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# Robust Output Feedback Stabilization and Optimization of Discrete-Time Control Systems

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**Abstract:** The paper is devoted to the problems of output feedback stabilization, robust stabilization, quadratic optimization and generalized $H_\infty$-control for a class of affine discrete-time systems. The solution of robust stabilization problem and evaluation of the quadratic performance criterion for a family of nonlinear nonautonomous control systems are proposed. Methods for construction of control laws providing a robust stability and specified evaluation of the weighted damping level of input signals and initial perturbations are proposed for linear systems with controllable and observable outputs. The application of the main results reduces to solving the systems of linear matrix inequalities.

**Keywords:** *discrete-time system, output feedback; robust stability; linear matrix inequality; quadratic Lyapunov function, $H_\infty$-control.*

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## 1 Introduction

State and output feedback controllers design for dynamic systems with the prescribed and desired properties is a key problem of control theory. At the same time, the properties of control systems such as asymptotic stability, robustness and optimality of the performance indexes are in the foreground. The main problem in $H_\infty$-control theory for continuous systems is connected with suppression of external and initial perturbations (see, e.g., [1–6] as well as review papers [7–9]). Practical applications of many modern methods for control systems design reduce to solving the linear matrix inequalities (LMI) [10, 11].

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In this paper, we consider classes of linear and affine discrete-time control systems for which closed loop systems can be represented in the pseudolinear form

$$x_{t+1} = M(x_t, t)\, x_t, \quad x_t \in \mathbb{R}^n, \quad t \in \mathcal{T} = \{0, 1, 2, \dots\},$$

besides, a matrix function $M(x, t)$ can contain uncertain quantities belonging to certain sets. Intervals, polytopes, affine families of matrices and other objects may serve as the uncertainty sets. To define uncertainties and robust stability conditions for systems in semiordered spaces one can use cone inequalities and intervals [6, 12, 13]. The applied control laws are of the form of static or dynamic output feedback. It should be noted that at the solution of many control problems the dynamic controllers have great potential as compared with the static controllers.

Our consideration includes the following types of problems:

- output feedback stabilization of discrete-time control systems (Section 2);
- robust stabilization and optimization of discrete-time control systems with polyhedral uncertainties (Section 3);
- robust stabilization and weighted perturbation suppression in discrete-time control systems (Section 4).

Throughout the paper, the following notations are used: $I_n$ is the identity $n \times n$ matrix; $0_{n\times m}$ is the $n \times m$ null matrix; $X = X^T > 0$ $(\geq 0)$ is the symmetric positive definite (semidefinite) matrix $X$; $\mathrm{i}(X) = \{\mathrm{i}_+, \mathrm{i}_-, \mathrm{i}_0\}$ is the inertia of Hermitian matrix $X = X^*$ consisting of the numbers of positive ($\mathrm{i}_+(X)$), negative ($\mathrm{i}_-(X)$) and zero ($\mathrm{i}_0(X)$) eigenvalues (taking into account the multiplicities); $\sigma(A)$ and $\rho(A)$ are the spectrum and the spectral radius of $A$, respectively; $\lambda_{\max}(X)$ and $\lambda_{\min}(X)$ are the maximum and the minimum eigenvalue of the Hermitian matrix $X$, respectively; $A^+$ is the pseudoinverse matrix; $W_A$ is a matrix whose columns make up the bases of the kernel $\mathrm{Ker}\, A$; $\|x\|$ denotes the Euclidean norm of the vector $x \in \mathbb{R}^n$; $\|w\|_P$ denotes the weighted $l_2$-norm of a vector sequence $w_t$, $t \in \mathcal{T}$; $\mathrm{Co}\{A_1, \dots, A_\nu\}$ stands for a polytope in a matrix space described as a convex hull of the set $\{A_1, \dots, A_\nu\}$, i.e.

$$\mathrm{Co}\{A_1, \dots, A_\nu\} = \{\alpha_1 A_1 + \dots + \alpha_\nu A_\nu : \ \alpha_1 + \dots + \alpha_\nu = 1, \ \ \alpha_i \geq 0, \ i = \overline{1, \nu}\}.$$

Note that matrix intervals and affine sets are described in terms of polytopes.

## 2 Output Feedback Stabilization of Nonlinear Systems

Consider the affine discrete-time control system

$$x_{t+1} = A(x_t)x_t + B(x_t)u_t, \quad y_t = C(x_t)x_t + D(x_t)u_t, \tag{1}$$

where $x_t \in \mathbb{R}^n$ is a state vector, $u_t \in \mathbb{R}^m$ and $y_t \in \mathbb{R}^l$ are input and output vectors, respectively, $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are continuous matrix functions in some neighborhood $\mathcal{S}_0$ of the zero state $x_t = 0$, $t \in \mathcal{T}$. Assume that $\mathrm{rank}\, B(x) \equiv m$ and $\mathrm{rank}\, C(x) \equiv l$ in $\mathcal{S}_0$.

Along with (1), consider the linear system

$$x_{t+1} = Ax_t + Bu_t, \quad y_t = Cx_t + Du_t, \tag{2}$$

with $A = A(0)$, $B = B(0)$, $C = C(0)$ and $D = D(0)$. Let $B^\perp$ and $C^\perp$ be the orthogonal complements of $B$ and $C$, respectively, i.e. $B^T B^\perp = 0$, $\det\,[B, B^\perp] \neq 0$, $C^\perp C^T = 0$, $\det\,[C^T, C^{\perp T}] \neq 0$.

### 2.1 Static controllers

Formulate stabilizability conditions of the zero state $x_t = 0$ for systems (1) and (2) through the static output-feedback controller

$$u_t = Ky_t, \quad K \in \mathcal{K}_D, \tag{3}$$

where $\mathcal{K}_D = \{K \in \mathbb{R}^{m\times l} : \det(I_m - KD) \neq 0\}$. Closed loop system (2), (3) has the form

$$x_{t+1} = Mx_t, \quad M = A + B\mathbf{D}(K)C, \tag{4}$$

where $\mathbf{D}(K) = (I_m - KD)^{-1}K$ is a nonlinear operator with the following properties:
• if $K \in \mathcal{K}_D$, then $\mathbf{D}(K) \equiv K(I_l - DK)^{-1}$ and $I_l + D\mathbf{D}(K) \equiv (I_l - DK)^{-1}$;
• if $K_1 \in \mathcal{K}_D$ and $K_2 \in \mathcal{K}_{D_1}$, then $K_1 + K_2 \in \mathcal{K}_D$ and

$$\mathbf{D}(K_1 + K_2) = \mathbf{D}(K_1) + (I_m - K_1D)^{-1}\mathbf{D}_1(K_2)\,(I_l - DK_1)^{-1}, \tag{5}$$

where $\mathbf{D}_1(K_2) = (I_m - K_2D_1)^{-1}K_2$, $D_1 = (I_l - DK_1)^{-1}D$;
• if $-K_0 \in \mathcal{K}_D$, then $K = -\mathbf{D}(-K_0) \in \mathcal{K}_D$ and $\mathbf{D}(K) = K_0$.

