i=1}^{4} \alpha_i(t) \left(\sum_{j=1}^{4} b_{ij} y_j(t) + g_i(Y(t)) \right), \tag{10}$$

$$R_2 = \sum_{j=1}^{4} b_{2j} y_j(t) + g_2(Y(t)) - \sum_{j=1}^{3} \dot{\beta}_j(t) x_j(t) - \sum_{j=1}^{3} \beta_j(t) \dot{x}_j(t), \tag{11}$$

$$R_3 = f_3(X(t)) - \sum_{j=1}^{4} \dot{\gamma}_j(t)\, y_j(t) - \sum_{i=1}^{4} \gamma_i(t) \left(\sum_{j=1}^{4} b_{ij} y_j(t) + g_i(Y(t)) \right),$$

$$R_4 = \sum_{j=1}^{4} b_{4j} y_j(t) + g_4(Y(t)) - \sum_{j=1}^{3} \dot{\theta}_j(t)\, x_j(t) - \sum_{j=1}^{3} \theta_j(t)\, \dot{x}_j(t).$$

Proof. The error system (7) can be differentiated as follows:

$$\begin{aligned} \dot{e}_1(t) &= \dot{x}_1(t) - \sum_{j=1}^{4} \dot{\alpha}_j(t)\, y_j(t) - \sum_{j=1}^{4} \alpha_j(t)\, \dot{y}_j(t), \\ \dot{e}_2(t) &= \dot{y}_2(t) - \sum_{j=1}^{3} \dot{\beta}_j(t)\, x_j(t) - \sum_{j=1}^{3} \beta_j(t)\, \dot{x}_j(t), \\ \dot{e}_3(t) &= \dot{x}_3(t) - \sum_{j=1}^{4} \dot{\gamma}_j(t)\, y_j(t) - \sum_{j=1}^{4} \gamma_j(t)\, \dot{y}_j(t), \\ \dot{e}_4(t) &= \dot{y}_4(t) - \sum_{j=1}^{3} \dot{\theta}_j(t)\, x_j(t) - \sum_{j=1}^{3} \theta_j(t)\, \dot{x}_j(t). \end{aligned} \tag{12}$$

Furthermore, the error system (12) can be written as

$$\begin{aligned} \dot{e}_1(t) &= \sum_{j=1}^{4} \alpha_j(t)\, u_j + R_1, \\ \dot{e}_2(t) &= u_2 + R_2, \\ \dot{e}_3(t) &= \sum_{j=1}^{4} \gamma_j(t)\, u_j + R_3, \\ \dot{e}_4(t) &= u_4 + R_4, \end{aligned} \tag{13}$$

where R_i, $i = 1, 2, 3, 4$, were described by (10). By substituting the control law (8) into (13), the error system can be described as

$$\dot{e}_i(t) = \sum_{j=1}^{4} (b_{ij} - c_{ij})\, e_j(t), \quad i = 1, 2, 3, 4, \tag{14}$$

or in the compact form

$$\dot{e}(t) = (B - C)\, e(t), \tag{15}$$

where $B = (b_{ij})_{4\times 4}$ and $C = (c_{ij})_{4\times 4}$ is the control matrix. If we select the control matrix C such that all the eigenvalues of $B - C$ are strictly negative, it is immediate that all solutions of the error system (15) go to zero as $t \to \infty$. Therefore, the systems (5) and (6) are globally synchronized in 4D.

4 Scheme 2

Now, the master and the slave systems can be described in the following forms

$$\dot{x}_i(t) = \textstyle\sum_{j=1}^{3} a_{ij} x_j(t) + f_i(X(t)), \quad i = 1, 2, 3, \tag{16}$$

$$\dot{y}_i(t) = g_i(Y(t)) + u_i, \quad i = 1, 2, 3, 4, \tag{17}$$

where $X(t) = (x_i)_{1\leq i\leq 3}$, $Y(t) = (y_i)_{1\leq i\leq 4}$ are the states of the master system and the slave system, respectively, $(a_{ij}) \in \mathbb{R}^{3\times 3}$, $f_i : \mathbb{R}^3 \to \mathbb{R}$ are nolinear functions, $g_i : \mathbb{R}^4 \to \mathbb{R}$ and $u_i, = 1, 2, 3, 4$, are controllers to be constructed.

