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Entropy Solutions of Nonlinear $p(x)$-Parabolic Inequalities

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Abstract: In this paper we prove the existence of entropy solutions for weighted $p(x)$-parabolic problem associated with the equation:

$$\frac{\partial u}{\partial t} + Au = g(u)\omega(x)\Big|\nabla u\Big|^{p(x)} + f \quad \text{in} \quad \Omega \times (0,T),$$

where the operator $Au = -\text{div}\Big(\omega(x)\Big|\nabla u\Big|^{p(x)-2}\nabla u\Big)$ and on the right-hand side f belongs to $L^1(\Omega \times (0,T))$ and $\omega(x)$ is a weight function.

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1 Introduction

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, T be a positive real number and $Q = \Omega \times (0,T)$, while the variable exponent $p: \bar{\Omega} \to (1,\infty)$ is a continuous function, the data $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$. The objective of this paper is to study the existence of an entropy solution for the obstacle parabolic problems of the type:

$$\begin{cases} u \geq \psi, & \text{a.e. in } \Omega \times (0,T), \\ \frac{\partial u}{\partial t} - \text{div}\Big(\omega(x)\Big|\nabla u\Big|^{p(x)-2}\nabla u\Big) = \omega(x)g(u)\Big|\nabla u\Big|^{p(x)} + f, & \text{in } \Omega \times (0,T), \\ u(x,0) = u_0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{1}$$

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The operator $-div\Big(\omega(x)\Big|\nabla u\Big|^{p(x)-2}\nabla u\Big)$ is a Leray-Lions operator defined on $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))$ which is coercive.

In recent years, the study of partial differential equations and variational problems with variable exponent has received considerable attention in many models coming from various branches of mathematical physics, such as elastic mechanics,electro-rheological fluid dynamics and image processing, etc. We refer the readers to [12, 22]. Degenerate phenomena appear in the area of oceanography, turbulent fluid flows, induction heating and electrochemical problems.The notion of entropy solutions has been proposed by Bènilan et al. in [8] for the nonlinear elliptic problems.

Recently, when $\omega(x)\equiv 1$, the existence and uniqueness of entropy solutions of $p(x)$-Laplace equation with L^1 data were proved in [24]by Sanchón and Urbano. This notion was adapted to the study of the entropy solutions for nonlinear elliptic equations with variable exponents by Chao Zhang in [26] and the existence of solutions of some unilateral problems in the framework of Orlicz spaces has been established by M. Kbiri Alaoui, D. Meskine, A. Souissi in [17] in terms of the penalization method. E. Azroul, H. Redwane and M. Rhoudaf [5] have proved the existence of renormalized solution in Orlicz spaces in the case where $b(u)=u$. Fortunately, Kim, Wang and Zhang [18]have shown good properties of a function space and the so-called weighted variable exponent Lebesque-Sobolev spaces, and the existence and some properties of solutions for degenerate elliptic equations with exponent variable have been proved by Ky Ho, Inbo Sim [16].Other work in this direction can be found in [4] by Y. Akdim, C. Allalou, N. El gorch.

Now we review some definitions and basic properties of the weighted variable exponent Lebesgue spaces $L^{p(x)}(\Omega,\omega)$ and the weighted variable exponent Sobolev spaces $W^{1,p(x)}(\Omega,\omega)$.

Let ω be a mesurable positive and a.e. finite function in $\mathbb{R}^N$. Set

$$C_+(\overline{\Omega})=\{h\in C(\overline{\Omega}):\min_{x\in\overline{\Omega}}h(x)>1\}.$$

For any $h\in C_+(\overline{\Omega})$, we define $h^+=\max_{x\in\overline{\Omega}}h(x)$, $h^-=\min_{x\in\overline{\Omega}}h(x)$.

For any $p\in C_+(\overline{\Omega})$, we introduce the weighted variable exponent Lebesgue space $L^{p(x)}(\Omega,\omega)$ which consists of all measurable real-valued functions u such that

$$\int_\Omega\Big|u(x)\Big|^{p(x)}\omega(x)dx<\infty,$$

endowed with the Luxemburg norm

$$||u||_{L^{p(x)}(\Omega,\omega)}=\inf\Big\{\lambda>0:\int_\Omega\Big|\frac{u(x)}{\lambda}\Big|^{p(x)}\omega(x)dx\leq 1\Big\}$$

becomes a normed space. When $\omega(x)\equiv 1$, we have $L^{p(x)}(\Omega,\omega)\equiv L^{p(x)}(\Omega)$ and we use the notation $||u||_{L^{p(x)}(\Omega)}$ instead of $\|u\|_{L^{p(x)}(\Omega,w)}$. The following Hölder type inequality is useful for the next sections. The weighted variable exponent Sobolev space $W^{1,p(x)}(\Omega,\omega)$ is defined by

$$W^{1,p(x)}(\Omega,\omega)=\Big\{u\in L^{p(x)}(\Omega);\Big|\nabla u\Big|\in L^{p(x)}(\Omega,\omega)\Big\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega,\omega)}=||u||_{L^{p(x)}(\Omega)}+||\nabla u||_{L^{p(x)}(\Omega,\omega)} \tag{2}$$

or, equivalently

$$\|u\|_{W^{1,p(x)}(\Omega,\omega)} = \inf\Big\{\lambda > 0 : \int_\Omega \Big|\frac{u(x)}{\lambda}\Big|^{p(x)} + \omega(x)\Big|\frac{\nabla u(x)}{\lambda}\Big|^{p(x)} dx \le 1\Big\}$$

for all $u \in W^{1,p(x)}(\Omega,\omega)$.

It is significant that smooth functions are not dense in $W^{1,p(x)}(\Omega)$ without additional assumptions on the exponent $p(x)$. This feature was observed by Zhikov [27] in connection with the Lavrentiev phenomenon. However, when the exponent $p(x)$ is log-Hölder continuous, i.e., there is a constant C such that

$$|p(x) - p(y)| \le \frac{C}{-\log|x-y|} \tag{3}$$

for every x, y with $|x-y| \le \frac{1}{2}$, then smooth functions are dense in variable exponent Sobolev spaces and there is no confusion in defining the Sobolev space with zero boundary values, $W_0^{1,p(x)}(\Omega)$, as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{1,p(x)}(\Omega)}$ (see [15]). $W_0^{1,p(x)}(\Omega,\omega)$ is defined as the completion of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega,\omega)$ with respect to the norm $\|u\|_{W^{1,p(x)}(\Omega,\omega)}$. Throughout the paper, we assume that $p \in C_+(\overline{\Omega})$ and ω is a measurable positive and a.e. finite function in Ω.

The plan of the paper is as follows. Section 2 presents the mathematical preliminaries. In Section 3 we make precise all the assumptions on A, g, f and u_0,and give the definition of an entropy solution of $(\mathcal{P})$. In Section 4 we establish the existence of such a solution in Theorem 4.1.

2 Preliminaries

In this section, we state some elementary properties for the weighted variable exponent Lebesgue-Sobolev spaces which will be used in the next sections. The basic properties of the variable exponent Lebesgue-Sobolev spaces, that is when $\omega(x) \equiv 1$, can be found in [13, 19].

Lemma 2.1 *(See [13, 19])(Generalised Hölder inequality).*

i) *For any functions $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*
$\Big|\int_\Omega uv dx\Big| \le (\frac{1}{p^-} + \frac{1}{p'^-})||u||_{p(x)}||v||_{p'(x)} \le 2||u||_{p(x)}||v||_{p'(x)}$.

ii) *For all p, $q \in C_+(\bar{\Omega})$ such that $p(x) \le q(x)$ a.e. in Ω, we have $L^{q(.)} \hookrightarrow L^{p(x)}$ and the embedding is continuous.*

Lemma 2.2 *(See [18]) Denote $\rho(u) = \int_\Omega \omega(x)\Big|u(x)\Big|^{p(x)} dx$ for all $u \in L^{p(x)}(\Omega,\omega)$. Then*

$$||u||_{L^{p(x)}(\Omega,\omega)} < 1 (= 1; > 1) \ if \ and \ only \ if \ \rho(u) < 1 (= 1; > 1), \tag{4}$$

$$if \ ||u||_{L^{p(x)}(\Omega,\omega)} > 1 \ then \ ||u||^{p^-}_{L^{p(x)}(\Omega,\omega)} \le \rho(u) \le ||u||^{p^+}_{L^{p(x)}(\Omega,\omega)}, \tag{5}$$

$$if \ ||u||_{L^{p(x)}(\Omega,\omega)} < 1 \ then \ ||u||^{p^+}_{L^{p(x)}(\Omega,\omega)} \le \rho(u) \le ||u||^{p^-}_{L^{p(x)}(\Omega,\omega)}. \tag{6}$$

Remark 2.1 ([23].) We set

$$I(u) = \int_\Omega \Big|u(x)\Big|^{p(x)} + \omega(x)|\nabla u(x)|^{p(x)}dx.$$

Then, following the same argumen, we have
$\min\Big\{\|u\|^{p^-}_{W^{1,p(x)}(\Omega,\omega)}, \|u\|^{p^+}_{W^{1,p(x)}(\Omega,\omega)}\Big\} \le I(u) \le \max\Big\{\|u\|^{p^-}_{W^{1,p(x)}(\Omega,\omega)}, \|u\|^{p^+}_{W^{1,p(x)}(\Omega,\omega)}\Big\}.$

Throughout the paper, we assume that ω is a measurable positive and a.e. finite function in Ω satisfying the following relations:
$(\mathbf{W}_1)$ $\omega \in L^1_{loc(\Omega)}$ and $\omega^{-\frac{1}{(p(x)-1)}} \in L^1_{loc}(\Omega)$;
$(\mathbf{W}_2)$ $\omega^{-s(x)} \in L^1(\Omega)$ with $s(x) \in (\frac{N}{p(x)}, \infty) \cap [\frac{1}{p(x)-1}, \infty)$.
The reasons why we assume $(\mathbf{W}_1)$ and $(\mathbf{W}_2)$ can be found in [18].