**Definition 2.1** System (4) is $\rho$-stable if the spectrum $\sigma(M)$ lies inside the circle $\{\lambda : |\lambda| < \rho\}$, where $0 < \rho \leq 1$.

**Theorem 2.1** *Let* $\operatorname{rank} B = m < n$ *and* $\operatorname{rank} C = l < n$. *Then the following statements are equivalent:*
1) *There exists a static controller* (3) *ensuring* $\rho$*-stability of system* (4).
2) *There exists a matrix* $X = X^T > 0$ *satisfying the relations*

$$B^{\perp T}(AXA^T - \rho^2 X)B^{\perp} < 0, \tag{6}$$

$$\mathrm{i}(H) = \{l, m, 0\}, \quad H = \begin{bmatrix} H_0 & H_1^T \\ H_1 & H_2 \end{bmatrix}, \tag{7}$$

*where* $H_0 = B^+(L - LRL)B^{+T}$, $H_1 = CXA^T(I_n - RL)B^{+T}$, $H_2 = C(X - XA^TRAX)C^T$, $L = AXA^T - \rho^2 X$, $R = B^{\perp}S^{-1}B^{\perp T}$, $S = B^{\perp T}LB^{\perp}$;
3) *There exists a matrix* $X = X^T > 0$ *satisfying the matrix inequalities* (6) *and*

$$AXA^T - \rho^2 X < AXC^T(CXC^T)^{-1}CXA^T. \tag{8}$$

4) *There exist mutually inverse matrices* $X = X^T > 0$ *and* $Y = Y^T > 0$ *satisfying the relations* (6) *and*

$$C^{\perp}(A^TYA - \rho^2 Y)C^{\perp T} < 0. \tag{9}$$

5) *There exists a matrix* $Y = Y^T > 0$ *satisfying the matrix inequalities* (9) *and*

$$A^TYA - \rho^2 Y < A^TYB(B^TYB)^{-1}B^TYA. \tag{10}$$

*When one of the statements* 2) – 4) *is true, then the controller*

$$u_t = Ky_t, \quad K = -\mathbf{D}(-K_0) \in \mathcal{K}_D, \tag{11}$$

*where* $K_0$ *is a solution of one of the equivalent LMI*

$$P_1^TK_0Q_1 + Q_1^TK_0^TP_1 < \begin{bmatrix} \rho^2 X & AX \\ XA^T & X \end{bmatrix}, \quad P_2^TK_0Q_2 + Q_2^TK_0^TP_2 < \begin{bmatrix} -H_0 & 0 \\ 0 & H_2^{-1} \end{bmatrix}, \tag{12}$$

*with* $P_1 = \big[-B^T, 0\big]$, $Q_1 = \big[0, CX\big]$, $P_2 = \big[I_m, 0\big]$ *and* $Q_2 = \big[H_1, I_l\big]$ *ensures* $\rho$*-stability of closed loop system* (4).

For the equivalence of the statements 1) and 2) in Theorem 2.1, see [6]. Equivalence of the statements 2) and 3) follows from the correlations (see [12, p. 147])

$$H = \widehat{H}_0 - \widehat{H}_1^T \widehat{H}_2^{-1} \widehat{H}_1, \ \ \mathrm{i}_+(\widehat{H}) = \mathrm{i}_+(H) = \mathrm{i}_+(\Delta), \ \ \mathrm{i}_-(\widehat{H}) = \mathrm{i}_-(H) + n - m = \mathrm{i}_-(\Delta),$$

where

$$\widehat{H} = \begin{bmatrix} \widehat{H}_0 & \widehat{H}_1^T \\ \widehat{H}_1 & \widehat{H}_2 \end{bmatrix} = \left[\begin{array}{cc|c} B^+ L B^{+T} & B^+ A X C^T & B^+ L B^\perp \\ C X A^T B^{+T} & C X C^T & C X A^T B^\perp \\ \hline B^{\perp T} L B^{+T} & B^{\perp T} A X C^T & S \end{array}\right] = W \Delta W^T,$$

$$\Delta = \begin{bmatrix} A X A^T - \rho^2 X & A X C^T \\ C X A^T & C X C^T \end{bmatrix}, \quad W^T = \begin{bmatrix} B^{+T} & 0 & B^\perp \\ 0 & I_l & 0 \end{bmatrix}, \quad \det W \neq 0.$$

For the equivalence of the statements 1) and 4), see also [5] and [6, Therem 6.1.2].

**Theorem 2.2** *Let one of the statements* 2) – 4) *of Theorem* 2.1 *hold for linear system* (2)*. Then relations* (11) *and* (12) *determine static controller ensuring asymptotic stability of the state* $x \equiv 0$ *and quadratic Lyapunov function* $v(x) = x^T X^{-1} x$ *of nonlinear closed loop system* (1)*,* (11)*.*

### 2.2 Dynamic controllers

The dynamic output feedback stabilization problem for system (1) is to find, if possible, a dynamic control law described by

$$\xi_{t+1} = Z\xi_t + V y_t, \quad u_t = U\xi_t + K y_t, \quad t \in \mathcal{T}, \tag{13}$$

where $\xi_t \in \mathbb{R}^r$ and $r \leq n$, such that the zero state of closed loop system is asymptotically stable. Equations (1) and (13) may be represented by control system in the extended phase space $\mathbb{R}^{n+r}$ with static controller

$$\widehat{x}_{t+1} = \widehat{A}(\widehat{x}_t)\widehat{x}_t + \widehat{B}(\widehat{x}_t)\widehat{u}_t, \quad \widehat{y}_t = \widehat{C}(\widehat{x}_t)\widehat{x}_t + \widehat{D}(\widehat{x}_t)\widehat{u}_t, \quad \widehat{u}_t = \widehat{K}\widehat{y}_t, \tag{14}$$

where

$$\widehat{x}_t = \begin{bmatrix} x_t \\ \xi_t \end{bmatrix}, \quad \widehat{y}_t = \begin{bmatrix} y_t \\ \xi_t \end{bmatrix}, \quad \widehat{u}_t = \begin{bmatrix} u_t \\ \xi_{t+1} \end{bmatrix}, \quad \widehat{K} = \begin{bmatrix} K & U \\ V & Z \end{bmatrix},$$

$$\widehat{A}(\widehat{x}) = \begin{bmatrix} A(x) & 0 \\ 0 & 0 \end{bmatrix}, \widehat{B}(\widehat{x}) = \begin{bmatrix} B(x) & 0 \\ 0 & I_r \end{bmatrix}, \widehat{C}(\widehat{x}) = \begin{bmatrix} C(x) & 0 \\ 0 & I_r \end{bmatrix}, \widehat{D}(\widehat{x}) = \begin{bmatrix} D(x) & 0 \\ 0 & 0 \end{bmatrix}.$$