Definition 4.1 Let $(\lambda_j(t))_{1\leq j\leq 3}$, $(\mu_j(t))_{1\leq j\leq 4}$ and $(\sigma_j(t))_{1\leq j\leq 3}$ be continuously differentiable and boundary functions, it is said that IFSHFPS and FSHFPS coexist in the synchronization of the master system (16) and the slave system (17), if there exist controllers $u_i, = 1, 2, 3$, such that the synchronization errors

$$\begin{aligned} e_1(t) &= y_1(t) - \sum_{j=1}^{3} \lambda_j(t)\, x_j(t), \qquad (18)\\ e_2(t) &= x_2(t) - \sum_{j=1}^{4} \mu_j(t)\, y_j(t), \\ e_3(t) &= y_3(t) - \sum_{j=1}^{3} \sigma_j(t)\, x_j(t), \end{aligned}$$

satisfy $\lim_{t\longrightarrow+\infty} e_i(t) = 0,\ \ i = 1, 2, 3.$

Hence, we have the following result.

Theorem 4.1 *To achieve the coexistence of IFSHFPS and FSHFPS between the master system (16) and the slave system (17), we assume that $\mu_2(t) \neq 0$ and the control law is constructed as follows:*

$$\begin{aligned} u_1 &= \sum_{j=1}^{3} (a_{1j} - l_{1j})\, e_j(t) - R_1, \qquad (19)\\ u_2 &= -\frac{\mu_1(t)}{\mu_2(t)} \left(\sum_{j=1}^{3} (a_{1j} - l_{1j})\, e_j(t) - R_1 \right) - \frac{1}{\mu_2(t)} \left(\sum_{j=1}^{3} (a_{2j} - l_{2j})\, e_j(t) - R_2 \right) \\ &\quad -\frac{\mu_3(t)}{\mu_2(t)} \left(\sum_{j=1}^{3} (a_{3j} - l_{3j})\, e_j(t) - R_3 \right), \\ u_3 &= \sum_{j=1}^{3} (a_{3j} - l_{3j})\, e_j(t) - R_3, \\ u_4 &= 0, \end{aligned}$$

where $(l_{ij})_{3\times 3}$ are control constants to be determined, whereas R_1, R_2 and R_3 are chosen *as follows*

$$R_1 = g_1(Y(t)) - \sum_{j=1}^{3}(a_{1j} - l_{1j})\, e_j(t) - \sum_{j=1}^{3}\dot{\lambda}_j\,(t)\, x_j\,(t) \tag{20}$$

$$- \sum_{i=1}^{3}\lambda_i\,(t)\left(\sum_{j=1}^{3} a_{ij}x_j(t) + f_i(X(t))\right), \tag{21}$$

$$R_2 = \sum_{j=1}^{3} a_{2j}x_j(t) + f_2(X(t)) - \sum_{j=1}^{3}(a_{2j} - l_{2j})\, e_j(t) \tag{22}$$

$$- \sum_{j=1}^{4}\dot{\mu}_j\,(t)\, y_j\,(t) - \sum_{j=1}^{4}\mu_j\,(t)\, g_j(Y(t)),$$

$$R_3 = g_3(Y(t)) - \sum_{j=1}^{3}(a_{3j} - l_{3j})\, e_j(t) - \sum_{j=1}^{3}\dot{\sigma}_j\,(t)\, x_j\,(t) \tag{23}$$

$$- \sum_{i=1}^{3}\sigma_i\,(t)\left(\sum_{j=1}^{3} a_{ij}x_j(t) + f_i(X(t))\right).$$