Remark 2.2 *([18].)* (i) If ω is a positive measurable and finite function, then $L^{p(x)}(\Omega,\omega)$ is a reflexive Banach space.
(ii) Moreover, if $(\mathbf{W}_1)$ holds, then $W^{1,p(x)}(\Omega,\omega)$ is a reflexive Banach space.

For $p, s \in C_+(\overline{\Omega})$, denote

$$p_s(x) = \frac{p(x)s(x)}{s(x)+1} < p(x),$$

where $s(x)$ is given in $(\mathbf{W}_2)$. Assume that we fix the variable exponent restrictions

$$p^*_s(x) = \begin{cases} \frac{p(x)s(x)N}{(s(x)+1)N - p(x)s(x)}, & \text{if } N > p_s(x), \\ \text{arbitrary}, & \text{if } N \le p_s(x), \end{cases}$$

for almost all $x \in \Omega$. These definitions play a key role in our paper. We shall frequently make use of the following (compact) imbedding lemma for the weighted variable exponent Lebesgue-Sobolev space in the next sections.

Lemma 2.3 *([18].) Let $p, s \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (3), and let ($\boldsymbol{W}_1$) and ($\boldsymbol{W}_2$) be satisfied. If $r \in C_+(\overline{\Omega}))$ and $1 < r(x) \le p^*_s$, then we obtain the continuous imbedding*

$$W^{1,p(x)}(\Omega,\omega) \hookrightarrow L^{r(x)}(\Omega).$$

Moreover, we have the compact imbedding

$$W^{1,p(x)}(\Omega,\omega) \hookrightarrow L^{r(x)}(\Omega),$$

provided that $1 < r(x) < p^*_{s(x)}$ for all $x \in \overline{\Omega}$.

From Lemma 2.3, we have Poincaré-type inequality immediately.

Corollary 2.1 *([18].) Let $p \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (3). If ($\boldsymbol{W}_1$) and ($\boldsymbol{W}_2$) hold, then the estimate*

$$\|u\|_{L^{p(x)}(\Omega)} \le C\|\nabla u\|_{L^{p(x)}(\Omega,\omega)}$$

holds for every $u \in C^\infty_0(\Omega)$ with a positive constant C independent of u.

Throughout this paper, let $p \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (3) and $X := W_0^{1,p(x)}(\Omega,\omega)$ be the weighted variable exponent Sobolev space that consists of all real valued functions u from $W^{1,p(x)}(\Omega,\omega)$ which vanish on the boundary $\partial\Omega$, endowed with the norm

$$\|u\|_X = \inf\Big\{\lambda > 0 : \int_\Omega \Big|\frac{\nabla u(x)}{\lambda}\Big|^{p(x)} \omega(x)dx \leq 1\Big\},$$

which is equivalent to the norm (2) due to Corollary 2.1. The following proposition gives the characterization of the dual space $(W_0^{k,p(x)}(\Omega,\omega))^*$, which is analogous to [[19], Theorem 3.16]. We recall that the dual space of weighted Sobolev spaces $W_0^{1,p(x)}(\Omega,\omega)$ is equivalent to $W^{-1,p'(x)}(\Omega,\omega)$, where $\omega^* = \omega^{1-p'(x)}$.

We will also use the standard notation for Bochner spaces, i.e., if $q \geq 1$ and X is a Banach space, then $L^q(0,T;X)$ denotes the space of strongly measurable function $u : (0,T) \to X$ for which $t \to \|u(t)\|_X \in L^q(0,T)$ Moreover, $C([0;T];X)$ denotes the space of continuous function $u : [0;T] \to X$ endowed with the norm $\|u\|_{C([0;T];X)} = max_{t\in[0;T]}\|u\|_X$,

$$L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega)) = \Big\{u : (0,T) \to W_0^{1,p(x)}(\Omega,\omega) \text{ measurable}; \\ (\int_0^T \|u(t)\|^{p^-}_{W_0^{1,p(x)}(\Omega,\omega)}\Big)^{1/p^-} < \infty\Big\}$$

and we define the space

$$L^\infty(0,T;X) = \Big\{u : (0,T) \to X \text{ measurable}; \exists C > 0/\|u(t)\|_X \leq C \text{ a.e.}\Big\},$$

where the norm is defined by:

$$\|u\|_{L^\infty(0,T;X)} = \inf\Big\{C > 0; \|u(t)\|_X \leq C \text{ a.e.}\Big\}.$$

We introduce the functional space (see [6])

$$V = \Big\{f \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega)); |\nabla f| \in L^{p(x)}(Q,\omega)\Big\}, \tag{7}$$

endowed with the norm

$$\|f\|_V = \|\nabla f\|_{L^{p(x)}(Q,\omega)}$$

or the equivalent norm

$$|||f|||_V = \|f\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))} + \|\nabla f\|_{L^{p(x)}(Q,\omega)},$$

which is a separable and reflexive Banach space. The equivalence of the two norms is an easy consequence of the continuous embedding $L^{p(x)}(Q) \hookrightarrow L^{p^-}(0,T;L^{p(x)}(\Omega))$ and the Poincaré inequality. We state some further properties of V in the following lemma.

Lemma 2.4 *Let V be defined as in* (7) *and its dual space be denoted by V^*. Then i) We have the following continuous dense embeddings:*

$$L^{p^+}(0,T;W_0^{1,p(x)}(\Omega,\omega)) \hookrightarrow V \hookrightarrow L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega)).$$

In particular, since $D(Q)$ is dense in $L^{p^+}(0,T;W_0^{1,p(x)}(\Omega,\omega)$, it is dense in V and for the corresponding dual spaces, we have

$$L^{(p^-)'}(0,T;(W_0^{1,p(x)}(\Omega,\omega))^*) \hookrightarrow V^* \hookrightarrow L^{(p^+)'}(0,T;W_0^{1,p(x)}(\Omega,\omega))^*).$$

Note that we have the following continuous dense embeddings

$$L^{p^+}(0,T;L^{p(x)}(\Omega,\omega)) \hookrightarrow L^{p(x)}(Q,\omega) \hookrightarrow L^{p^-}(0,T;L^{p(x)}(\Omega,\omega)).$$

ii) One can represent the elements of V^* as follows: if $T \in V^*$, then there exists $F = (f_1, ..., f_N) \in (L^{p'(x)}(Q))^N$ such that $T = div_X F$ and

$$\langle T, \xi\rangle_{V^*,V} = \int_0^T \int_\Omega F \cdot \nabla\xi dxdt$$

for any $\xi \in V$. Moreover, we have

$$\|T\|_{V^*} = \max\Big\{\|f_i\|_{L^{p(\cdot)}(Q,\omega)},\ i = 1, ..., n\Big\}.$$

Remark 2.3 The space $V \cap L^\infty(Q)$, endowed with the norm

$$\|v\|_{V\cap L^\infty(Q)} = \max\Big\{\|v\|_V, \|v\|_{L^\infty(Q)}\Big\},\ v \in V \cap L^\infty(Q),$$

is a Banach space. In fact, it is the dual space of the Banach space $V + L^1(Q)$ endowed with the norm

$$\|v\|_{V^*+L^1(Q)} := \inf\Big\{\|v_1\|_{V^*} + \|v_2\|_{L^1(Q)}\Big\};\ v = v_1 + v_2,\ v_1 \in V^*, v_2 \in L^1(Q).$$

2.1 Some technical results

This subsection introduces some basic technical lemmas and results that will be needed throughout this paper. For some details concerning the related issues, the reader can consult papers [7, 9].