If $K \in \mathcal{K}_D$, then linear closed loop system (2), (13) has the form

$$\widehat{x}_{t+1} = \widehat{M}\,\widehat{x}_t, \quad \widehat{M} = \widehat{A} + \widehat{B}\widehat{\mathbf{D}}(\widehat{K})\widehat{C}, \tag{15}$$

where $\widehat{A} = \widehat{A}(0)$, $\widehat{B} = \widehat{B}(0)$, $\widehat{C} = \widehat{C}(0)$, $\widehat{D} = \widehat{D}(0)$, $\widehat{\mathbf{D}}(\widehat{K}) = (I_{m+r} - \widehat{K}\widehat{D})^{-1}\widehat{K}$, and

$$\widehat{\mathbf{D}}(\widehat{K}) = \left[\begin{array}{c|c} \mathbf{D}(K) & (I_m - KD)^{-1}U \\ \hline V(I_l - DK)^{-1} & Z + VD(I_m - KD)^{-1}U \end{array}\right],$$

$$\widehat{M} = \left[\begin{array}{c|c} M & B(I_m - KD)^{-1}U \\ \hline V(I_l - DK)^{-1}C & Z + VD(I_m - KD)^{-1}U \end{array}\right].$$

**Theorem 2.3** *The following statements are equivalent:*

1) *There exists a dynamic controller* (13) *of order $r \le n$ ensuring $\rho$-stability of closed loop system* (15).

2) *There exist matrices $X$ and $X_0$ satisfying the relations* (6) *and*

$$X \ge X_0 > 0, \;\; \operatorname{rank}(X - X_0) \le r, \;\; AX_0A^T - \rho^2 X_0 < AX_0C^T(CX_0C^T)^{-1}CX_0A^T. \quad (16)$$

3) *There exist matrices $X$ and $Y$ satisfying the relations* (6), (9) *and*

$$W = \begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \ge 0, \quad \operatorname{rank} W \le n + r. \quad (17)$$

Proof of Theorem 2.3 follows from the corresponding statements of Theorem 2.1 taking into account the structure of block matrices in (15) (see [6]).

**Remark 2.1** The coefficient matrices of stabilizing controller (13) in Theorem 2.3 may be defined in the form

$$\begin{gathered} K = (I_m + K_0 D)^{-1} K_0, \quad U = (I_m + K_0 D)^{-1} U_0, \\ V = V_0 (I_l + D K_0)^{-1}, \quad Z = Z_0 - V_0 (I_l + D K_0)^{-1} D U_0, \end{gathered} \quad (18)$$

using the solution $\widehat{K}_0$ of the LMI

$$\widehat{P}^T \widehat{K}_0 \widehat{Q} + \widehat{Q}^T \widehat{K}_0^T \widehat{P} < \widehat{F}, \quad (19)$$

where $\widehat{P} = \big[-\widehat{B}^T, 0\big]$, $\widehat{Q} = \big[0, \widehat{C}\widehat{X}\big]$, $X - X_0 = X_1^T X_2^{-1} X_1 \ge 0$, $K_0 \in \mathcal{K}_D$, $0 < \rho \le 1$,

$$\widehat{F} = \begin{bmatrix} \rho^2 \widehat{X} & \widehat{A}\widehat{X} \\ \widehat{X}\widehat{A}^T & \widehat{X} \end{bmatrix}, \quad \widehat{K}_0 = \begin{bmatrix} K_0 & U_0 \\ V_0 & Z_0 \end{bmatrix}, \quad \widehat{X} = \begin{bmatrix} X & X_1^T \\ X_1 & X_2 \end{bmatrix} > 0.$$

For example, one can use the Cholesky factorization $X - X_0 = X_1^T X_1 \ge 0$ with $X_2 = I_r$.

**Remark 2.2** Note, that matrices $X$ and $X_0$ satisfy statement 2) iff matrices $X$ and $Y = X_0^{-1}$ satisfy statement 3). From (17) it follows that matrices $X$ and $Y$ are positive definite. The rank restriction in (17) always holds in case of full order $r = n$ dynamic regulator.

**Theorem 2.4** *Let one of the statements* 2) *or* 3) *of Theorem* 2.3 *hold for linear system* (2). *Then relations* (18) *and* (19) *determine dynamic controller* (13) *ensuring asymptotic stability of the state $x \equiv 0$ and quadratic Lyapunov function $v(\widehat{x}) = \widehat{x}^T \widehat{X}^{-1} \widehat{x}$ of nonlinear closed loop system* (1), (13).

## 3 Robust Stabilization and Optimization of Nonlinear Systems

We formulate an auxiliary statement that will be used in the proofs of our main results. Consider a nonlinear operator

$$\mathbf{F}(K) = W + U^T \mathbf{D}(K) V + V^T \mathbf{D}^T(K) U + V^T \mathbf{D}^T(K)\, R\, \mathbf{D}(K) V \quad (20)$$

with $\mathbf{D}(K) = (I_m - KD)^{-1} K$ and an ellipsoidal set of matrices

$$\mathcal{K} = \big\{K \in \mathbb{R}^{m \times l} : \, K^T P K \le Q\big\}, \quad (21)$$

where $P = P^T > 0$, $Q = Q^T > 0$, $R = R^T \ge 0$, $W = W^T \le 0$, $U$, $V$ and $D$ are matrices of suitable sizes.

**Lemma 3.1** [14] *Suppose that the following matrix inequalities hold:*

$$D^TQD + R < P, \quad \Omega = \begin{bmatrix} W & U^T & V^T \\ U & R-P & D^T \\ V & D & -Q^{-1} \end{bmatrix} \le 0 \ (< 0). \tag{22}$$

*Then* $\mathbf{F}(K) \le 0$ $(< 0)$ *for every matrix* $K \in \mathcal{K}$.

Note that Lemma 3.1 is a generalization of the sufficiency statement for a criterion known as Petersen's lemma on matrix uncertainty [15] (see also [16]).

Consider a nonlinear control system in the vector-matrix form

$$x_{t+1} = A(x_t,t)x_t + B(x_t,t)u_t, \;\; y_t = C(x_t,t)x_t + D(x_t,t)u_t, \tag{23}$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$ and $y_t \in \mathbb{R}^l$. We construct a set of the static controllers

$$u_t = K(x_t,t)\, y_t, \quad K(x_t,t) = K_*(x_t,t) + \widetilde{K}(x_t,t), \quad \widetilde{K}(x_t,t) \in \mathcal{K}, \tag{24}$$

where $\mathcal{K}$ is an ellipsoidal set of matrices of the form (21). We assume that the matrices $A$, $B$, $C$, $D$, $K$ and $K_*$ depend on $x_t$ and $t$ continuously and the equilibrium state $x_t \equiv 0$ is isolated, i.e., the neighborhood $\mathcal{S}_0 = \{x \in \mathbb{R}^n : \|x\| \le h\}$ does not contain other equilibrium states of this system. If $K \in \mathcal{K}_D$, then the closed loop system (23), (24) can be represented as

$$x_{t+1} = M(x_t,t)x_t, \quad M(x_t,t) = A + B\mathbf{D}(K)C. \tag{25}$$

Let the zero state of this system for $K \equiv K_*$ be asymptotically stable. When looking for the stabilizing matrix $K_*$ in the class of autonomous systems (1), one can use Theorem 2.1 and its special cases. The problem is to construct conditions under which the zero state of system (25) is asymptotically Lyapunov stable for every matrix $\widetilde{K}(x_t,t) \in \mathcal{K}$. We find a solution for our problem in terms of a quadratic Lyapunov function (see [6, 14]).