Proof. Error system (18), between master system (16) and the slave system (17), can be derived as

$$\begin{aligned} \dot{e}_1(t) &= \dot{y}_1\,(t) - \sum_{j=1}^{3}\dot{\lambda}_j\,(t)\, x_j\,(t) - \sum_{j=1}^{3}\lambda_j\,(t)\,\dot{x}_j\,(t)\,, \\ \dot{e}_2(t) &= \dot{x}_2\,(t) - \sum_{j=1}^{4}\dot{\mu}_j\,(t)\, y_j\,(t) - \sum_{j=1}^{4}\mu_j\,(t)\,\dot{y}_j\,(t)\,, \\ \dot{e}_3(t) &= \dot{y}_3\,(t) - \sum_{j=1}^{3}\dot{\sigma}_j\,(t)\, x_j\,(t) - \sum_{j=1}^{3}\sigma_j\,(t)\,\dot{x}_j\,(t)\,. \end{aligned} \tag{24}$$

Error system (24), after some algebraic manipulations, becomes

$$\begin{aligned} \dot{e}_1(t) &= \sum_{j=1}^{3}(a_{1j} - l_{1j})\, e_j(t) + u_1 + R_1, \\ \dot{e}_2(t) &= \sum_{j=1}^{3}(a_{2j} - l_{2j})\, e_j(t) - \sum_{j=1}^{4}\mu_j\,(t)\, u_j + R_2, \\ \dot{e}_3(t) &= \sum_{j=1}^{3}(a_{3j} - l_{3j})\, e_j(t) + u_3 + R_3, \end{aligned} \tag{25}$$

where R_i, $i = 1, 2, 3$, were given by (21). By considering the control law (19), it follows that the error dynamics between systems (16) and (17) are described by

$$\dot{e}_i(t) = \sum_{j=1}^{4}(b_{ij} - l_{ij})\, e_j(t), \quad i = 1, 2, 3, \tag{26}$$

or in the compact form

$$\dot{e}(t) = (A - L)\, e(t), \tag{27}$$

where $e(t) = (e_i(t))_{1\leq i\leq 3}$, $A = (a_{ij})_{3\times 3}$, $L = (l_{ij})_{3\times 3}$. Construct the candidate Lyapunov function in the form: $V(e(t)) = e^T(t)e(t)$, we obtain

$$\begin{aligned} \dot{V}(e(t)) &= \dot{e}^T(t)e(t) + e^T(t)\dot{e}(t) \\ &= e^T(t)(A-L)^T e(t) + e^T(t)(A-L)\, e(t) \\ &= e^T(t)\left[(A-L)^T + (A-L)\right] e(t). \end{aligned}$$

If the control matrix L is chosen such that $(A-L)^T + (A-L)$ is a negative definite matrix, we get $\dot{V}(e(t)) < 0$. Thus, from the Lyapunov stability theory, the zero solution of the error system (27) is globally asymptotically stable, i.e,

$$\lim_{t\to\infty} e_i(t) = 0, \quad i = 1, 2, 3. \tag{28}$$

Therefore, systems (16) and (17) are globally synchronized in 3D.

5 Numerical Examples

This section provides several examples of coexistence of FSHFPS and IFSHFPS between 3D chaotic systems and 4D hyperchaotic systems in 4D and 3D, respectively. Each numerical example is related to one of the theorems developed in previous sections.

5.1 Example 1

In this example, the master system is defined by the following new 3D system [20]

$$\begin{aligned} \dot{x}_1 &= a_1(x_2 - x_1), \\ \dot{x}_2 &= x_1x_3, \\ \dot{x}_3 &= 50 - a_2x_1^2 - a_3x_3. \end{aligned} \tag{29}$$

When $a_1 = 2.9$, $a_2 = 0.7$, $a_3 = 0.6$ and the initial conditions are taken as $(x_1(0), x_2(0), x_3(0)) = (0.6, 0.5, 0.4)$, system (29) exhibits chaotic attractors as shown in Figures 1 and 2.

