Nonlinear Dynamics and Systems Theory, **18** (1) (2018) 80–91

Homoclinic Orbits for Damped Vibration Systems with Small Forcing Terms

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Received: May 10, 2016; Revised: December 23, 2017

Abstract: We study the existence of homoclinic orbits for second order non-autonomous damped vibration system

$$\ddot{q}(t) + B\dot{q}(t) + V'(t, q(t)) = f(t),$$

where B is a skew-symmetric constant matrix, $t \in \mathbb{R}$, $q \in \mathbb{R}^N$ and $\mathrm{V} \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, $V(t, q) = -K(t, q) + W(t, q)$ is T-periodic with respect to t, $T > 0$. We assume that $W(t, q)$ satisfies an assumption weaker than the global Ambrosetti-Rabinowitz condition and that the norm of B is sufficiently small. This homoclinic orbit is obtained as a limit of $2kT$-periodic solutions of a certain sequence of second order differential equations. This result generalizes and improves some existing findings in the known literature.

Keywords: *vector field; homoclinic orbits; damped vibration systems; mountain pass theorem; critical points; minimax methods.*

Mathematics Subject Classification (2010): 34C37.

1 Introduction and Main Results

We consider the following system

$$\ddot{q}(t) + B\dot{q}(t) + V'(t, q(t)) = f(t), \qquad (DS)$$

where B is a skew-symmetric constant matrix, $V : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $(t, x) \to V(t, x)$ is a continuous function, T-periodic in the first variable with $T > 0$ and differentiable with respect to the second variable such that $V'(t, x) = \frac{\partial V}{\partial x}(t, x)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$

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and f: $\mathbb{R} \to \mathbb{R}^N$ is a continuous and bounded function. We say that a solution $x(t)$ of (DS) is a nontrivial homoclinic(to 0) if $x \not\equiv 0$ and $x(t) \to 0$ as $t \to \pm\infty$. The importance of the study of the existence of homoclinic orbits for damped vibration systems has been recognized by Poincaré at the end of the 19th century. Therefore, the existence of homoclinic orbits has become one of the most important problems in the research of damped vibration systems. Firstly, when $B \equiv 0$ and $f \equiv 0$ the system (DS) is just the following second order non-autonomous Hamiltonian system:

$$\ddot{q}(t) + V'(t, q(t)) = 0. \tag{1}$$

In 1990, Rabinowitz [14] showed the existence of homoclinic orbits for system (1) by taking the limit of $2kT$-periodic solutions of approximating problems under the well known Ambrosetti-Rabinowitz condition: there exists a constant $\mu > 2$ such that for every $t \in \mathbb{R}$ and $q \in \mathbb{R}^N \backslash \{0\}$

$$0 < \mu V(t, q) \leq V'(t, q).q.$$

By using the same approach, the existence of homoclinic orbits for the system (1) has been intensively studied by many mathematicians via variational methods in critical point theory, see([4], [5], [6], [8], [9], [13], [14], [16]) and the references therein. Particularly, in [10], Izydorek and Janczewska considered a more general Hamiltonian system

$$\ddot{q}(t) + V'(t, q(t)) \quad = \quad f(t), \tag{2}$$

where $V(t, q) = -K(t, q) + W(t, q)$. If V is neither autonomous nor periodic in t, the problem of the existence of homoclinic orbits of (1) is more complicated because the compactness arguments derived from Sobolev imbedding theorem are not available for the study of (1), see, for example, ([1], [4], [5], [6], [8], [10], [11], [15]). Secondly, if $B \not\equiv 0$, $f \not\equiv 0$ and $V = -K + W$ the existence of homoclinic orbits for system (DS) has not been previously studied. Our aim in this paper is to study the existence of homoclinic orbits for the system (DS), where K is a quadratic growth function and W satisfies an assumption weaker than the global Ambrosetti-Rabinowitz condition. Here and subsequently, $(.,.)$: $\mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ denotes the standard inner product and $|.|$ is the induced norm in $\mathbb{R}^N$.