**Theorem 3.1** *Let for some matrix functions* $X_t = X_t^T$ *and* $K_*(x,t)$ *the correlations*

$$\varepsilon_1 I_n \le X_t \le \varepsilon_2 I_n, \quad 0 < \varepsilon_1 \le \varepsilon_2, \tag{26}$$

$$\begin{bmatrix} M_*^T X_{t+1} M_* - X_t + \varepsilon_0 I_n & M_*^T X_{t+1} B_* & C_*^T \\ B_*^T X_{t+1} M_* & B_*^T X_{t+1} B_* - P & D_*^T \\ C_* & D_* & -Q^{-1} \end{bmatrix} < 0, \tag{27}$$

*hold with* $\varepsilon_0 > 0$, $M_* = A + B\mathbf{D}(K_*)C$, $B_* = B(I_m - K_*D)^{-1}$, $C_* = (I_l - DK_*)^{-1}C$ *and* $D_* = D(I_m - K_*D)^{-1}$, $x_t = 0$ *and* $t \in \mathcal{T}$. *Then any control* (24) *ensures asymptotic stability of the zero state* $x_t \equiv 0$ *for system* (25) *and a common Lyapunov function* $v(x,t) = x^T X_t x$.

Consider control system (23) with quadratic quality functional

$$J_u(x_0) = \sum_0^\infty \varphi_t, \quad \varphi_t = \begin{bmatrix} x_t^T \; u_t^T \end{bmatrix} \Phi_t \begin{bmatrix} x_t \\ u_t \end{bmatrix}, \tag{28}$$

where

$$\Phi_t = \begin{bmatrix} S & N \\ N^T & R \end{bmatrix}, \quad S \ge NR^{-1}N^T + \eta\, I_n, \quad R > 0, \quad \eta > 0, \quad t \in \mathcal{T}.$$

**Theorem 3.2** *Let for some matrix functions $X_t = X_t^T$ and $K_*(x_t, t)$ the correlations* (26) *and*

$$\begin{bmatrix} M_*^T X_{t+1} M_* - X_t + \Phi_* + \varepsilon_0 I_n & M_*^T X_{t+1} B_* + N_* + C^T K_*^T R_* & C_*^T \\ B_*^T X_{t+1} M_* + N_*^T + R_* K_* C & B_*^T X_{t+1} B_* + R_* - P & D_*^T \\ C_* & D_* & -Q^{-1} \end{bmatrix} < 0 \quad (29)$$

*hold with $\Phi_* = L_*^T \Phi L_*$, $L_*^T = \left[I_n, C^T \mathbf{D}^T(K_*)\right]$, $R_* = (I_m - K_* D)^{-1T} R\, (I_m - K_* D)^{-1}$, $N_* = N(I_m - K_* D)^{-1}$, $\varepsilon_0 > 0$, $x_t = 0$ and $t \in \mathcal{T}$. Then any control* (24) *ensures asymptotic stability of the zero state $x_t \equiv 0$ for system* (25)*, a common Lyapunov function $v(x,t) = x^T X_t x$ and a bound on the functional $J_u(x_0) \le v(x_0, 0) = x_0^T X_0 x_0$.*

**Corollary 3.1** *Let for some matrix $X_t = X_t^T > 0$ and $K_*$ the system of LMI*

$$\begin{bmatrix} M_{ijk}^T X_{t+1} M_{ijk} - X_t + \Phi_k + \varepsilon_0 I_n & M_{ijk}^T X_{t+1} B_{*j} + N_* + C_k^T K_*^T R_* & C_{*k}^T \\ B_{*j}^T X_{t+1} M_{ijk} + N_*^T + R_* K_* C_k & B_{*j}^T X_{t+1} B_{*j} + R_* - P & D_*^T \\ C_{*k} & D_* & -Q^{-1} \end{bmatrix} < 0,$$

*hold with $M_{ijk} = A_i + B_j \mathbf{D}(K_*) C_k$, $B_{*j} = B_j (I_m - K_* D)^{-1}$, $\Phi_k = L_k^T \Phi L_k$, $L_k^T = \left[I_n, C_k^T \mathbf{D}^T(K_*)\right]$, $C_{*k} = (I_l - DK_*)^{-1} C_k$, $i = \overline{1,\alpha}$, $j = \overline{1,\beta}$, $k = \overline{1,\gamma}$, $\varepsilon_0 > 0$, $x_t = 0$, $t \in \mathcal{T}$. Then any control* (24) *ensures asymptotic stability of the zero state $x_t \equiv 0$ for system* (25) *with uncertainties $A(0,t) \in \mathrm{Co}\{A_1, \ldots, A_\alpha\}$, $B(0,t) \in \mathrm{Co}\{B_1, \ldots, B_\beta\}$ and $C(0,t) \in \mathrm{Co}\{C_1, \ldots, C_\gamma\}$, a common Lyapunov function $v(x,t) = x^T X_t x$ and a bound on the functional $J_u(x_0) \le v(x_0, 0) = x_0^T X_0 x_0$.*

Note that the proofs of Theorems 3.1 and 3.2 follow directly from Lemma 3.1 and the Lyapunov theorem on asymptotic stability taking into account representation of the first difference of Lyapunov function $v(x,t)$ with respect to system (25) in the form of a quadratic function with matrix of the form (20) and application of formula (5) (see [6,14]).

## 4 Generalized $H_\infty$-control

### 4.1 Weighted level of perturbation suppression

Consider a dynamical system with external perturbations

$$x_{t+1} = f(x_t, w_t, t), \quad y_t = g(x_t, w_t, t), \quad t \in \mathcal{T}, \tag{30}$$

where $x_t \in \mathbb{R}^n$, $w_t \in \mathbb{R}^s$ and $y_t \in \mathbb{R}^l$ are the state, the $l_2$-norm-limited external perturbations and the output vector, respectively.

**Definition 4.1** The dynamical system (30) is called *nonexpansive*, if for all square-integrable functions $w_t$ and $\tau > 0$

$$\sum_{t=0}^{\tau} y_t^T Q y_t \le \sum_{t=0}^{\tau} w_t^T P w_t + x_0^T X_0 x_0,$$

where $Q$, $P$ and $X_0$ are weight symmetric positive definite matrices.