The salve system is described by

$$\begin{aligned} \dot{y}_1 &= b_1(y_2 - y_1) + y_2y_3 + y_4 + u_1, \\ \dot{y}_2 &= b_2y_1 + y_4 - b_3y_1y_3 + u_2, \\ \dot{y}_3 &= -b_4y_3 + b_5y_1y_2 + u_3, \\ \dot{y}_4 &= -y_1 - y_2 + u_4. \end{aligned} \tag{30}$$

When the controllers $u_1 = u_2 = u_3 = u_4 = 0$, $(b_1,b_2,b_3,b_4,b_5) = (18,40,5,-3,4)$ and the initial conditions are given as $(y_1(0), y_2(0), y_3(0), y_4(0)) = (0.5, 0.8, 0.2, 1.3)$, system (30) exhibits hyperchaotic attractors as shown in Figure 2 [21].

Based on the notations used in Section 3, the linear part B and the nonlinear part g of the slave system (30) are given as follows

$$B = \begin{pmatrix} -18 & 18 & 0 & 1 \\ 40 & 0 & 0 & 1 \\ 0 & 0 & -3 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix} \text{ and } g = \begin{pmatrix} y_2y_3 \\ -5y_1y_3 \\ 4y_1y_2 \\ 0 \end{pmatrix}.$$

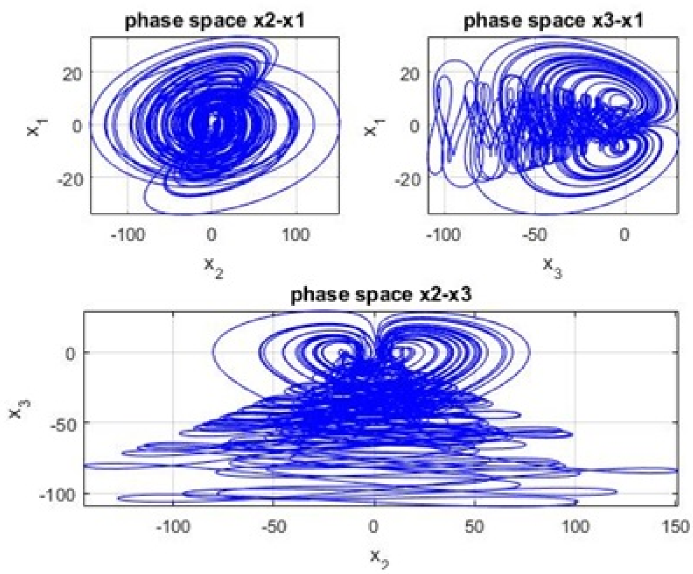

Figure 1: Phase portraits of the master system (25) in 2D.

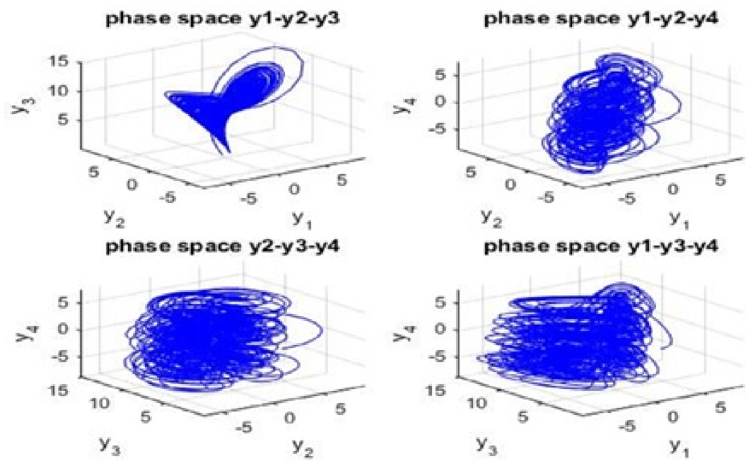

Figure 2: Phase portraits of the slave system without control (26) in 3D.