Definition 1.1 A vector field v defined on $\mathbb{R}^N$ is called positive if $v(x).x > 0$ for all $x \in \mathbb{R}^N \backslash \{0\}$. We call v a normalized positive vector field if v is positive, linear and satisfies the following condition:

$$v(x).x = x.x, \quad \forall x \in \mathbb{R}^N. \qquad (v_1)$$

Our basic hypotheses on V and f are the following:
(V_1) There exist normalized positive vector field v and constant $b_1, b_2 > 0$ such that

$$b_1|x|^2 \leq K(t, x) \leq b_2|x|^2, \quad K(t, x) \leq K'(t, x).v(x) \leq 2K(t, x)$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$,
(V_2) $W'(t, x) = o(|x|)$ as $|x| \to 0$ uniformly in $t \in \mathbb{R}$,
(V_3) There exists a constant $\mu > 2$ such that for every $t \in \mathbb{R}$ and $x \in \mathbb{R}^N \backslash \{0\}$

$$0 < \mu W(t, x) \leq W'(t, x).v(x),$$

(V_4) $W(t,x) \leq M|x|^{\mu}$, for all $t \in \mathbb{R}$ and $|x| \leq 1$, where $M = \sup_{t\in\mathbb{R},|x|=1} W(t,x)$.

(V_5) $\bar{b}_1 = \min\{1, 2b_1\} > 2M$ and $\left(\int_{\mathbb{R}} |f(t)|^2 dt\right)^{1/2} \leq \frac{\beta}{2C}$, where $0 < \beta < \bar{b}_1 - 2M$, and C is a positive constant defined in [10].

Remark 1.1 We see that if $v(x) = x$, then (V_1) becomes (H_3) and (V_3) becomes (H_5) in [10]. From (V_2)-(V_4) we see that $W(t,x) = o(|x|^2)$ as $|x| \to 0$ uniformly in t and $W(t,0) = 0, W'(t,0) = 0$. Moreover, from (V_1) we conclude that $K(t,0) = 0$, $K'(t,0) = 0$. Example 1.1 below shows that (V_3) is weaker than the global Ambrosetti-Rabinowitz condition.

In addition, we need the following hypothesis on B.

(V_6) $\|B\| < \min\left\{\bar{b}_1 - \beta - 2M, \frac{\mu-2}{\mu+2b}\bar{b}_1, \frac{1}{b}, \frac{b_1}{b}\right\}$, where $b = \|v\|$ is the norm of the operator v.

Now, we state our existence result of homoclinic orbits for problem (DS).

Theorem 1.1 *Suppose that K and W are T-periodic with respect to t, $T > 0$ satisfying $(V_1) - (V_6)$, then the system (DS) possesses a nontrivial homoclinic solution $q \in W^{1,2}(\mathbb{R}, \mathbb{R}^N)$ such that $\dot{q}(t) \to 0$, as $t \to \pm\infty$.*

Example 1.1 Let $\theta(x)$ be the argument of $x = (\xi_1, \xi_2) \in \mathbb{R}^2\backslash\{0\}$ defined by

$$\theta(x) = \begin{cases} \arctan(\xi_2/\xi_1), if \ \ \xi_1 > 0, \xi_2 \geq 0, \\ \frac{\pi}{2}, \ \ if \ \xi_1 = 0, \xi_2 > 0, \\ \arctan(\xi_2/\xi_1) + \pi, \ \ if \ \ \xi_1 < 0, \\ \frac{3\pi}{2}, \ \ if \ \xi_1 = 0, \xi_2 < 0, \\ \arctan(\xi_2/\xi_1) + 2\pi, if \ \ \xi_1 > 0, \xi_2 < 0. \end{cases}$$

Define a function $K \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ as follows:

$$K(t,x) = \begin{cases} \frac{|x|^2}{\exp(2\sin 4(\ln|x| + \theta(x)))}, \ \ if \ x \neq 0, \\ 0, \ \ if \ x = 0. \end{cases}$$

Define a normalized positive vector field v by

$$v(x) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} x.$$

An easy computation shows that K satisfies (V_1).

For any $\mu > 2$, define a function $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ as follows:

$$W(t,x) = \begin{cases} \frac{|x|^{\mu}}{\exp(\mu(2\sin 4(\ln|x| + \theta(x))))}, \ \ if \ \ x \neq 0, \\ 0, \ \ if \ \ x = 0. \end{cases}$$

A direct computation (see [3]) shows that W satisfies (V_2), (V_3) and (V_4). Moreover, W does not satisfy the global Ambrosetti-Rabinowitz condition.