We introduce the performance criterion of system (30) with respect to output $y$:

$$J = \sup_{0 < \|w\|_P^2 + x_0^T X_0 x_0 < \infty} \varphi(w, x_0), \tag{31}$$

where

$$\varphi(w,x_0)=\frac{\|y\|_Q}{\sqrt{\|w\|_P^2+x_0^TX_0x_0}},\quad \|y\|_Q^2=\sum_{t=0}^{\infty}y_t^TQy_t,\quad \|w\|_P^2=\sum_{t=0}^{\infty}w_t^TPw_t.$$

In case of $x_0=0$, we denote $J$ by $J_0$. It is obvious that $J_0\le J$ and $J\le 1$ for a nonexpansive system. The value $J$ describes the weighted level of external and initial perturbation suppression in system (30). If $P=I_s$, $Q=I_l$ and $X_0=\rho I_n$, then $J$ and $J_0$ coincide with known performance criteria of dynamical systems [17].

Consider the class of linear systems

$$x_{t+1}=Ax_t+Bw_t,\quad y_t=Cx_t+Dw_t,\quad t\in\mathcal{T}. \tag{32}$$

**Lemma 4.1** *Let $\rho(A)<1$. Then an evaluation $J_0<\gamma$ for system (32) holds iff the LMI*

$$\Psi_\gamma=\begin{bmatrix} A^TXA-X+C^TQC & A^TXB+C^TQD \\ B^TXA+D^TQC & B^TXB+D^TQD-\gamma^2P\end{bmatrix}<0 \tag{33}$$

*has a solution $X=X^T>0$. To perform the evaluation $J<\gamma$ it is necessary and sufficient that LMI (33) has a solution $X$ such that*

$$0<X<\gamma^2X_0. \tag{34}$$

The sufficiency assertion of Lemma 4.1 follows from the relation

$$\Delta v(x_t)+y_t^TQy_t-\gamma^2w_t^TPw_t=\left[x_t^T,w_t^T\right]\Psi_\gamma\begin{bmatrix}x_t\\ w_t\end{bmatrix}<0,$$

where $\Delta v(x_t)=v(x_{t+1})-v(x_t)$ is the first difference of Lyapunov function $v(x)=x^TXx$ with respect to system (32). The necessity assertion of Lemma 4.1 may be established via representation of functional $\varphi(w,x_0)$ by similar expression with the identity weight matrices (see the proof of Lemma 5.1.1 in [6] and [17]).

**Remark 4.1** If $J_0<\gamma$, then system (32) with a structured uncertainty

$$w_t=\frac{1}{\gamma}\Theta y_t,\quad \Theta^TP\Theta\le Q,\quad t\in\mathcal{T}, \tag{35}$$

is robust stable and has a common Lyapunov function $v(x)=x^TXx$. This fact follows from Lemma 4.1 and Theorem 3.1. The functional $\varphi(w,x_0)$ on the set of functions (35) takes the minimum value, if $\Theta^TP\Theta=Q$.

It follows from Lemma 4.1 that the performance criteria $J$ and $J_0$ of system (32) may be computed as the solutions of the corresponding optimization problems:

$$J_0=\inf\left\{\gamma:\Psi_\gamma<0,\ X>0\right\},\quad J=\inf\left\{\gamma:\Psi_\gamma<0,\ 0<X<\gamma^2X_0\right\}. \tag{36}$$

Consider the affine system with norm-limited external perturbations

$$x_{t+1}=A(x_t)x_t+B(x_t)w_t,\quad y_t=C(x_t)x_t+D(x_t)w_t,\quad t\in\mathcal{T}, \tag{37}$$

where $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are continuous matrix functions in $\mathcal{S}_0$. We can formulate the following statement.

**Lemma 4.2** *Suppose that there exists a matrix $X = X^T > 0$ satisfying the matrix inequality*

$$\left[\begin{array}{cc} A^T(x)XA(x) - X + C^T(x)QC(x) & A^T(x)XB(x) + C^T(x)QD(x) \\ B^T(x)XA(x) + D^T(x)QC(x) & B^T(x)XB(x) + D^T(x)QD(x) - \gamma^2 P \end{array}\right] < 0 \tag{38}$$

*for all $x \in \mathcal{S}_0$. Then $J_0 \leq \gamma$ and the zero state $x_t \equiv 0$ of system* (37) *with a structured uncertainty* (35) *is robust stable with a common Lyapunov function $v(x) = x^T Xx$. In addition, if the restriction $0 < X \leq \gamma^2 X_0$ holds, then $J \leq \gamma$.*

### 4.2 Static controllers with perturbations

Consider control systems (1), (2) and the performance criteria $J$ and $J_0$ of the form (31). We are interested in control laws that ensure nonexpansivity property of close loop system and minimize $J$ and $J_0$. A control law is said to be $J$-*optimal*, if the corresponding close loop system has minimum performance criteria $J$.

Primarily, we consider the static output-feedback controller

$$u_t = K_* y_t + w_t, \quad t \in \mathcal{T}, \tag{39}$$

where $w_t \in \mathbb{R}^m$ is a vector of $l_2$-bounded perturbations and $K_* \in \mathcal{K}_D$ is an unknown matrix. Assuming that $\det\left[I_m - K_* D(x)\right] \neq 0$, $x \in \mathcal{S}_0$, we rewrite the corresponding close loop systems in the form

$$x_{t+1} = A_*(x_t)x_t + B_*(x_t)w_t, \quad y_t = C_*(x_t)x_t + D_*(x_t)w_t, \tag{40}$$

$$x_{t+1} = A_* x_t + B_* w_t, \quad y_t = C_* x_t + D_* w_t, \tag{41}$$

where $A_*(x) = A(x) + B(x)\left[I_m - K_* D(x)\right]^{-1} K_* C(x)$, $B_*(x) = B(x)\left[I_m - K_* D(x)\right]^{-1}$, $C_*(x) = \left[I_l - D(x)K_*\right]^{-1} C(x)$, $D_*(x) = \left[I_l - D(x)K_*\right]^{-1} D(x)$, $A_* = A_*(0)$, $B_* = B_*(0)$, $C_* = C_*(0)$, $D_* = D_*(0)$.