Lemma 2.5 *(see [3]) Assume (9) and let $(u_n)_n$ be a sequence in $L^{p^-}(0,T,W_0^{1,p(x)}(\Omega,\omega))$ such that $u_n \rightharpoonup u$ weakly in $L^{p^-}(0,T,W_0^{1,p(x)}(\Omega,\omega))$ and*

$$\int_Q \Big(a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)\Big) \cdot \nabla(u_n - u)dxdt \to 0. \tag{8}$$

Then $u_n \to u$ strongly in $L^{p^-}(0,T,W_0^{1,p(x)}(\Omega,\omega))$.

Besides, $a(x,t,u,\nabla u) = \Big(\Big|\nabla u\Big|^{p(x)-2}\nabla u\Big)$ in our case.

Lemma 2.6 *([6]) Let $g \in L^{p(x)}(Q,\omega)$ and let $g_n \in L^{p(x)}(Q,\omega)$, with $\|g_n\|_{L^{p(x)}(Q,\omega)} \le c$, $1 < r(x) < \infty$. If $g_n(x) \to g(x)$ a.e. in Q, then $g_n \rightharpoonup g$ in $L^{p(\cdot)}(Q,\omega)$, where $\rightharpoonup$ denotes weak convergence and ω is a weight function on Q.*

Lemma 2.7 *(See [23]) $W := \Big\{u \in V; u_t \in V^* + L^1(Q)\Big\} \hookrightarrow C([0,T];L^1(\Omega))$ and $W \cap L^\infty(Q) \hookrightarrow C([0,T];L^2(\Omega))$.*

3 Assumptions and Definition

Throughout this paper, we assume that the following assumptions hold true.

3.1 Basic assumptions

Let Ω be an open bounded subset of $\mathbb{R}^N, N \geq 2$, $T > 0$ be a positive real number and let us set $Q = \Omega \times (0,T)$ and let $p \in C_+(\bar{\Omega})$ and assume that $p(x)$ satisfies the log-Hölder condition (3) with $1 < p^- \leq p(x) \leq p^+ < \infty$. The differential operator $A : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$Au = -\mathrm{div}\Big(\omega(x)\Big|\nabla u\Big|^{p(x)-2}\nabla u\Big), \tag{9}$$

is a Leray-Lions operator which is coercive and

$$g : \quad \mathbb{R} \to \mathbb{R}^+ \tag{10}$$

is a bounded and continuous positive function that belongs to $L^\infty(\mathbb{R})$,

$$f \text{ is an element of } L^1(Q), u_0 \in L^1(\Omega), \quad u_0 \geq 0 \; and \; p \in C_+(\bar{\Omega}). \tag{11}$$

Let ψ be a measurable function with values in $\overline{\mathbb{R}}$ such that $\psi \in W_0^{1,p(x)}(\Omega,\omega) \cap L^\infty(\Omega)$, (see [17]), K is defined by: $K = \Big\{u \in W_0^{1,p(x)}(\Omega,\omega); \quad u(x) \geq \psi(x) \quad a.e. \quad in \quad \Omega\Big\}$ and consider the convex set

$$K_\psi = \Big\{u \in V, \; u(t) \in K\Big\}.$$

We recall that, for $k > 0$ and $s \in \mathbb{R}$, the truncation function $T_k(.)$ is defined by
$$T_k(s) = \begin{cases} s, & if \quad |s| \leq k, \\ k\frac{s}{|s|}, & if \quad |s| \geq k. \end{cases}$$

3.2 Definition of entropy solution

Definition 3.1 Let $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$. A measurable function u defined on Q is a unilateral entropy solution of problem $(\mathcal{P})$ if

$$u \geq \psi \text{ a.e. in } Q, \tag{12}$$

$$T_k(u) \in L^{p^-}(0,T; W_0^{1,p(x)}(\Omega,\omega)), \; for \; all \; k \geq 0 \; and \quad u \in C(0,T; L^1(\Omega)), \tag{13}$$

$$\begin{aligned} &\int_\Omega \Big[S_k(u-v)\Big]_0^T dx + \int_Q \frac{\partial v}{\partial t} T_k(u-v)\,dxdt \\ &+ \int_Q \omega(x)\Big|\nabla u\Big|^{p(x)-2}\nabla u \nabla T_k(u-v)\,dxdt \\ &\leq \int_Q \omega(x) g(u)\Big|\nabla u\Big|^{p(x)} T_k(u-v)\,dxdt \\ &+ \int_Q f T_k(u-v)\,dxdt, \end{aligned} \tag{14}$$

for all $v \in K_\psi \cap \; L^\infty(Q), \frac{\partial v}{\partial t} \in L^{(p^-)'}(0,T; (W_0^{1,p(x)}(\Omega,\omega))^*)$, where $S_k(s) = \displaystyle\int_0^s T_k(r)\,dr \quad \forall k > 0.$

4 The Principal Result

The aim of the present work is to prove the following result.

Theorem 4.1 *Under assumptions* (9)-(11)*, there exists at least one unilateral entropy solution of problem* (1)*.*

Proof of Theoreme 4.1. Existence of entropy solutions.

We first introduce the approximate problems. Find two sequences of functions $\{f_n\} \subset L^{p'(x)}(Q)$ and $\{u_{0n}\} \subset D(\Omega)$ strongly converging with respct to f in $L^1(Q)$ and to u_0 in $L^1(\Omega)$ such that

$$\|f_n\|_{L^1(Q)} \leq \|f\|_{L^1(Q)} \quad \textit{and} \quad \|u_{0n}\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)}. \tag{15}$$

Then, we consider the approximate problem of

$$\begin{cases} \frac{\partial u_n}{\partial t} - \operatorname{div}\Big(\omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n\Big) - nT_n\Big((u_n-\psi)^-\Big) \\ \qquad\qquad = \omega(x)g(u_n)\Big|\nabla u_n\Big|^{p(x)} + f_n, & \text{in } D'(Q), \\ u_n = 0, & \text{on } \partial\Omega \times (0,T), \\ u_n(t=0) = u_{0n}, & \text{in } \Omega. \end{cases} \tag{16}$$

Moreover, since $f_n \in L^{(p^-)'}(0,T;(W_0^{1,p(x)}(\Omega,\omega))^*)$, proving the existence of weak solution $u_n \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))$ of (16) is an easy task (see [4]).

Our aim is to prove that a subsequence of these approximate solution u_n converges to a measurable function u, which is an entropy solution of the problem.

Step 1: A priori estimates. The estimate derived in this step relies on standard techniques for problems of the type (16).

Proposition 4.1 *Assume that* (9)-(11) *hold true and let* u_n *be a solution of the approximate problem* (16)*. Then for all* $k > 0$*, we have*

$$\|T_k(u_n)\|_{L^{p^-}(0,T,W_0^{1,p(x)}(\Omega,\omega))} \leq C\,k \quad \textit{for all}\, n \in \mathbb{N},$$

where C *is a constant independent of* n*.*

Proof. Let $h > k > 0$ and consider the test function $\varphi = T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n)) \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega)) \cup L^\infty(Q)$ in the approximate problem (16), where $G(s) = \displaystyle\int_0^s g(r)dr$, we have

$$\begin{aligned} &\Big\langle\Big\langle \frac{\partial u_n}{\partial t}, T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))\Big\rangle\Big\rangle \\ &+ \int_{\{k\leq|u_n|\leq k+h\}} \Big(\omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n\Big)\nabla u_n \exp(G(u_n))dxdt \\ &+ \int_Q \Big(\omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n\Big)\nabla u_n T_h\Big(u_n - T_k(u_n)\Big)g(u_n)\exp(G(u_n))dxdt \\ &- \int_Q nT_n\Big((u_n-\psi)^-\Big)T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))dxdt \end{aligned}$$

$$= \int_Q \omega(x)g(u_n)\Big|\nabla u_n\Big|^{p(x)} T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))dxdt$$
$$+ \int_Q f_n T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))dxdt,$$

then

$$\Big\langle\Big\langle \frac{\partial u_n}{\partial t}, T_h(u_n - T_k(u_n))\exp(G(u_n))\Big\rangle\Big\rangle + \int_{\{k\le|u_n|\le k+h\}} \omega(x)\Big|\nabla u_n\Big|^{p(x)}\exp(G(u_n))dxdt$$
$$- \int_Q nT_n\Big((u_n-\psi)^-\Big)T_h(u_n - T_k(u_n))\exp(G(u_n))dxdt$$
$$= \int_Q f_n T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))dxdt.$$