2 Variational Setting and Preliminaries

Similarly to [10] and [14], we will prove the existence of homoclinic orbits for (DS) as the limit of $2kT$-periodic solutions of the following systems of differential equations:

$$\ddot{q}(t) + B\dot{q}(t) + V'(t, q(t)) = f_k(t), \qquad (DS_k)$$

where $f_k : \mathbb{R} \to \mathbb{R}^N$ is a bounded continuous function, $2kT$-periodic extension of f to the interval $[-kT, kT]$, $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, let L^2_{2kT} be the Hilbert space of $2kT$-periodic functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ equipped with the norm

$$\|q\|_{L^2_{2kT}} = \left(\int_{-kT}^{kT} |q(t)|^2 dt \right)^{\frac{1}{2}},$$

and L^∞_{2kT} be the space of $2kT-$periodic essentially bounded functions from $\mathbb{R}$ into $\mathbb{R}^N$ equipped with the norm

$$\|q\|_{L^\infty_{2kT}} = esssup \left\{ |q(t)| : t \in [-kT, kT] \right\}.$$

Denote by $E_k := W^{1,2}_{2kT}$ the Hilbert space of $2kT$-periodic functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norm

$$\|q\|_{E_k} = \left[\int_{-kT}^{kT} |q(t)|^2 dt + \int_{-kT}^{kT} |\dot{q}(t)|^2 dt \right]^{1/2}.$$

Next, we need the following lemma.

Lemma 2.1 *([10]). There is a positive constant C such that for each $k > 0$ and $q \in E_k$ the following inequality holds:*

$$\|q\|_{L^\infty_{2kT}} \leq C \|q\|_{E_k}. \tag{3}$$

Let $\eta_k : E_k \to [0, +\infty[$ be given by

$$\eta_k(q) = \left(\int_{-kT}^{kT} [|\dot{q}(t)|^2 + 2K(t, q(t))] dt \right)^{1/2}. \tag{4}$$

By using (V_1), we have

$$\bar{b}_1 \|q\|^2_{E_k} \leq \eta^2_k(q) \leq \bar{b}_2 \|q\|^2_{E_k}, \tag{5}$$

where $\bar{b}_2 = \max\{1, 2b_2\}$. Let $I_k : E_k \to \mathbb{R}$ be the functional defined by

$$\begin{aligned} I_k(q) &= \int_{-kT}^{kT} \left[\frac{1}{2}|\dot{q}(t)|^2 + \frac{1}{2}Bq(t).\dot{q}(t) + K(t, q(t)) - W(t, q(t)) + f_k(t).q(t) \right] dt \\ &= \frac{1}{2}\eta^2_k(q) + \int_{-kT}^{kT} \left[\frac{1}{2}Bq(t).\dot{q}(t) - W(t, q(t)) + f_k(t).q(t) \right] dt. \end{aligned} \tag{6}$$

It is easy to check that $I_k \in C^1(E_k, \mathbb{R})$ and by using the skew-symmetry of B, we have for every $q, v \in E_k$

$$I'_k(q)v = \int_{-kT}^{kT} \left[\dot{q}(t).\dot{v}(t) - B\dot{q}(t).v(t) - V'(t, q(t)).v(t) + f_k(t).v(t) \right] dt. \tag{7}$$

It is known that the critical points of I_k in E_k are the classical $2kT$-periodic solution of (DS_k). We will obtain a critical point of I_k by using a standard version of the mountain pass theorem:

Lemma 2.2 *([13]). Let H be a real Banach space and $I \in C^1(H,\mathbb{R})$ satisfying the Palais-Smale condition. If I satisfies the following conditions:*
(i) $I(0) = 0$,
(ii) there exist constants ρ, $\alpha > 0$ such that $I_{|\partial B_\rho(0)} \geq \alpha$,
(iii) there exists $e \in H\backslash \overline{B}_\rho(0)$ such that $I(e) \leq 0$.
Then I possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I(g(s)),$$

where

$$\Gamma = \{g \in C([0,1],H) : g(0) = 0, g(1) = e\}.$$

Lemma 2.3 *([4]). There exist $a_1, a_2 > 0$ such that*

$$W(t,x) \geq a_1|x|^\mu - a_2,\ \forall\, t \in \mathbb{R},\ \forall x \in \mathbb{R}^N. \tag{8}$$

Let v be the normalized positive vector field in (V_1) and (V_3) of Theorem 1.1. Then v is an invertible linear operator from $\mathbb{R}^N$ to $\mathbb{R}^N$. Let $a = \frac{1}{\|v^{-1}\|}$, $b = \|v\|$, where $\|v\|$ and $\|v^{-1}\|$ are operator norms. For any $x \in \mathbb{R}^N$, one has

$$a|x| \leq |v(x)| \leq b|x|. \tag{9}$$

Define a vector field $\tilde{v}$ on E_k by

$$(\tilde{v}(x))(t) = v(x(t)). \tag{10}$$

Using condition (v_1) and Fourier series, we perform direct computation to show the following lemma.