**Theorem 4.1** *For linear system* (2)*, there exists an output-feedback controller* (39) *such that $J < \gamma$ iff the following correlations are feasible:*

$$W_R^T \left[\begin{array}{cc} A^T XA - X + C^T QC & A^T XB + C^T QD \\ B^T XA + D^T QC & B^T XB + D^T QD - \gamma^2 P \end{array}\right] W_R < 0, \tag{42}$$

$$W_L^T \left[\begin{array}{cc} AYA^T - Y + BP^{-1}B^T & AYC^T + BP^{-1}D^T \\ CYA^T + DP^{-1}B^T & CYC^T + DP^{-1}D^T - \gamma^2 Q^{-1} \end{array}\right] W_L < 0, \tag{43}$$

$$0 < X < \gamma^2 X_0, \quad XY = \gamma^2 I_n, \tag{44}$$

*where $R = [C, D]$, $L = [B^T, D^T]$. Herewith, the zero states $x \equiv 0$ of systems* (40) *and* (41) *with uncertainty* (35) *are robust stable with common Lyapunov function $v(x) = x^T Xx$.*

**Remark 4.2** The gain matrix $K_*$ in Theorem 4.1 may be constructed in the form

$$K_* = K_0(I_l + DK_0)^{-1}, \quad -K_0 \in \mathcal{K}_D, \tag{45}$$

Here $K_0$ is an arbitrary solution of the LMI

$$L_0^T K_0 R_0 + R_0^T K_0^T L_0 + \Omega < 0, \tag{46}$$

where

$$\Omega = \begin{bmatrix} -X & 0 & A^T & C^T \\ 0 & -\gamma^2 P & B^T & D^T \\ A & B & -X^{-1} & 0 \\ C & D & 0 & -Q^{-1} \end{bmatrix}, \quad R_0^T = \begin{bmatrix} C^T \\ D^T \\ 0 \\ 0 \end{bmatrix}, \quad L_0^T = \begin{bmatrix} 0 \\ 0 \\ B \\ D \end{bmatrix}.$$

**Lemma 4.3** [3] *LMI* (46) *has a solution* $K_0$ *if and only if*

$$W_{L_0}^T \Omega W_{L_0} < 0, \quad W_{R_0}^T \Omega W_{R_0} < 0, \tag{47}$$

*where* $W_{L_0}$ $(W_{R_0})$ *is a matrix whose columns make up the bases of the kernel* $\operatorname{Ker} L_0$ $(\operatorname{Ker} R_0)$.

### 4.3 Dynamic controllers with perturbations

Consider control systems (1) and (2) with the dynamic output-feedback controller

$$\xi_{t+1} = Z\xi_t + Vy_t, \quad u_t = U\xi_t + Ky_t + w_t, \quad t \in \mathcal{T}, \tag{48}$$

where $\xi_0 = 0$, $w_t \in \mathbb{R}^m$ is a vector of bounded perturbations, $Z$, $V$, $U$ and $K$ are unknown coefficient matrices. If $K \in \mathcal{K}_D$, then linear close loop system (2), (48) reduces to the form

$$\widehat{x}_{t+1} = \widehat{A}_* \widehat{x}_t + \widehat{B}_* w_t, \quad y_t = \widehat{C}_* \widehat{x}_t + \widehat{D}_* w_t, \tag{49}$$

where

$$\widehat{x}_t = \begin{bmatrix} x_t \\ \xi_t \end{bmatrix}, \quad \widehat{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \widehat{B} = \begin{bmatrix} B & 0 \\ 0 & I_r \end{bmatrix}, \quad \widehat{C} = \begin{bmatrix} C & 0 \\ 0 & I_r \end{bmatrix},$$

$$\widehat{A}_* = \widehat{A} + \widehat{B}\widehat{K}_0\widehat{C}, \quad \widehat{B}_* = \widehat{B}_1 + \widehat{B}\widehat{K}_0\widehat{D}_1, \quad \widehat{C}_* = \widehat{C}_1 + \widehat{D}_2\widehat{K}_0\widehat{C}, \quad \widehat{D}_* = D + \widehat{D}_2\widehat{K}_0\widehat{D}_1,$$

$$\widehat{B}_1 = \begin{bmatrix} B \\ 0 \end{bmatrix}, \ \widehat{C}_1 = \begin{bmatrix} C & 0 \end{bmatrix}, \ \widehat{D}_1 = \begin{bmatrix} D \\ 0 \end{bmatrix}, \ \widehat{D}_2 = \begin{bmatrix} D & 0 \end{bmatrix}, \quad \widehat{K}_0 = \begin{bmatrix} K_0 & U_0 \\ V_0 & Z_0 \end{bmatrix},$$

$K_0 = \mathbf{D}(K)$, $U_0 = (I_m - KD)^{-1}U$, $V_0 = V(I_l - DK)^{-1}$, $Z_0 = Z + VD(I_m - KD)^{-1}U$.

We give the following auxiliary statement (see also [18] in the case of $\gamma = 1$).

**Lemma 4.4** *Given the matrices* $X > 0$, $Y > 0$ *and the number* $\gamma > 0$, *there are matrices* $X_1 \in \mathbb{R}^{r\times n}$, $X_2 \in \mathbb{R}^{r\times r}$, $Y_1 \in \mathbb{R}^{r\times n}$ *and* $Y_2 \in \mathbb{R}^{r\times r}$ *such that*

$$\widehat{X} = \begin{bmatrix} X & X_1^T \\ X_1 & X_2 \end{bmatrix} > 0, \quad \widehat{Y} = \begin{bmatrix} Y & Y_1^T \\ Y_1 & Y_2 \end{bmatrix} > 0, \quad \widehat{X}\widehat{Y} = \gamma^2 I_{n+r}, \tag{50}$$

*if and only if*

$$W = \begin{bmatrix} X & \gamma I_n \\ \gamma I_n & Y \end{bmatrix} \geq 0, \quad \operatorname{rank} W \leq n + r. \tag{51}$$

Applying Lemmas 4.3, 4.4 and Theorem 4.1 to system (49), we get the following result.

**Theorem 4.2** *There exists a dynamic controller* (48) *such that the evaluation* $J < \gamma$ *holds for linear system* (49), *iff the LMI system* (34), (42), (43) *and* (51) *is solvable with respect to* $X = X^T > 0$ *and* $Y = Y^T > 0$. *In addition, a close loop system* (49) *with a structured uncertainty* (35) *is robust stable.*

**Remark 4.3** The coefficient matrices of dynamic controller (48) in Theorem 4.2 may be constructed in the form (18) by solving LMI with respect to $\widehat{K}_0$:

$$\widehat{L}^T \widehat{K}_0 \widehat{R} + \widehat{R}^T \widehat{K}_0^T \widehat{L} + \widehat{\Omega} < 0, \tag{52}$$

where

$$\widehat{\Omega} = \begin{bmatrix} -\widehat{X} & 0 & \widehat{A}^T & \widehat{C}_1^T \\ 0 & -\gamma^2 P & \widehat{B}_1^T & D^T \\ \widehat{A} & \widehat{B}_1 & -\widehat{X}^{-1} & 0 \\ \widehat{C}_1 & D & 0 & -Q^{-1} \end{bmatrix}, \quad \widehat{R}^T = \begin{bmatrix} \widehat{C}^T \\ \widehat{D}_1^T \\ 0 \\ 0 \end{bmatrix}, \quad \widehat{L}^T = \begin{bmatrix} 0 \\ 0 \\ \widehat{B} \\ \widehat{D}_2 \end{bmatrix}.$$

Here $\widehat{X}$ is a block matrix determined in Lemma 4.4 for $X$ and $Y$ satisfying Theorem 4.2.