On the one hand, we have

$$\Big\langle\Big\langle \frac{\partial u_n}{\partial t}, T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))\Big\rangle\Big\rangle = \int_\Omega S_h^k(u_n(T))dx - \int_\Omega S_h^k(u_{0n})dx,$$

where $S_h^k(s) = \int_0^s T_h\Big(q - T_k(q)\Big)\exp(G(q))dq$, and by using the fact that $\int_\Omega S_h^k(u_n(T))dx \ge 0$ and $\int_\Omega S_h^k(u_{0n})dx \le h\exp(\|g\|_{L^1(\mathbb{R})})\|u_{0n}\|_{L^1(\Omega)}$, we get

$$\int_{\{k\le|u_n|\le k+h\}} \omega(x)\Big|\nabla u_n\Big|^{p(x)}\exp(G(u_n))dxdt$$
$$- \int_Q nT_n\Big((u_n-\psi)^-\Big)T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))dxdt$$
$$\le h\exp(\|g\|_{L^1(\mathbb{R})})\Big[\|f_n\|_{L^1(Q)} + \|u_{0n}\|_{L^1(\Omega)}\Big].$$

We have

$$\int_{\{k\le|u_n|\le k+h\}} \omega(x)\Big|\nabla u_n\Big|^{p(x)}\exp(G(u_n))dxdt$$
$$- \int_Q nT_n\Big((u_n-\psi)^-\Big)T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))dxdt$$
$$\le C_1 h. \tag{17}$$

We obtain

$$\int_{\{k\le|u_n|\le k+h\}} \omega(x)\Big|\nabla u_n\Big|^{p(x)}\exp(G(u_n))dxdt$$
$$\le C_1 h + (h+k)\|g\|_\infty \exp(\|g\|_{L^1(\mathbb{R})})\int_Q nT_n\Big((u_n-\psi)^-\Big)dxdt,$$

then

$$\int_{\{k\le|u_n|\le k+h\}} \omega(x)\Big|\nabla u_n\Big|^{p(x)}\exp(G(u_n))dxdt \le C_2 h\int_Q nT_n\Big((u_n-\psi)^-\Big)dxdt.$$

Let us take $\rho_1(u_n) = \displaystyle\int_0^{u_n} g(s)\chi_{\{|s|\leq k\}} ds \exp(G(u_n))$ as a test function of (16) , we obtain

$$\Big[\int_\Omega \varphi_2(u_n)dx\Big]_0^T + \int_Q \omega(x)\Big|\nabla u_n\Big|^{p(x)} g(u_n)\exp(G(u_n))\chi_{\{|u_n|\leq k\}}dxdt$$
$$- \int_Q nT_n\Big((u_n-\psi)^-\Big)\rho_1(u_n)dxdt$$
$$\leq \Big(\int_0^\infty g(s)ds\Big)\exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\|f_n\|_{L^1(Q)},$$

where $\varphi_2(r) = \displaystyle\int_0^r \rho_1(s)ds$, which implies, in view of $\varphi_2(r) \geq 0$, that

$$\int_{\{|u_n|\leq k\}} \omega(x)\Big|\nabla u_n\Big|^{p(x)} g(u_n)\exp(G(u_n))dxdt$$
$$\leq \|g\|_\infty \exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\Big[\|f_n\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\Big]$$
$$+(h+k)\|g\|_\infty \exp(\|g\|_{L^1(\mathbb{R})})\int_Q nT_n\Big((u_n-\psi)^-\Big)dxdt.$$

Then

$$\int_{\{|u_n|\leq k\}} \omega(x)\Big|\nabla u_n\Big|^{p(x)} g(u_n)\exp(G(u_n))dxdt$$
$$\leq hC_3\int_Q nT_n\Big((u_n-\psi)^-\Big)dxdt.$$

Similarly, taking $\rho_2 = \displaystyle\int_0^{u_n} g(s)\chi_{\{|s|\geq k+h\}} ds \exp(G(u_n))$ as a test function of (16) , we conclude that

$$\int_{\{|u_n|\geq k+h\}} \Big|\nabla u_n\Big|^{p(x)} g(u_n)\exp(G(u_n))dxdt \leq hC_4\int_Q nT_n((u_n-\psi)^-)dxdt.$$

Consequently, we have :

$$\begin{cases} \displaystyle\int_Q \omega(x)\Big|\nabla u_n\Big|^{p(x)} g(u_n)\exp(G(u_n))dxdt \\ \leq \displaystyle\int_{\{|u_n|\geq k+h\}} \omega(x)\Big|\nabla u_n\Big|^{p(x)} g(u_n)\exp(G(u_n))dxdt \\ + \displaystyle\int_{\{|u_n|\leq k\}} \omega(x)\Big|\nabla u_n\Big|^{p(x)} g(u_n)\exp(G(u_n))dxdt \\ + \displaystyle\int_{\{k\leq|u_n|\leq k+h\}} \omega(x)\Big|\nabla u_n\Big|^{p(x)} g(u_n)\exp(G(u_n))dxdt \\ \leq hC_5\displaystyle\int_Q nT_n((u_n-\psi)^-)dxdt, \quad where\ \ C_5 = Max(C_2,C_3,C_4). \end{cases}$$

Using (17), we have

$$\int_Q \omega(x)\Big|\nabla u_n\Big|^{p(x)} g(u_n)\exp(G(u_n))dxdt - \int_Q nT_n\Big((u_n-\psi)^-\Big)T_h\Big(u_n-T_k(u_n)\Big)\exp(G(u_n))dxdt$$
$$\leq hC_5\int_Q nT_n\Big((u_n-\psi)^-\Big)dxdt,$$

we obtain

$$-\int_Q nT_n\Big((u_n-\psi)^-\Big)T_h\Big(u_n-T_k(u_n)\Big)\exp(G(u_n))dxdt \le hC_5\int_Q nT_n\Big((u_n-\psi)^-\Big)dxdt$$

so that

$$-\int_Q nT_n\Big((u_n-\psi)^-\Big)\frac{T_h\Big(u_n-T_k(u_n)\Big)}{h}\exp(G(u_n))dxdt \le C_5\int_Q nT_n\Big((u_n-\psi)^-\Big)dxdt.$$

Let us now fix $k>\|\psi\|_\infty$, by the fact that $nT_n((u_n-\psi)(u_n-k)\chi_{\{u_n\le\psi;k\le u_n\le k+h\}}\ge 0$ and letting $h\to 0$, one has

$$\int_Q nT_n\Big((u_n-\psi)^-\Big)dxdt \le C_6. \tag{18}$$

Let use $v=T_k(u_n)\exp(G(u_n))\chi(0,\tau)$ as a test function of (16)

$$\begin{aligned}\Big[\int_\Omega \varphi_3(u_n)dx\Big]_0^T &+ \int_{Q^\tau}\omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n\nabla T_k(u_n)\exp(G(u_n))dxdt\\ &+\int_{Q^\tau}\omega(x)\Big|\nabla u_n\Big|^{p(x)}T_k(u_n)g(u_n)\exp(G(u_n))dxdt\\ &-\int_{Q^\tau} nT_n\Big((u_n-\psi)^-\Big)T_k(u_n)\exp(G(u_n))dxdt\\ &=\int_{Q^\tau}\omega(x)\Big|\nabla u_n\Big|^{p(x)}g(u_n)T_k(u_n)\exp(G(u_n))dxdt\\ &+\int_{Q^\tau} f_nT_k(u_n)\exp(G(u_n))dxdt,\end{aligned}$$

where $\varphi_3(r)=\displaystyle\int_0^r T_k(s)\exp(G(s))ds$. Due to the definition of φ_3 and the fact that $|G(u_n)|\le\exp(\|g\|_{L^1(\mathbb{R})})\|u_{0n}\|_{L^1(\Omega)}$, we have $0\le\displaystyle\int_\Omega\varphi_3(u_{0n})dx\le k\exp(\|g\|_{L^1(\mathbb{R})})\|u_{0n}\|_{L^1(\Omega)}$, and by using (18) we arrive at

$$\begin{aligned}&\int_{Q^\tau}\omega(x)\Big|\nabla T_k(u_n)\Big|^{p(x)}\exp(G(u_n))dxdt\\ &\qquad\le k\exp(\|g\|_{L^1(\mathbb{R})})\Big[\|f_n\|_{L^1(Q)}+\|u_0\|_{L^1(\Omega)+C_7}\Big].\end{aligned}$$

Let us take $\rho_4(u_n)=\displaystyle\int_0^{u_n} g(s)\chi_{\{s\ge0\}}ds\exp(G(u_n))$ as a test function of 16, we obtain

$$\begin{aligned}\Big[\int_\Omega\varphi_4(u_n)dx\Big]_0^T &+\int_Q\omega(x)\Big|\nabla u_n\Big|^{p(x)}g(u_n)\exp(G(u_n))\chi_{\{u_n\ge0\}}dxdt\\ &-\int_Q nT_n\Big((u_n-\psi)^-\Big)\rho_4(u_n)dxdt\\ &\le\Big(\int_0^\infty g(s)ds\Big)\exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\|f_n\|_{L^1(Q)},\end{aligned}$$

where $\varphi_4(r) = \int_0^r \rho_4(s)ds$, which implies, in view of $\varphi_4(r) \geq 0$, that

$$\int_{\{u_n \geq 0\}} \omega(x)\Big|\nabla u_n\Big|^{p(x)} g(u_n)\exp(G(u_n))dxdt$$
$$\leq \|g\|_\infty \exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\Big[\|f_n\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\Big] + C_8,$$

then

$$\int_{\{u_n \geq 0\}} \omega(x)\Big|\nabla u_n\Big|^{p(x)} g(u_n)\exp(G(u_n))dxdt \leq C_9.$$