Lemma 2.4 *([4]). For any $x \in E_k$,*

$$\int_{-kT}^{kT} |\dot{x}(t)|^2 dt = \int_{-kT}^{kT} \dot{x}(t).\overbrace{v(x(t))}^{\cdot}dt. \tag{11}$$

$$a\|x\|_{E_k} \leq \|\tilde{v}(x)\|_{E_k} \leq b\|x\|_{E_k}. \tag{12}$$

From (V_1), (7), (10) and (11) we have

$$\begin{aligned} I_k'(q).\tilde{v}(q) &\leq \eta_k^2(q) - \int_{-kT}^{kT} \left[B\dot{q}(t).v(q(t)) - W'(t,q(t)).v(q(t))\right] dt \\ &+ \int_{-kT}^{kT} f_k(t).v(q(t))dt. \end{aligned} \tag{13}$$

Lemma 2.5 *Under the assumptions (V_1)–(V_6), for every $k \in \mathbb{N}$ the system (DS_k) possesses a $2kT$-periodic solution $q_k \in E_k$.*

Proof. **Step 1.** We will show that I_k satisfies the Palais-Smale condition. Assume that $\{q_j\}_{j\in\mathbb{N}} \subset E_k$, $\{q_j\}_{j\in\mathbb{N}}$ has a convergent subsequence if $\{I_k(q_j)\}_{j\in\mathbb{N}}$ is bounded and $I_k'(q_j) \to 0$ as $j \to +\infty$. Then there exists a constant $M_k > 0$ such that

$$|I_k(q_j)| \leq M_k, \quad \|I_k'(q_j)\|_{E_k^*} \leq M_k \tag{14}$$

for every $j \in \mathbb{N}$. We firstly prove that $\{q_j\}_{j\in\mathbb{N}}$ is bounded in E_k. Without loss of generality, we may assume that $\|q_j\|_{E_k} \neq 0$. Then by (V_3) and (6), it follows that

$$\begin{aligned}\eta_k^2(q_j) &\leq 2I_k(q_j) + \int_{-kT}^{kT} B\dot{q}_j(t).q_j(t)dt + \frac{2}{\mu}\int_{-kT}^{kT} W'(t,q_j(t)).v(q_j(t))dt \\ &- 2\int_{-kT}^{kT} f_k(t).q_j(t)dt. \end{aligned}\tag{15}$$

From (13) and (15) we obtain

$$\begin{aligned}(1-\frac{2}{\mu})\eta_k^2(q_j) \leq 2I_k(q_j) - \frac{2}{\mu}I_k'(q_j).\tilde{v}(q_j(t)) + \int_{-kT}^{kT} B\dot{q}_j(t).q_j(t)dt + \frac{2}{\mu}\int_{-kT}^{kT} Bq_j(t).\dot{\overbrace{v(q_j(t))}}dt \\ -2\int_{-kT}^{kT} f_k(t).q_j(t)dt + \frac{2}{\mu}\int_{-kT}^{kT} f_k(t).v(q_j(t))dt. \end{aligned}\tag{16}$$

Moreover, by (5), (9) and (16) it follows that

$$\begin{aligned}\left[\left(1-\frac{2}{\mu}\right)\bar{b}_1 - \left(1+\frac{2b}{\mu}\right)\|B\|\right]\|q_j\|_{E_k}^2 \leq 2I_k(q_j) + \frac{2b}{\mu}\|I_k'(q_j)\|_{E_k^*}\|q_j\|_{E_k} \\ +2\left(\int_{-kT}^{kT}|f_k(t)|^2dt\right)^{\frac{1}{2}}\|q_j\|_{E_k} + \frac{2b}{\mu}\left(\int_{-kT}^{kT}|f_k(t)|^2dt\right)^{\frac{1}{2}}\|q_j\|_{E_k}. \end{aligned}\tag{17}$$

By (14), (17) and (V_5) we get

$$\left[\left(1-\frac{2}{\mu}\right)\bar{b}_1 - \left(1+\frac{2b}{\mu}\right)\|B\|\right]\|q_j\|_{E_k}^2 \leq 2M_k + \left(\frac{2bM_k}{\mu} + \frac{\beta}{C}(1+\frac{b}{\mu})\right)\|q_j\|_{E_k}. \tag{18}$$