If $K \in \mathcal{K}_D$, then $\det \left[ I_m - KD(x) \right] \neq 0$ for all $x \in \mathcal{S}_0$, where $\mathcal{S}_0$ is some neighbourhood of the point $x = 0$, and nonlinear close loop system (1), (48) reduces to the form

$$\widehat{x}_{t+1} = \widehat{A}_*(\widehat{x}_t)\widehat{x}_t + \widehat{B}_*(\widehat{x}_t) w_t, \quad y_t = \widehat{C}_*(\widehat{x}_t)\widehat{x}_t + \widehat{D}_*(\widehat{x}_t) w_t, \tag{53}$$

where all coefficient matrices are continuous in $\mathcal{S}_0$. Therefore, the dynamic controller (48), (18) ensures robust stability of the zero state $\widehat{x}_t \equiv 0$ of system (53) with uncertainty (35) and a common Lyapunov function $v(\widehat{x}) = \widehat{x}^T \widehat{X} \widehat{x}$. To evaluate characteristics $J_0$ and $J$ of system (53), we can apply Lemma 4.2.

### 4.4 Control systems with controlled and observed outputs

Consider the linear control system

$$\begin{aligned} x_{t+1} &= Ax_t + B_1 w_t + B_2 u_t, \\ z_t &= C_1 x_t + D_{11} w_t + D_{12} u_t, \\ y_t &= C_2 x_t + D_{21} w_t + D_{22} u_t, \end{aligned} \tag{54}$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$, $w_t \in \mathbb{R}^s$, $z_t \in \mathbb{R}^k$ and $y_t \in \mathbb{R}^l$ are the state, the control, the norm-limited external perturbations, the controlled and observed outputs, respectively, and $t \in \mathcal{T}$. We are interested in static and dynamic control laws that ensure nonexpansivity property of close loop system and minimize the performance criteria $J$ and $J_0$ with respect to controlled output $z$ of the form (31), where

$$\varphi(w, x_0) = \frac{\|z\|_Q}{\sqrt{\|w\|_P^2 + x_0^T X_0 x_0}}, \quad \|z\|_Q^2 = \sum_{t=0}^{\infty} z_t^T Q z_t, \quad \|w\|_P^2 = \sum_{t=0}^{\infty} w_t^T P w_t.$$

#### 4.4.1 Static controllers

If we use the static output feedback controller

$$u_t = K y_t, \quad K \in \mathcal{K}_{D_{22}}, \quad t \in \mathcal{T}, \tag{55}$$

then closed loop system (54), (55) has the form

$$x_{t+1} = A_* x_t + B_* w_t, \quad z_t = C_* x_t + D_* w_t, \tag{56}$$

where $A_* = A + B_2K_0C_2$, $B_* = B_1 + B_2K_0D_{21}$, $C_* = C_1 + D_{12}K_0C_2$, $D_* = D_{11} + D_{12}K_0D_{21}$ and $K_0 = (I_m - KD_{22})^{-1}K$. To formulate an analog of Theorem 4.1 we construct the following LMI

$$W_R^T \begin{bmatrix} A^TXA - X + C_1^TQC_1 & A^TXB_1 + C_1^TQD_{11} \\ B_1^TXA + D_{11}^TQC_1 & B_1^TXB_1 + D_{11}^TQD_{11} - \gamma^2 P \end{bmatrix} W_R < 0, \tag{57}$$

$$W_L^T \begin{bmatrix} AYA^T - Y + B_1P^{-1}B_1^T & AYC_1^T + B_1P^{-1}D_{11}^T \\ C_1YA^T + D_{11}P^{-1}B_1^T & C_1YC_1^T + D_{11}P^{-1}D_{11}^T - \gamma^2 Q^{-1} \end{bmatrix} W_L < 0, \tag{58}$$

where $R = \begin{bmatrix} C_2, D_{21} \end{bmatrix}$, $L = \begin{bmatrix} B_2^T, D_{12}^T \end{bmatrix}$.

**Theorem 4.3** *For system* (54)*, there exists an output feedback controller* (55) *such that* $J < \gamma$ *iff the system of correlations* (44)*,* (57) *and* (58) *is feasible. Herewith, closed loop system* (56) *with a structured uncertainty*

$$w_t = \frac{1}{\gamma}\Theta z_t, \quad \Theta^T P \Theta \le Q, \quad t \in \mathcal{T}, \tag{59}$$

*is robust stable with common Lyapunov function* $v(x) = x^TXx$.

If we use a static state feedback $u_t = Kx_t$, then $C_2 = I_n$, $D_{21} = 0$ and $D_{22} = 0$. In this case the correlations (44) and (57) can be written as

$$\begin{bmatrix} X_0 & I_n \\ I_n & Y \end{bmatrix} > 0, \quad \begin{bmatrix} P - \gamma^{-2}D_{11}^TQD_{11} & B_1^T \\ B_1 & Y \end{bmatrix} > 0. \tag{60}$$

**Corollary 4.1** *For system* (54)*, there exists a state feedback controller* $u_t = Kx_t$ *such that* $J < \gamma$ *iff the LMI system* (58) *and* (60) *is solvable for some matrix* $Y = Y^T > 0$*. Herewith, closed loop system* (56) *with uncertainty* (59) *is robust stable with common Lyapunov function* $v(x) = \gamma^2 x^TY^{-1}x$.

**Remark 4.4** The gain matrix $K$ in Theorem 4.3 and Corollary 4.1 may be constructed as

$$K = K_0(I_l + D_{22}K_0)^{-1}, \quad -K_0 \in \mathcal{K}_{D_{22}}, \tag{61}$$

where $K_0$ is an arbitrary solution of LMI:

$$L_0^TK_0R_0 + R_0^TK_0^TL_0 + \Omega < 0,$$

where

$$\Omega = \begin{bmatrix} -X & 0 & A^T & C_1^T \\ 0 & -\gamma^2P & B_1^T & D_{11}^T \\ A & B_1 & -X^{-1} & 0 \\ C_1 & D_{11} & 0 & -Q^{-1} \end{bmatrix}, \quad R_0^T = \begin{bmatrix} C_2^T \\ D_{21}^T \\ 0 \\ 0 \end{bmatrix}, \quad L_0^T = \begin{bmatrix} 0 \\ 0 \\ B_2 \\ D_{12} \end{bmatrix}.$$