Similarly, taking $\rho_5 = \int_{u_n}^0 g(s)\chi_{\{s\leq 0\}}ds\exp(G(u_n))$ as a test function of (16), we conclude that

$$\int_{\{u_n \leq 0\}} \omega(x)\Big|\nabla u_n\Big|^{p(x)} g(u_n)\exp(G(u_n))dxdt \leq C_{10}.$$

Consequently,

$$\int_Q \omega(x)\Big|\nabla u_n\Big|^{p(x)} g(u_n)\exp(G(u_n))dxdt \leq C_{11}.$$

As $C_1, ..., C_{11}$ are constants independent of n, we deduce that

$$\int_Q \omega(x)\Big|\nabla T_k(u_n)\Big|^{p(x)} dxdt \leq k\ C_{12}$$

$$\Rightarrow \|T_k(u_n)\|_{L^{p^-}(0,T,W_0^{1,p(x)}(\Omega,\omega))} \leq C_{13}\ k. \tag{19}$$

Then, we conclude that $T_k(u_n)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))$, independently of n for any $k > 0$.

Now we turn to proving that $(u_n)_n$ is a Cauchy sequence in measures. Let $k > 0$ be large enough and B_R be a ball of Ω. Using (19) and applying Hölder's inequality and Poincarè's inequality, we obtain that

$$\begin{aligned}
k\ \ meas\Big(\{\Big|u_n\Big| > k\} \cap B_R \times [0,T]\Big) &= \int_0^T \int_{\{|u_n|>k\}\cap B_R} \Big|T_k(u_n)\Big|dxdt \\
&\leq \int_0^T \int_{B_R} \Big|T_k(u_n)\Big|dxdt \\
&\leq C\|\nabla T_k(u_n)\|_{L^{p(x)}(\Omega,\omega)} \\
&\leq C\left(\int_Q \Big|\nabla T_k(u_n)\Big|^{p(x)} wdxdt\right)^{\frac{1}{\theta}} \\
&\leq Ck^{\frac{1}{\theta}},
\end{aligned}$$

where

$$\theta = \begin{cases} p^-, & if \quad \|T_k(u_n)\|_{L^{p(x)}(\Omega,\omega)} \leq 1 \\ p^+, & if \quad \|T_k(u_n)\|_{L^{p(x)}(\Omega,\omega)} > 1. \end{cases}$$

This implies that

$$meas\Big(\{\Big|u_n\Big| > k\} \cap (B_R \times [0,T]\Big) \leq \frac{c_1}{k^{1-\frac{1}{\theta}}}, \quad \forall k \geq 1. \tag{20}$$

So, we have

$$\lim_{k\to+\infty}\Big(meas\Big(\{\Big|u_n\Big| > k\} \cap (B_R \times [0,T]\Big)\Big) = 0.$$

Then, we obtain for all $\delta > 0$

$$\begin{aligned} meas\Big(\{\Big|u_n - u_m\Big| > \delta\} \cap (B_R \times [0,T]\Big) \leq\ & meas\Big(\{\Big|u_n\Big| > k\} \cap (B_R \times [0,T]\Big) \\ & + meas\Big(\{\Big|u_m\Big| > k\} \cap (B_R \times [0,T]\Big) \\ & + meas\Big(\{\Big|T_k(u_n) - T_k(u_m)\Big| > \delta\}\Big). \end{aligned}$$

Since $T_k(u_n)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))$, it is clear that $T_k(u_n) \to v_k$ strongly in $L^{p(x)}(Q,\omega)$ and almost everywhere in Q. Hence $(T_k(u_n))_n$ is a Cauchy sequence in measure in Q.

Let $\epsilon > 0$, then by (20)), there exists a $k(\epsilon) > 0$ such that

$$meas\Big(\{\Big|u_n - u_m\Big| > \delta\} \cap (B_R \times [0,T]\Big) < \epsilon \quad \forall n, m \geq n_0(k(\epsilon), \delta, R).$$

This proves that $(u_n))_n$ is a Cauchy sequence in measures in B_R.

Consider a non-decreasing function $g_k \in C^2(\mathbb{R})$ such that

$$g_k(s) = \begin{cases} s, & if\ |s| \leq \frac{k}{2}, \\ k, & if\ |s| \geq k. \end{cases}$$

Multiplying the approximate equation by $g_k'(u_n)$, we get

$$\begin{aligned} \frac{\partial g_k(u_n)}{\partial t} & - \operatorname{div}\Big(\omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n)g_k'(u_n)\Big) + \omega(x)\Big|\nabla u_n\Big|^{p(x)} g_k''(u_n) \\ & -nT_n\Big((u_n - \psi)^-\Big)g_k'(u_n) = \omega(x)g(u_n)\Big|\nabla u_n\Big|^{p(x)} g_k'(u_n) + f_n g_k'(u_n) \end{aligned} \tag{21}$$

in the sense of distributions. This implies, thanks to the fact that g_k' has compact support, that $g_k(u_n)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))$, while its time derivative $\frac{\partial g_k(u_n)}{\partial t}$ is bounded in $L^1(Q) + V^*$ Due to the choice of g_k, we conclude that for each k, the sequence $T_k(u_n)$ converges almost everywhere in Q, which implies that the sequence u_n converges almost everywhere to some measurable function v in Q. Thus, by using the same argument as in [9], [10], [11], we can show the following lemma.

Lemma 4.1 *Let* u_n *be a solution of* (16). *Then*

$$u_n \quad \to \quad u \qquad a.e. \quad in \quad Q. \tag{22}$$

We can deduce from (19) *that*

$$T_k(u_n) \rightharpoonup T_k(u) \qquad in \quad L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega)). \tag{23}$$

Lemma 4.2 *[2] Let* u_n *be a solution of* (16). *Then*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,\nabla(u_n))\nabla u_n dxdt = 0. \tag{24}$$

Step 3: **Almost everywhere convergence of the gradients.** This step is devoted to introducing, for a fixed $k \geq 0$, a time regularization of the function $T_k(u)$ in order to apply the monotonicity method. This specific time regularization of $T_k(u)$ (for fixed $k \geq 0$) is defined as follows. Let $(\upsilon_0^\mu)_\mu$ be a sequence of functions defined on Ω such that

$$\upsilon_0^\mu \in L^\infty(\Omega)\cap W_0^{1,p(x)}(\Omega,\omega) \quad for\ all\ \mu > 0, \tag{25}$$

$$\|\upsilon_0^\mu\|_{L^\infty(\Omega)} \leq k \quad for\ all\ \mu > 0, \tag{26}$$

$$\upsilon_0^\mu \to T_k(u_0)\ a.e.\ in\ \Omega\ and\ \frac{1}{\mu}\|\upsilon_0^\mu\|_{L^{p(x)}(\Omega,\omega)} \to 0,\ as\ \mu\to\infty. \tag{27}$$

For fixed k, $\mu > 0$, let us consider the unique solution $(T_k(u))_\mu \in L^\infty(Q)\cap L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))$ of the monotone problem:

$$\frac{\partial(T_k(u))_\mu}{\partial t} + \mu\Big((T_k(u))_\mu - T_k(u)\Big) = 0 \quad in\ D'(Q), \tag{28}$$

$$(T_k(u))_\mu(t=0) = \upsilon_0^\mu \quad in \quad \Omega. \tag{29}$$

Note that due to (28), we have for $\mu > 0$ and $k \geq 0$

$$\frac{\partial(T_k(u))_\mu}{\partial t} \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega)). \tag{30}$$

We just recall here that (28)–(29) imply that

$$(T_k(u))_\mu \to T_k(u)\ a.e.\ in\ Q, \tag{31}$$

as well as weakly in $L^\infty(Q)$ and strongly in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))$ as $\mu\to\infty$. Note that for any μ and any $k \geq 0$, we have

$$\|(T_k(u))_\mu\|_{L^\infty(Q,\omega)} \leq \max\Big(\|T_k(u)\|_{L^\infty(Q,\omega)};\ \|\upsilon_0^\mu\|_{L^\infty(\Omega,\omega)}\Big) \leq k. \tag{32}$$