Since $\mu > 2$ and (V_6) imply that $\left[(1-\frac{2}{\mu})\bar{b}_1 - (1+\frac{2b}{\mu})\|B\|\right] > 0$, inequality (18) shows that $\{q_j\}_{j\in\mathbb{N}}$ is bounded in E_k. Going if necessary to a subsequence, we can assume that there exists $q \in E_k$ such that $q_j \rightharpoonup q$, as $j \to +\infty$ in E_k, which implies that $q_j \to q$ as $j \to +\infty$ uniformly on $[-kT, kT]$. By Proposition 4.3 in [17], we can prove that $\{q_j\}_{j\in\mathbb{N}}$ has a convergent subsequence in E_k. Hence, I_k satisfies the Palais-Smale condition.

Step 2. We prove that there exist constants $\rho, \alpha > 0$ independent of k such that I_k satisfies the assumption (ii) of Lemma 2.2. Letting $\rho = \frac{1}{C}$ and $\|q\|_{E_k} = \rho$, we have $\|q\|_{L^\infty_{2kT}} \leq 1$, where $C > 0$ is defined in (3). It follows from (V_4) that

$$\int_{-kT}^{kT} W(t,q)dt \leq M\int_{-kT}^{kT}|q(t)|^2dt \leq M\|q\|_{E_k}^2. \tag{19}$$

In consequence, combining this with (5), (V_5) and Hölder's inequality, we obtain

$$\begin{aligned} I_k(q) &= \int_{-kT}^{kT}\left[\frac{1}{2}|\dot{q}(t)|^2+\frac{1}{2}Bq(t).\dot{q}(t)+K(t,q(t))-W(t,q(t))+f_k(t).q(t)\right]dt \\ &\geq \frac{1}{2}\bar{b}_1\|q\|_{E_k}^2-\frac{1}{2}\|B\|\|q\|_{E_k}^2-M\|q\|_{E_k}^2-\|f_k\|_{L^2_{2kT}}\|q\|_{L^2_{2kT}} \\ &\geq \left(\frac{1}{2}\bar{b}_1-\frac{1}{2}\|B\|-M\right)\|q\|_{E_k}^2-\frac{\beta}{2C}\|q\|_{E_k} \\ &\geq \frac{1}{2}\left(\bar{b}_1-\beta-2M-\|B\|\right)\|q\|_{E_k}^2+\frac{\beta}{2}\|q\|_{E_k}^2-\frac{\beta}{2C}\|q\|_{E_k}. \end{aligned} \tag{20}$$

Note that (V_6) implies $\left(\bar{b}_1-\beta-2M-\|B\|\right)>0$.

We set $\alpha=\frac{\bar{b}_1-\beta-2M-\|B\|}{2C^2}$, than the inequality (20) implies that

$$I_{k_{|\partial B_\rho}}\geq\alpha>0 \;\; for \;\; k\in\mathbb{N}.$$

Step 3. It remains to show that I_k satisfies assumption (iii) of Lemma 2.2. By (5), (6) and (8), for every $s\in\mathbb{R}\backslash\{0\}$ and $q\in\mathbb{R}^N\backslash\{0\}$, we have

$$\begin{aligned} I_k(sq) &\leq \frac{\bar{b}_2s^2}{2}\|q\|_{E_k}^2+s^2\|B\|\|q\|_{E_k}^2-a_1|s|^\mu\int_{-kT}^{kT}|q(t)|^\mu dt \\ &+ |s|\|f_k\|_{L^2_{2kT}}\|q\|_{L^2_{2kT}}+2kTa_2. \end{aligned} \tag{21}$$

Take $Q\in E_1$ such that $Q(T)=Q(-T)=0$. Since $\mu>2$ and $a_1>0$, (21) implies that there exists $s_0\in\mathbb{R}\backslash\{0\}$ such that $\|s_0Q\|_{E_1}>\rho$ and $I_1(s_0Q)<0$. Set $e_1(t)=s_0Q(t)$ and

$$e_k(t)=\begin{cases} e_1(t) & for \;\; |t|\leq T, \\ 0 & for \;\; T<|t|\leq kT, \end{cases} \tag{22}$$

for $k>0$. Then $e_k\in E_k$, $\|e_k\|_{E_k}=\|e_1\|_{E_1}>\rho$ and $I_k(e_k)=I_1(e_1)<0$ for every $k\in\mathbb{N}$.