According to the approach developed in Section 3, the synchronization errors between the master system (29) and the slave system (30) are defined as:

$$\begin{aligned} e_1 &= x_1 - \alpha_1(t)\, y_1 - \alpha_2(t)\, y_2 - \alpha_3(t)\, y_3 - \alpha_4(t)\, y_4, \\ e_2 &= y_2 - \beta_1(t)\, x_1 - \beta_2(t)\, x_2 - \beta_3(t)\, x_3, \\ e_3 &= x_3 - \gamma_1(t)\, y_1 - \gamma_2(t)\, y_2 - \gamma_3(t)\, y_3 - \gamma_4(t)\, y_4, \\ e_4 &= y_4 - \theta_1(t)\, x_1 - \theta_2(t)\, x_2 - \theta_3(t)\, x_3, \end{aligned} \tag{31}$$

where $\alpha_1(t) = \sin t$, $\alpha_2(t) = 1$, $\alpha_3(t) = \frac{1}{t+1}$, $\alpha_4(t) = 2$, $\beta_1(t) = 3$, $\beta_2(t) = \cos t$, $\beta_3(t) = 4$, $\gamma_1(t) = e - t$, $\gamma_2(t) = 2$, $\gamma_3(t) = 0$, $\gamma_4(t) = \frac{1}{t^2+1}$, $\theta_1(t) = \frac{t}{t+1}$, $\theta_2(t) = 0$, $\theta_3(t) = \sin 3t$. So,

$$\alpha_3(t)\,\gamma_1(t) - \alpha_1(t)\,\gamma_3(t) = \frac{1}{e^t(t+1)} \neq 0. \tag{32}$$

The coexistence of IFSHFPS and FSHFPS, in this example, is achieved when the control matrix C is selected as

$$C = \begin{pmatrix} 0 & 18 & 0 & 1 \\ 40 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix}, \tag{33}$$

and the controllers u_i, $1 \leq i \leq 4$, are constructed according to (8) as follows:

$$\begin{aligned} u_1 &= -2e^t(-e_2 - R_2) + e^t(-3e_3 - R_3) - \frac{e^t}{t^2+1}(-e_4 - R_4), && (34)\\ u_2 &= -e_2 + 5y_1y_3 - 40y_1 - y_4 - R_2, \\ u_3 &= -(t+1)(-18e_1 - R_1) + e^t(t+1) && (35)\\ &\quad \left[-(2 + 2e_2 + 2R_2 + 3e_3 + R_3)\sin t + \left(\frac{\sin t}{t^2+1} - e^{-t}\right)(-e_4 - R_4)\right], \\ u_4 &= -e_4 + y_1 + y_2 - R_4, \end{aligned}$$

where

$$\begin{aligned} R_1 &= 2.9(x_2 - x_1) - y_1\cos t + \frac{1}{(t+1)^2}y_3 - \sin t\,(18(y_2 - y_1) + y_2y_3) && (36)\\ &\quad + \frac{1}{t+1}(4y_1y_2 - 3y_3) - y_1 - y_2, && (37)\\ R_2 &= -5y_1y_3 + 40y_1 + y_4 + x_2\sin t - 8.7(x_2 - x_1) - x_1x_3\cos t, \\ R_3 &= 50 - 0.7x_1^2 - 0.6x_3 + e^{-t}y_1 + \frac{2t}{(t^2+1)^2}y_4 - e^{-t}(18(y_2 - y_1) + y_2y_3) && (38)\\ &\quad + 10y_1y_3 + 80y_1 - 2y_4 + \frac{1}{t^2+1}(y_1 + y_2), \\ R_4 &= -y_1 - y_2 - \frac{t+1-t^2}{(t+1)^2}x_1 - 3x_3\cos 3t - \frac{2.9t}{t+1}(x_2 - x_1) && (39)\\ &\quad - \left(50 - 0.7x_1^2 - 0.6x_3\right)\sin 3t. \end{aligned}$$