We introduce a sequence of increasing $C^\infty(\mathbb{R})$–functions S_m such that

$$S_m(r) = r\ for\ |r|\leq m,\ supp(S_m') \subset [-(m+1), m+1], \|S_m''\|_{L^\infty(\mathbb{R})} \leq 1,$$

for any $m \geq 1$, and we denote by $\omega(n,\mu,\eta,m)$ the quantities such that

$$\lim_{m\to\infty}\lim_{\eta\to 0}\lim_{\mu\to\infty}\lim_{n\to\infty}\omega(n,\mu,\eta,m) = 0.$$

Lemma 4.3 *([2, 11]). We have*

$$\int_0^T \left\langle \frac{\partial u_n}{\partial t}, T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+ \exp(G(u_n))S_m'(u_n)\right\rangle dt \geq \omega(n,\mu,\eta)\ \forall m \geq 1. \tag{33}$$

Taking now $v = T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+ S'_m(u_n)\exp(G(u_n))$ of (16), we get

$$\int_0^T \Big\langle \frac{\partial u_n}{\partial t}, T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+ \exp(G(u_n))S'_m(u_n)\Big\rangle dt$$
$$+\int_Q \Big(\omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n\Big)\nabla\Big(T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+\Big)\exp(G(u_n))S'_m(u_n)dxdt$$
$$+\int_{\{m\le|u_n|\le m+1\}} \Big(\omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n\Big)T_\eta(u_n - (T_k(u))_\mu)^+ \exp(G(u_n))S''_m(u_n)\nabla u_n dxdt$$
$$+\int_Q \Big(\omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n\Big)g(u_n)\nabla u_n \exp(G(u_n))T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+ S'_m(u_n)dxdt$$
$$-n\int_Q T_n\Big((u_n-\psi)^-\Big)T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+ \exp(G(u_n))S'_m(u_n)dxdt \tag{34}$$
$$=\int_Q \omega(x)g(u_n)\Big|\nabla u_n\Big|^{p(x)}T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+ S'_m(u_n)\exp(G(u_n))dxdt$$
$$+\int_Q f_n T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+ S'_m(u_n)\exp(G(u_n))dxdt.$$

From (19),(24),(33),(34) it follows that

$$\int_Q \Big(\omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n\Big)\nabla\Big(T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+\Big)\exp(G(u_n))S'_m(u_n)dxdt$$
$$\le C\eta + \omega(n,\mu,\eta,m), \tag{35}$$

where C is a constant independent of n and m. On the other hand, let $A = \{0 \le T_k(u_n) - (T_k(u))_\mu < \eta\}$ and $B = \{0 \le u_n - (T_k(u))_\mu < \eta\}$. Then, we have

$$\int_Q \Big(\omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n\Big)\nabla\Big(T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+\Big)\exp(G(u_n))S'_m(u_n)dxdt$$
$$=\int_B \Big(\omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n\Big)\Big(\nabla u_n - \nabla(T_k(u))_\mu\Big)\exp(G(u_n))S'_m(u_n)dxdt \tag{36}$$
$$=\int_A \Big(\omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n)\Big)\Big(\nabla T_k(u_n) - \nabla(T_k(u))_\mu\Big)\exp(G(u_n))S'_m(u_n)dxdt$$
$$+\int_{\{|u_n|>k\}\cap B} \Big(\omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n)\Big(\nabla u_n - \nabla(T_k(u))_\mu\Big)\exp(G(u_n))S'_m(u_n)dxdt.$$

Given the definition of $S'_m[S'_m(u_n) = 1$ a.e. in $\{|u_n| \le k\}$ if $k \le m]$, it is possible to obtain from (35) and (36), that

$$\int_A \Big(\omega(x)\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u_n))\Big(\nabla T_k(u_n) - \nabla(T_k(u))_\mu\Big)\exp(G(u_n)S'_m(u_n)dxdt$$
$$\le \int_{\{|u_n|>k\}\cap B} \Big(\omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n)\nabla(T_k(u))_\mu \exp(G(u_n))S'_m(u_n)dxdt$$
$$+C\eta + \omega(n,\mu,\eta,m). \tag{37}$$

Since $\nabla T_{k+\eta}(u_n)$ is bounded in $(L^{p'(x)}(Q,\omega))^N$ and $u_n \to u$ a.e. in Q,one has $\nabla T_{k+\eta}(u_n) \rightharpoonup \nabla T_{k+\eta}(u)$ weakly in $(L^{p'(x)}(Q,\omega))^N$. Consequently,

$$\int_{\{|u_n|>k\}\cap B} \omega(x)\Big|\nabla T_{k+\eta}(u_n)\Big|^{p(x)-2}\nabla T_{k+\eta}(u_n)|\nabla(T_k(u))_\mu|\exp(G(u_n))S'_m(u_n)dxdt$$
$$=\int_{\{|u|>k\}\cap\{0\le u-(T_k(u))_\mu<\eta\}} \omega(x)\Big|\nabla T_{k+\eta}(u_n)\Big|^{p(x)-2}\nabla(T_k(u))_\mu \exp(G(u))S'_m(u)dxdt + \omega(n).$$

Thanks to (31) one easily has

$$\int_{\{|u|>k\}\cap\{0\leq u-(T_k(u))_\mu<\eta\}} \omega(x)\Big|\nabla T_{k+\eta}(u)\Big|^{p(x)-2}\nabla(T_k(u))_\mu \exp(G(u))S'_m(u)dxdt=\omega(\mu).$$

Hence,

$$\int_A \Big(\omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n\Big)\nabla\Big(T_\eta\Big(u_n-(T_k(u))_\mu\Big)^+\Big)\exp(G(u_n))S'_m(u_n)dxdt \\ \leq C\eta+\omega(n,\mu,\eta,m). \tag{38}$$

On the other hand, note that

$$\int_A \Big(\omega(x)\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u_n)\Big)\Big(\nabla T_k(u_n)-\nabla(T_k(u))_\mu\Big)\exp(G(u_n))dxdt \\ = \int_A \Big(\omega(x)\Big|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)\Big)\Big(\nabla T_k(u_n)-\nabla T_k(u)\Big)\exp(G(u_n))dxdt \\ +\int_A \Big(\omega(x)\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u_n)\Big)\Big(\nabla T_k(u)-\nabla(T_k(u))_\mu\Big)\exp(G(u_n))dxdt, \tag{39}$$

and the last integral tends to 0 as $n\to\infty$ and $\mu\to\infty$. Indeed, we have that

$$\int_A \Big(\omega(x)\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u_n))\Big(\nabla T_k(u)-\nabla(T_k(u))_\mu\Big)\exp(G(u_n))dxdt \\ \to \int_{\{0\leq T_k(u)-(T_k(u))_\mu<\eta\}} \Big(\omega(x)\Big|\nabla T_k(u)\Big|^{p(x)-2}\nabla T_k(u))\Big(\nabla T_k(u)-\nabla(T_k(u))_\mu\Big)\exp(G(u))dxdt$$

as $n\to\infty$.
Using (31) and Lebesgue's theorem, we have

$$\int_{\{0\leq T_k(u)-(T_k(u))_\mu<\eta\}} \Big(\omega(x)\Big|\nabla T_k(u)\Big|^{p(x)-2}\nabla T_k(u))\Big(\nabla T_k(u)-\nabla(T_k(u))_\mu\Big)\exp(G(u))dxdt\to 0$$

as $\mu\to\infty$. We deduce then that

$$\int_A \Big(\omega(x)\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u_n))\Big(\nabla T_k(u_n)-\nabla T_k(u)\Big)\exp(G(u_n)dxdt \\ \leq C\eta+\omega(n,\mu,\eta,m). \tag{40}$$

Let $M_n=\Big(\Big[\Big(\omega(x)\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u_n)\Big]\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]\Big)\times\Big(\exp(G(u_n))\Big)$.
Then, for any $0<\theta<1$, we write

$$I_n = \int_{\{|u_n-(T_k(u))_\mu|\geq 0\}} M_n^\theta dxdt=\int_{\{|T_k(u_n)-(T_k(u))_\mu|\leq\eta,\ u_n-T_k(u)_\mu\geq 0\}} M_n^\theta dxdt \\ +\int_{\{|T_k(u_n)-(T_k(u))_\mu|>\eta,\ u_n-(T_k(u))_\mu\geq 0\}} M_n^\theta dxdt.$$

Since $\nabla T_k(u_n)$ is bounded in $(L^{p(x)}(Q,\omega))^N$, we obtain by applying Hölder's inequality that

$$I_n \leq C_1\Big(\int_{\{0\leq T_k(u_n)-(T_k(u))_\mu<\eta\}} M_n dxdt\Big)^\theta \tag{41}$$
$$+C_2 \text{ meas}\Big\{(x,t)\in Q:\Big|T_k(u_n)-(T_k(u))_\mu\Big|>\eta\ ,\ u_n-(T_k(u))_\mu\geq 0\Big\}^{1-\theta}.$$