### 4.4.2 Dynamic controllers

If we use the dynamic output feedback

$$\xi_{t+1} = Z\xi_t + Vy_t, \quad u_t = U\xi_t + Ky_t, \quad t \in \mathcal{T}, \tag{62}$$

with $\xi_0 = 0$ and $K \in \mathcal{K}_{D_{22}}$, then closed loop system (54), (62) has the form

$$\widehat{x}_{t+1} = \widehat{A}_*\widehat{x}_t + \widehat{B}_*w_t, \quad z_t = \widehat{C}_*\widehat{x}_t + \widehat{D}_*w_t, \tag{63}$$

where

$$\widehat{x}_t = \begin{bmatrix} x_t \\ \xi_t \end{bmatrix}, \quad \widehat{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \widehat{B}_2 = \begin{bmatrix} B_2 & 0 \\ 0 & I_r \end{bmatrix}, \quad \widehat{C}_2 = \begin{bmatrix} C_2 & 0 \\ 0 & I_r \end{bmatrix},$$

$$\widehat{A}_* = \widehat{A} + \widehat{B}_2\widehat{K}_0\widehat{C}_2, \ \widehat{B}_* = \widehat{B}_1 + \widehat{B}_2\widehat{K}_0\widehat{D}_{21}, \ \widehat{C}_* = \widehat{C}_1 + \widehat{D}_{12}\widehat{K}_0\widehat{C}_2, \ \widehat{D}_* = D_{11} + \widehat{D}_{12}\widehat{K}_0\widehat{D}_{21},$$

$$\widehat{B}_1 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \ \widehat{C}_1 = [C_1, 0], \ \widehat{D}_{21} = \begin{bmatrix} D_{21} \\ 0 \end{bmatrix}, \ \widehat{D}_{12} = [D_{12}, 0], \ \widehat{K}_0 = \begin{bmatrix} K_0 & U_0 \\ V_0 & Z_0 \end{bmatrix}.$$

Here the blocks of matrix $\widehat{K}_0$

$$K_0 = (I_m - KD_{22})^{-1}K, \quad U_0 = (I_m - KD_{22})^{-1}U,$$
$$V_0 = V(I_l - D_{22}K)^{-1}, \quad Z_0 = Z + VD_{22}(I_m - KD_{22})^{-1}U,$$

are unknown, and

$$\begin{aligned} K &= (I_m + K_0D_{22})^{-1}K_0, \quad U = (I_m + K_0D_{22})^{-1}U_0, \\ V &= V_0(I_l + D_{22}K_0)^{-1}, \quad Z = Z_0 - V_0D_{22}(I_m + K_0D_{22})^{-1}U_0. \end{aligned} \tag{64}$$

Applying Lemmas 4.3, 4.4 and Theorem 4.1 to system (63), we get the following result.

**Theorem 4.4** *For linear system* (54)*, there exists a dynamic controller* (62) *such that $J < \gamma$ iff the system of correlations* (34), (51), (57) *and* (58) *is feasible. Herewith, closed loop system* (63) *with uncertainty* (59) *is robust stable.*

**Remark 4.5** The coefficient matrices of dynamic controller (62) in Theorem 4.4 may be constructed in the form (64) by solving the LMI

$$\widehat{L}^T\widehat{K}_0\widehat{R} + \widehat{R}^T\widehat{K}_0^T\widehat{L} + \widehat{\Omega} < 0, \tag{65}$$

where

$$\widehat{\Omega} = \begin{bmatrix} -\widehat{X} & 0 & \widehat{A}^T & \widehat{C}_1^T \\ 0 & -\gamma^2P & \widehat{B}_1^T & D_{11}^T \\ \widehat{A} & \widehat{B}_1 & -\widehat{X}^{-1} & 0 \\ \widehat{C}_1 & D_{11} & 0 & -Q^{-1} \end{bmatrix}, \quad \widehat{R}^T = \begin{bmatrix} \widehat{C}_2^T \\ \widehat{D}_{21}^T \\ 0 \\ 0 \end{bmatrix}, \quad \widehat{L}^T = \begin{bmatrix} 0 \\ 0 \\ \widehat{B}_2 \\ \widehat{D}_{12} \end{bmatrix}.$$

Herewith, system (63) with uncertainty (59) has common Lyapunov function $v(\widehat{x}) = \widehat{x}^T\widehat{X}\widehat{x}$. Here $\widehat{X}$ is a block matrix determined in Lemma 4.4 for $X$ and $Y$ satisfying Theorem 4.4.

We give the following algorithm for constructing stabilizing dynamic controller (62) satisfying Theorem 4.4.

**Algorithm 4.1** 1) calculate the matrices $W_R$ and $W_L$, where $R = \left[\, C_2, D_{21} \,\right]$ and $L = \left[\, B_2^T, D_{12}^T \,\right]$;
2) find the matrices $X = X^T > 0$ and $Y = Y^T > 0$ satisfying (34), (51), (57) and (58);
3) construct the expansion $Z = Y - \gamma^2 X^{-1} = S^T S$, $S \in \mathbb{R}^{r \times n}$, $\ker S = \ker Z$ and form the block matrix

$$\widehat{X} = \begin{bmatrix} X & X_1^T \\ X_1 & X_2 \end{bmatrix} > 0, \quad X_1 = \frac{1}{\gamma} SX, \quad X_2 = \frac{1}{\gamma^2} SXS^T + I_r;$$

4) solve the LMI (65) under restriction $\det(I_m + K_0 D_{22}) \neq 0$;
5) calculate the coefficient matrices of dynamic controller (62) by formula (64).

Static and dynamic output-feedback controllers (55) and (62) with $K \in \mathcal{K}_{D_{22}}$ may be applied to a class of affine systems

$$\begin{aligned} x_{t+1} &= A(x_t)x_t + B_1(x_t)w_t + B_2(x_t)u_t, \\ z_t &= C_1(x_t)x_t + D_{11}(x_t)w_t + D_{12}(x_t)u_t, \\ y_t &= C_2(x_t)x_t + D_{21}(x_t)w_t + D_{22}(x_t)u_t. \end{aligned} \tag{66}$$

So, close loop system (62), (66) reduces to the form

$$\widehat{x}_{t+1} = \widehat{A}_*(\widehat{x}_t)\widehat{x}_t + \widehat{B}_*(\widehat{x}_t)w_t, \quad z_t = \widehat{C}_*(\widehat{x}_t)\widehat{x}_t + \widehat{D}_*(\widehat{x}_t)w_t. \tag{67}$$

As a result, the dynamic controller (62), (64) ensures robust stability of the zero state $\widehat{x}_t \equiv 0$ of system (67) with uncertainty (59) and a common Lyapunov function $v(\widehat{x}) = \widehat{x}^T \widehat{X} \widehat{x}$. To evaluate characteristics $J_0$ and $J$ of system (67), we can apply Lemma 4.2.

**Remark 4.6** Note that we have necessary and sufficient conditions for an evaluation $J_0 < \gamma$ represented by the corresponding statements of Theorems 4.1 – 4.4 without using additional restriction $X < \gamma^2 X_0$. With the use of static state feedback or full order dynamic controllers the problems under consideration are reduced to the solution of LMI systems. We can formulate analogs of Theorems 4.1 – 4.4 for the corresponding control systems with uncertain coefficient matrices

$$A \in \mathrm{Co}\big\{A^1, \ldots, A^{\nu_1}\big\},\ B_1 \in \mathrm{Co}\big\{B_1^1, \ldots, B_1^{\nu_2}\big\},$$

$$C_1 \in \mathrm{Co}\big\{C_1^1, \ldots, C_1^{\nu_3}\big\},\ D_{11} \in \mathrm{Co}\big\{D_{11}^1, \ldots, D_{11}^{\nu_4}\big\}.$$

In addition, sufficient statements of these theorems may be generalized for the corresponding affine control systems with continuous coefficient matrices (see Lemma 4.2).

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