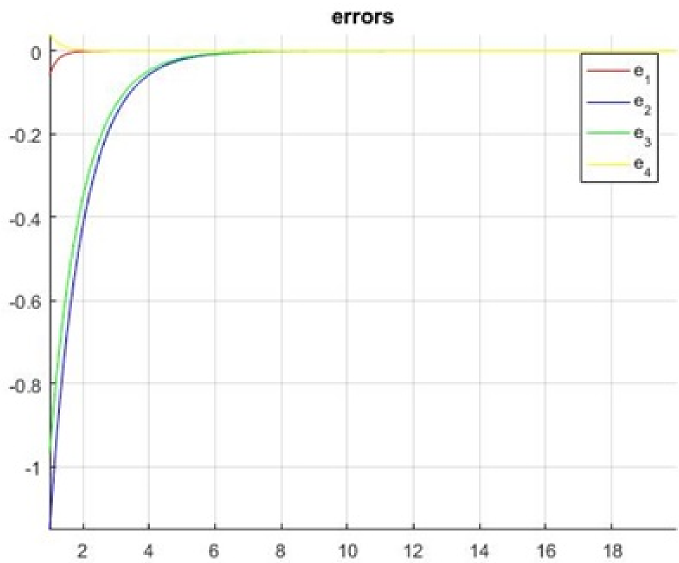

Figure 3: Time evolution of the errors e1, e2, e3 and e4.

We can show that all eigenvalues of $B-C$ have negative real parts. It can be seen that all conditions of Theorem 1 are satisfied. Consequently, the error functions between systems (29) and (30) are described by

$$\begin{aligned} \dot{e}_1 &= -18e_1, \\ \dot{e}_2 &= -e_2, \\ \dot{e}_3 &= -3e_3, \\ \dot{e}_4 &= -e_4. \end{aligned} \tag{40}$$

Numerical results plotted in Figure 3 are obtained, indicating that the coexistence of IFSHFPS and FSHFPS is effectively achieved in 4D.

5.2 Example 2

Herein, the master system is selected as a 3D chaotic system proposed in [22] by the following ODE system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= -c_1 x_1 (1-x_1) - x_2 + c_2 x_2^2. \end{aligned} \tag{41}$$

System (41), when $(c_1, c_2) = (0.2, 0.01)$ and $(x_1(0), x_2(0), x_3(0)) = (0.0.1, -0.0.1, 0.0.1)$, possesses chaotic attractors plotted in Figures 4.

Using the notations presented in Section 4, the linear part A and the nonlinear part

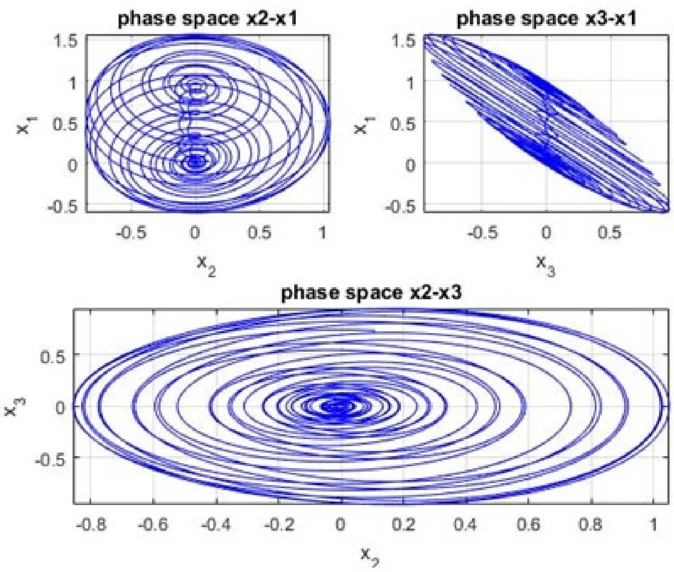

Figure 4: Phase portraits of the master system (33) in 2D.

f of the master system (41) are given as follows

$$A = (a_{ij})_{3\times 3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.2 & -1 & 0 \end{pmatrix} \text{ and } f = \begin{pmatrix} 0 \\ 2 \\ 0.2x_1^2 + 0.01x_1^2 \end{pmatrix}.$$