On the other hand, we have

$$\begin{aligned}&\int_{\{0\le T_k(u_n)-(T_k(u))_\mu<\eta\}} M_n\,dx\,dt\\ &=\int_{\{0\le T_k(u_n)-(T_k(u))_\mu<\eta\}}\Big(\omega(x)\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u_n)\Big)\times\\ &\qquad\times\Big(\nabla T_k(u_n)-\nabla T_k(u)\Big)\exp(G(u_n))\,dx\,dt\\ &-\int_{\{0\le T_k(u_n)-(T_k(u))_\mu<\eta\}}\Big(\omega(x)\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u)\Big)\times\\ &\qquad\times\Big(\nabla T_k(u_n)-\nabla T_k(u)\Big)\exp(G(u_n))\,dx\,dt\\ &=I_n^1+I_n^2.\end{aligned}\tag{42}$$

Using (40), we have

$$I_n^1\le C\,\eta+w(n,\mu,\eta,m).\tag{43}$$

Concerning I_n^2, that is the second term of the right-hand side of the (42), it is easy to see that

$$I_n^2=w(n,\mu).\tag{44}$$

Therefore, for all $i=1,...,N$, we have $\frac{\partial T_k(u_n)}{\partial x_i}\rightharpoonup\frac{\partial T_k(u)}{\partial x_i}$ in $L^{p(x)}(Q,\omega)$. Combining (41), (42), (43) and (44), we get

$$I_n\le C_1\,\Big(C\,\eta+w(n,\mu,\eta,m)\Big)^\theta+C_2\,\Big(w(n,\mu)\Big)^{1-\theta}$$

and by passing to the limit sup over n,μ and η

$$\begin{aligned}\int_{\{u_n-(T_k(u))_\mu\ge0\}}&\Big(\omega(x)\Big[\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u_n))-\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u))\Big]\times\\ &\times\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]\Big)^\theta\,dx\,dt=w(n).\end{aligned}\tag{45}$$

On the other hand, we choose $v=T_\eta\Big(u_n-(T_k(u))_\mu\Big)^-\exp(-G(u_n))$ in (16) and obtain:

$$\begin{aligned}\int_{\{u_n-T_k(u)_\mu\le0\}}&\Big(\Big[\omega(x)\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u_n))-\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u))\Big]\times\\ &\times\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]\Big)^\theta\,dx\,dt=w(n).\end{aligned}\tag{46}$$

Moreover, (45) and (46) imply that

$$\begin{aligned}\int_Q&\Big(\omega(x)\Big[\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u_n))-\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u))\Big]\times\\ &\times\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]\Big)^\theta\,dx\,dt=w(n),\end{aligned}\tag{47}$$

which implies that

$$T_k(u_n)\to T_k(u)\quad in\quad L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))\ \forall\ k\ge0.\tag{48}$$

According to [9, 10], there exists a subsequence also denoted by u_n such that

$$\nabla u_n\to\nabla u\ \text{ a.e. in }\ Q.\tag{49}$$

Proposition 4.2 *Let u_n be a solution of* (16). *Then* $u \geq \psi$ *a.e. in* Q.

Proof. Thanks to (18), we can write $\int_Q T_n\Big((u_n-\psi)^-\Big)dxdt \leq \frac{C}{n}$. So, by using Fatou's lemma as $n \to \infty$,we infer that $\int_Q (u-\psi)^- dxdt = 0$,which implies that $(u-\psi)^- = 0$ a.e. in Q. Consequently, we conclude that $u \geq \psi$ a.e. in Q.

Step 4: **Passing to the limit**

a) we claim that $u \in C(0,T;L^1(\Omega))$ .We will show that

$$u_n \to u \quad in \quad C(0,T;L^1(\Omega)).$$

Since $T_k(u) \in K_\psi$, for every $k \geq \|\psi\|_{L^\infty}$ there exists a sequence $v_j \in K_\psi \cap D(\bar{Q})$ such that

$$v_j \to T_k(u) \quad in \quad L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))$$

for the modular convergence.

Let $\omega_{j,\mu}^{i,l} = (T_l(v_j))_\mu + e^{-\mu t}T_l(\eta_i)$ with $\eta_i \geq 0$ converge to u_0 in $L^1(\Omega)$, where $(T_l(v_j))_\mu$ is the mollification of $T_l(v_j)$ with respect to time. Note that $\omega_{j,\mu}^{i,l}$ is a smooth function having the following properties:

$$\frac{\partial \omega_{j,\mu}^{i,l}}{\partial t} = \mu(T_l(v_j) - \omega_{j,\mu}^{i,l}), \quad \omega_{j,\mu}^{i,l}(0) = T_l(\eta_i), \quad |\omega_{j,\mu}^{i,l}| \leq l, \tag{50}$$

$$\omega_{j,\mu}^{i,l} \to T_l(v_j) \quad \text{in } L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega)) \quad \text{as } \mu \to \infty. \tag{51}$$

Choosing now $v = T_k(u_n - \omega_{j,\mu}^{i,l})\chi_{(0,\tau)}$ as a test function of (16), we get

$$\begin{aligned} &\Big\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega_{j,\mu}^{i,l})\Big\rangle_{Q^\tau} + \int_{Q^\tau} \omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n \nabla T_k(u_n - \omega_{j,\mu}^{i,l})dxdt \\ &\quad - \int_{Q^\tau} nT_n\Big((u_n-\psi)^-\Big)T_k(u_n - \omega_{j,\mu}^{i,l})dxdt \\ &\quad = \int_{Q^\tau} \omega(x)g(u_n)\Big|\nabla u_n\Big|^{p(x)}T_k(u_n - \omega_{j,\mu}^{i,l})dxdt + \int_{Q^\tau} f_n T_k(u_n - \omega_{j,\mu}^{i,l})dxdt. \end{aligned} \tag{52}$$

By using the fact that $-\int_{Q^\tau} nT_n(u_n-\psi)^- T_k(u_n - \omega_{j,\mu}^{i,l})dxdt \geq 0$, we deduce that:

$$\begin{aligned} &\Big\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega_{j,\mu}^{i,l})\Big\rangle_{Q^\tau} + \int_{Q^\tau} \omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n \nabla T_k(u_n - \omega_{j,\mu}^{i,l})dxdt \\ &\quad = \int_{Q^\tau} \omega(x)g(u_n)\Big|\nabla u_n\Big|^{p(x)}T_k(u_n - \omega_{j,\mu}^{i,l})dxdt + \int_{Q^\tau} f_n T_k(u_n - \omega_{j,\mu}^{i,l})dxdt. \end{aligned}$$

• On the one hand, we have

$$\begin{aligned} I &= \int_{Q^\tau} \omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n \nabla T_k(u_n - \omega_{j,\mu}^{i,l})dxdt \\ &= \int_{\{|T_k(u_n) - \omega_{j,\mu}^{i,l}| \leq k\}} \omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n[\nabla T_k(u_n) - \nabla \omega_{j,\mu}^{i,l}]dxdt. \end{aligned} \tag{53}$$

In the following, we pass to the limit in (53): By letting n and μ to infinity and by using Lebesgue theorem, we have

$$I = \int_{\{|T_k(u)-T_l(v_j)|\le k\}} \omega(x)\Big|\nabla T_k(u)\Big|^{p(x)-2}\nabla T_k(u)\Big[\nabla T_k(u) - \nabla T_l(v_j)\Big]dxdt + \epsilon(n,\mu),$$

consequently, by taking the limit as $j \to \infty$, we deduce that

$$I = \epsilon(n, \mu, j, l).$$

• On the other hand, we have

$$J = \int_{Q^\tau} \omega(x)g(u_n)\Big|\nabla u_n\Big|^{p(x)} T_k(u_n - \omega^{i,l}_{j,\mu})dxdt. \tag{54}$$

In the following, we pass to the limit in (54): Taking the limit as $n \to \infty$ in (54) and since $\omega(x)g(u_n)|\nabla u_n|^{p(x)} \to \omega(x)g(u)|\nabla u|^{p(x)}$ in $L^1(Q)$, and by using Lebesque theorem, we obtain $J = \int_{Q^\tau} g(u)|\nabla u|^{p(x)}T_k(u - \omega^{i,l}_{j,\mu})dxdt + \epsilon(n)$ and by letting μ and j to infinity, we have

$$J = \epsilon(n, \mu, j, l).$$

• Due to (15) , $u_n \to u_0$ and letting n , μ and j to infinity, we have

$$\int_{Q^\tau} f_n\Big[T_k(u_n - \omega^{i,l}_{j,\mu})\Big]dxdt = \epsilon(n, \mu, j, l)$$

and by using Vitali's theorem, we get

$$\limsup_{k\to\infty}\limsup_{i\to 0}\limsup_{j\to\infty}\limsup_{\mu\to\infty}\lim_{n\to\infty}\Big\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega^{i,l}_{j,\mu})\Big\rangle_{Q^\tau} \le 0. \tag{55}$$