As the slave master system, we consider a novel 4D hyperchaotic system introduced in [23] by the following ODE system

$$\begin{aligned} \dot{y}_1 &= d_1(y_2 - y_1) + y_2y_3 - y_4 + u_1, \\ \dot{y}_2 &= d_2y_2 - y_1y_3 + y_4 + u_2, \\ \dot{y}_3 &= y_1y_2 - d_3y_3 + u_3, \\ \dot{y}_4 &= -d_4(y_1 + y_2) + u_4. \end{aligned} \tag{42}$$

System (42), when $u_1 = u_2 = u_3 = u_4 = 0$, $(d_1, d_2, d_3, d_4) = (40, 20.5, 5, 2.5)$ and $(y_1(0), y_2(0), y_3(0), y_4(0)) = (0.5, 0.8, 0.6, 0.2)$, displays hyperchaotic attractors shown in Figure 5.

In this example, according to the control scheme presented in Section 4, the synchronization errors are given as

$$\begin{aligned} e_1 &= y_1 - \lambda_1(t)x_1 - \lambda_1(t)x_1 - \lambda_1(t)x_1, \\ e_2 &= \mu_1(t)y_1 + \mu_2(t)y_2 + \mu_3(t)y_3 + \mu_4(t)y_4 - x_2, \\ e_3 &= y_3 - \sigma_1(t)x_1 - \sigma_2(t)x_2 - \sigma_3(t)x_3, \end{aligned} \tag{43}$$

where $\lambda_1(t) = e^{-t}$, $\lambda_2(t) = \sin 2t$, $\lambda_3(t) = 0$, $\mu_1(t) = 0$, $\mu_2(t) = \frac{1}{\sqrt{t+1}}$, $\mu_3(t) = \frac{1}{1+\cos^2 t}$, $\mu_4(t) = 4$, $\sigma_1(t) = \frac{1}{\text{lin}(t+1)}$, $\sigma_2(t) = \frac{1}{1+\sin^2 t}$ and $\sigma_3(t) = 0$.

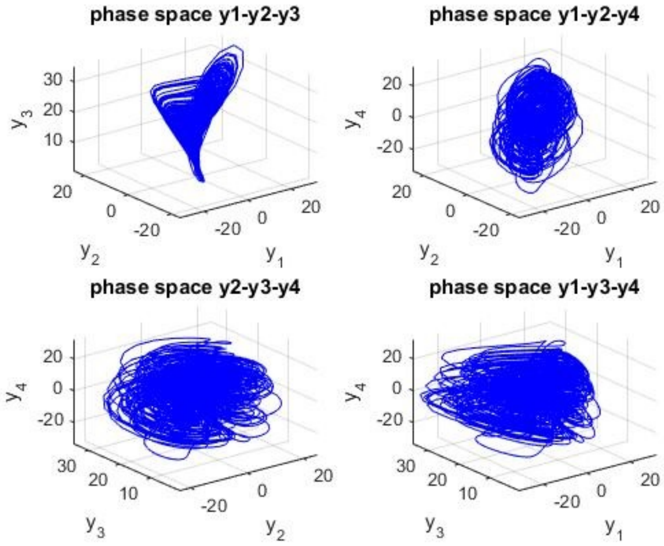

Figure 5: Phase portraits of the slave system (34) without control in 3D.