We have (see([1]))

$$\Big\langle \frac{\partial \omega^{i,l}_{j,\mu}}{\partial t}, T_k(u_n - \omega^{i,l}_{j,\mu})\Big\rangle_{Q^\tau} = \mu\int_{Q^\tau}(T_k(v_j) - \omega^{i,l}_{j,\mu}))T_k(u_n - \omega^{i,l}_{j,\mu}) \ge \epsilon(n, j, \mu, l) \tag{56}$$

uniformly on τ. Therefore, by writing

$$\begin{aligned}\int_\Omega S_k\Big(u_n(\tau) - \omega^{i,l}_{j,\mu}(\tau)\Big)dx &= \Big\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega^{i,l}_{j,\mu})\Big\rangle_{Q^\tau}\\ &-\Big\langle \frac{\partial \omega^{i,l}_{j,\mu}}{\partial t}, T_k(u_n - \omega^{i,l}_{j,\mu})\Big\rangle_{Q^\tau} + \int_\Omega S_k\Big(u_n(0) - T_l(\eta_i)\Big)dx\end{aligned} \tag{57}$$

and using (55) and (56)) and (57), we see that

$$\int_\Omega S_k\Big(u_n(\tau) - \omega^{i,l}_{j,\mu}(\tau)\Big)dx \le \epsilon(n, j, \mu, l), \tag{58}$$

which implies, by writing

$$\begin{aligned}\int_\Omega S_k\Big(\frac{u_n(\tau) - u_m(\tau)}{2}\Big)dx &\le \frac{1}{2}\Big(\int_\Omega S_k(u_n(\tau) - \omega^{i,l}_{j,\mu}(\tau)\Big)dx\\ &+ \int_\Omega S_k\Big(u_m(\tau) - \omega^{i,l}_{j,\mu}(\tau))dx\Big),\end{aligned} \tag{59}$$

that

$$\int_\Omega S_k\Big(\frac{u_n(\tau)-u_m(\tau)}{2}\Big)dx \le \epsilon_1(n,m).$$

We deduce then that

$$\int_\Omega |u_n(\tau)-u_m(\tau)|dx \le \epsilon_2(n,m), \text{ independently of } \tau \tag{60}$$

and thus (u_n) is a Cauchy sequence in $C(0,T;L^1(\Omega))$, and since $u_n \to u$, a.e. in Q, we deduce that

$$u_n \to u \quad in \quad C(0,T;L^1(\Omega)). \tag{61}$$

b) We prove that u satisfies (14)

Indeed, let $v \in K_\psi \cap L^\infty(Q)$, $\frac{\partial v}{\partial t} \in L^{(p^-)'}(0,T;(W_0^{1,p(x)}(\Omega,\omega))^*)$. By the pointwise multiplication of (16) by $T_k(u_n-v)$, we get

$$\begin{aligned}\int_\Omega S_k\Big(u_n(T)-v(T)\Big)dx - \int_\Omega S_k\Big(u_{0n}-v(0)\Big)dx\\ +\int_Q \frac{\partial v}{\partial t}T_k(u_n-v)dxdt + \int_Q \Big(\omega(x)\Big|\nabla u\Big|^{p(x)-2}\nabla u\Big)\nabla T_k(u_n-v)dxdt\\ -\int_Q nT_n\Big((u_n-\psi)^-\Big)T_k(u_n-v)dxdt\\ = \int_Q \omega(x)g(u_n)\Big|\nabla u_n\Big|^{p(x)}T_k(u_n-v)dxdt\\ +\int_Q f_nT_k(u_n-v)dxdt,\end{aligned}$$

where $S_k(s)=\int_0^s T_k(r)dr$.

Since $v\in K_\psi\cap L^\infty(Q)$, we have $-\int_Q nT_n(u_n-\psi)^-T_k(u_n-v)dxdt \ge 0$, we deduce that

$$\begin{aligned}\int_\Omega S_k\Big(u_n(T)-v(T)\Big)dx - \int_\Omega S_k\Big(u_{0n}-v(0)\Big)dx + \int_Q \frac{\partial v}{\partial t}T_k(u_n-v)dxdt\\ +\int_Q \Big(\omega(x)\Big|\nabla u\Big|^{p(x)-2}\nabla u\Big)\nabla T_k(u_n-v)dxdt\\ \le \int_Q \omega(x)g(u_n)\Big|\nabla u_n\Big|^{p(x)}T_k(u_n-v)dxdt\\ +\int_Q f_nT_k(u_n-v)dxdt.\end{aligned} \tag{62}$$

• Let us pass to the limit with $n\to\infty$ in each term in (62). We saw that $u_n \to u$ in $C(0,T,L^1(\Omega))$. Therefore $u_n(t)\to u(t)$ in $L^1(\Omega)$ for all $t\le T$.

As S_k is Lipschitz of coefficient k, when $n\to\infty$, we have

$$\int_\Omega S_k(u_n-v)(T)dx \to \int_\Omega S_k(u-v)(T)dx$$

$$and \int_\Omega S_k(u_n-v)(0)dx = \int_\Omega S_k(u_{0n}-v(0))dx \to \int_\Omega S_k(u_0-v(0))dx.$$

• Since $\frac{\partial v}{\partial t} \in L^{(p^-)'}(0,T;(W_0^{1,p(x)}(\Omega,\omega))^*)$, one has

$$\int_0^T \Big\langle \frac{\partial v}{\partial t}, T_k(u_n - v)\Big\rangle dt \to \int_0^T \Big\langle \frac{\partial v}{\partial t} T_k(u-v)\Big\rangle dt.$$

• On the other hand, we note $M = \|v\|_\infty$. Then, we get

$$\begin{aligned}
&\int_Q \Big(\omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n\Big)\nabla T_k(u_n - v)dxdt \\
&= \int_0^T\int_\Omega \Big(\omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n\Big)\nabla T_k\Big(T_{k+M}(u_n) - v\Big)dxdt \\
&= \int_0^T\int_\Omega \Big(\omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n\Big)\nabla T_{k+M}(u_n))\mathbf{1}_{\{|T_{k+M}(u_n)-v|\le k\}}dxdt \\
&- \int_0^T\int_\Omega \Big(\omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n\Big)\nabla v\mathbf{1}_{\{|T_{k+M}(u_n)-v|\le k\}}dxdt.
\end{aligned}$$

As $T_{k+M}(u_n)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$ and $\nabla u_n \to \nabla u$ a.e. in Q, then

$$\nabla T_{k+M}(u_n) \to \nabla T_{k+M}(u) \text{ almost everywhere},$$

and by using Lebesgue theorem, we deduce that

$$\begin{aligned}
&\int_Q \Big(\omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n\Big)\nabla T_{k+M}(u_n))\mathbf{1}_{\{|T_{k+M}(u_n)-v|\le k\}}dxdt \\
&\qquad \to \int_Q \Big(\omega(x)\Big|\nabla u\Big|^{p(x)-2}\nabla u\Big)\mathbf{1}_{\{|T_{k+M}(u-v)|\le k\}}dxdt
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T\int_\Omega \Big(\omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n\Big)\nabla v\mathbf{1}_{\{|T_{k+M}(u_n)-v|\le k\}}dxdt \to \\
&\quad \int_0^T\int_\Omega \Big(\omega(x)\Big|\nabla u\Big|^{p(x)-2}\nabla u\Big)\nabla v\mathbf{1}_{\{|T_{k+M}(u-v)|\le k\}}dxdt,
\end{aligned}$$

then

$$\int_Q \Big(\omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n)T_k(u_n - v)dxdt \to \int_Q \Big(\omega(x)\Big|\nabla u\Big|^{p(x)-2}\nabla u\Big)T_k(u-v)dxdt.$$

• Let us pass to the limit for other term. Due to (15) , $T_k(u_n) \to T_k(u)$ *in* $V\ \forall\ k \ge 0$ and $u_n \to u$ a.e. in Q, we have

$$f_nT_k(u_n - v) \to fT_k(u-v) \text{ strongly } \text{ in } L^1(Q)$$

and by Lebesgue theorem, we have

$$\int_Q f_nT_k(u_n - v) \to \int_Q fT_k(u-v) \text{ strongly } in\ L^1(Q).$$

• Similarly, since g is a bounded and continuous function belonging to $L^1(\mathbb{R})$ and $u_n \to u$ a.e. in Q, we obtain

$$\int_Q \omega(x)g(u_n)\Big|\nabla u_n\Big|^{p(x)-2}T_k(u_n - v) \to \int_Q \omega(x)g(u)\Big|\nabla u\Big|^{p(x)-2}T_k(u-v) \text{ strongly } in\ L^1(Q).$$

Then, we conclude that u satisfies (14).

As a conclusion of Step 1 to Step 4, the proof of Theorem 4.1 is complete. □

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