We selecte the control matrix L as

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \tag{44}$$

and by using (19), the controllers u_1, u_2, u_3 and u_4 are designed as follows

$$\begin{aligned} u_1 &= -e_1 - R_1, \\ u_2 &= -\left(\sqrt{t}+1\right)(-2e_2 - R_2) - \frac{\sqrt{t}+1}{1+\cos^2 t}(-3e_3 - R_3), \\ u_3 &= -3e_3 - R_3, \\ u_4 &= 0, \end{aligned} \tag{45}$$

where

$$R_1 = 40(y_2 - y_1) + y_2 y_3 - y_4 + e_1 + e^{-t} x_1 - 2x_2 \cos 2t - x_2 e^{-t} - x_3 \sin 2t, \tag{46}$$

$$\begin{aligned} R_2 = {} & x_3 + 2e_2 - \frac{y_2}{2\sqrt{t}\left(\sqrt{t}+1\right)^2} - y_3 \frac{2\sin t \cos t}{(1+\cos^2 t)} - \frac{1}{\sqrt{t}+1} \\ & (20.5y_2 - y_1 y_3 + y_4) - \frac{1}{1+\cos^2 t}(y_1 y_2 - 5.5y_3) + 10(y_1 + y_2), \end{aligned} \tag{47}$$

$$\begin{aligned} R_3 = {} & y_1 y_2 - 5.5y_3 + 3e_3 + \frac{1}{(t+1)\,\mathrm{lin}^2(t+1)} x_1 + \frac{2\sin t\cos t}{(1+\sin^2 t)} x_2 \\ & -\frac{x_2}{\mathrm{lin}(t+1)} - \frac{x_3}{1+\sin^2 t}. \end{aligned} \tag{48}$$

It is easy to see that $(A-L)^T + (A-L)$ is a negative definite matrix. It can be readily shown that all conditions of Theorem 2 are satisfied. Consequently, the error functions

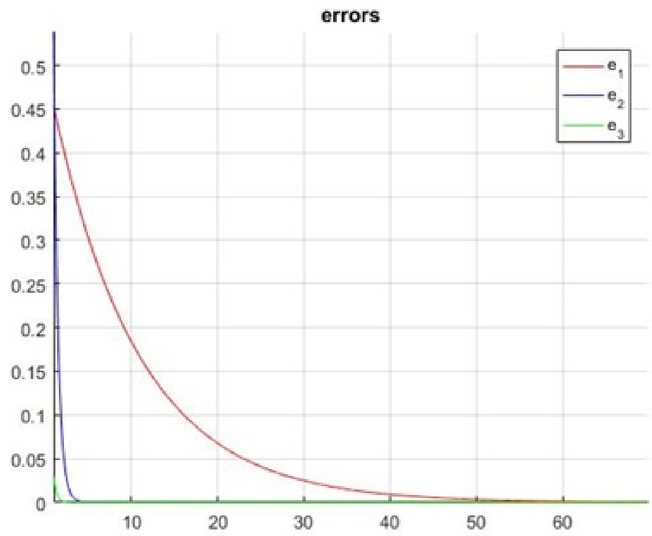

Figure 6: Time evolution of the errors e1, e2 and e3.

between systems (41) and (42) are described by

$$\begin{aligned} \dot{e}_1 &= -0.1e_1, \\ \dot{e}_2 &= -2e_2, \\ \dot{e}_3 &= -3e_2. \end{aligned} \tag{49}$$

According to numerical results obtained in Figure 6, it can be concluded that the coexistence of FSHFPS and IFSHFPS synchronization is effectively achieved in 3D.

6 Conclusion

When analyzing the synchronization of chaotic systems, an interesting phenomenon that may occur is the co-existence of some synchronization types. Based on these considerations, this paper has presented new results related to the co-existence of FSHFPS and IFSHFPS between non-identical and different dimensions chaotic systems characterized. Specifically, the manuscript has proposed new schemes, which assures the co-existence of FSHFPS and IFSHFPS between a three-dimensional master system and a four-dimensional slave system. Note that the approach developed herein enables to prove the co-existence of FSHFPS and IFSHFPS in several cases. Specifically, the approach can be applied to: i) wide classes of chaotic (hyperchaotic) master-slave systems; ii) non-identical systems with different dimensions; iii) schemes wherein the scaling factor of the linear combination can be any arbitrary differentiable function. Numerical examples, describing the co-existence of FSHFPS and IFSHFPS between chaotic and hyperchaotic systems, have clearly highlighted the effectiveness of the approach proposed herein.

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