By Lemma 2.2, I_k possesses a critical value $c_k\geq\alpha$ given by

$$c_k=\inf_{g\in\Gamma_k}\max_{s\in[0,1]}I_k(g(s)), \tag{23}$$

where

$$\Gamma_k=\left\{g\in C([0,1],E_k):g(0)=0,g(1)=e_k\right\}.$$

Hence for every $k\in\mathbb{N}$, there exists $q_k\in E_k$ such that

$$I_k(q_k)=c_k, \;\; I_k'(q_k)=0. \tag{24}$$

The function q_k is a desired classical $2kT-$periodic solution of (DS_k) for $k\in\mathbb{N}$. Since $c_k>0$, q_k is a nontrivial solution even if $f\equiv 0$. The proof of Lemma 2.5 is complete.

Lemma 2.6 *Let $(q_k)_{k\in\mathbb{N}}$ be the solution of system (DS_k) which satisfies (24) for $k\in\mathbb{N}$. Then there exists a positive constant M_1 independent of k such that*

$$\|q_k\|_{E_k}\leq M_1, \;\; \forall k\in\mathbb{N}. \tag{25}$$

Proof. For $k \in \mathbb{N}$, let $g_k : [0,1] \to E_k$ be a curve given by $g_k(s) = se_k$, where e_k is defined by (22). Then $g_k \in \Gamma_k$ and $I_k(g_k(s)) = I_1(g_1(s))$ for all $k \in \mathbb{N}$ and $s \in [0,1]$. Therefore, by (23)

$$c_k = \inf_{g \in \Gamma_k} \max_{s \in [0,1]} I_k(g(s)) \leq \max_{s \in [0,1]} I_k(g_1(s)) \equiv M_0, \ \ \forall k \in \mathbb{N}, \tag{26}$$

where M_0 is independent of k. Since $I'_k(q_k) = 0$, we get from (V_1), (V_3), (6) and (11)

$$\begin{aligned} c_k = I_k(q_k) - \frac{1}{2} I'_k(q_k).\tilde{v}(q_k) &\geq (\frac{\mu}{2} - 1) \int_{-kT}^{kT} W(t, q_k(t))dt + \frac{1}{2} \int_{-kT}^{kT} Bq_k(t).\dot{q}_k(t)dt \\ &+ \frac{1}{2} \int_{-kT}^{kT} B\dot{q}_k(t).v(q_k(t))dt + \int_{-kT}^{kT} f_k(t).q_k(t))dt - \frac{1}{2} \int_{-kT}^{kT} f_k(t).v(q_k(t))dt. \end{aligned}$$

Then we have

$$\begin{aligned} \int_{-kT}^{kT} W(t, q_k(t))dt \quad &\leq \quad \frac{1}{\mu - 2} \int_{-kT}^{kT} B\dot{q}_k(t).q_k(t)dt + \frac{1}{\mu - 2} \int_{-kT}^{kT} Bq_k.\overset{\cdot}{\overbrace{v(q_k(t))}} \\ &- \quad \frac{2}{\mu - 2} \int_{-kT}^{kT} f_k(t).q_k(t)dt + \frac{2c_k}{\mu - 2} \\ &+ \quad \frac{1}{\mu - 2} \int_{-kT}^{kT} f_k(t).v(q_k(t)dt. \end{aligned} \tag{27}$$

Combining (27) with (5), (6), (12), (26), (V_5) and (V_6) we obtain

$$\left(\frac{\bar{b}_1}{2} - \frac{1+b}{\mu - 2} \|B\| \right) \|q_k\|^2_{E_k} \leq \frac{\mu M_0}{\mu - 2} + \frac{\beta(\mu + b)}{2C(\mu - 2)} \|q_k\|_{E_k}. \tag{28}$$

Since (V_6) implies that $\frac{\bar{b}_1}{2} - \frac{1+b}{\mu-2}\|B\| > 0$ and all coefficients of (28) are independent of k, there exists a constant $M_1 > 0$ independent of k such that

$$\|q_k\|_{E_k} \leq M_1, \ \ \forall k \in \mathbb{N}. \tag{29}$$

The proof of Lemma 2.6 is complete. $\square$

Let $C^p_{loc}(\mathbb{R}, \mathbb{R}^N)$ $(p \in \mathbb{N})$ denote the space of C^p functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the topology of almost uniformly convergence on compact subintervals of $\mathbb{R}$ and all derivatives up to order p. Using the Arzelà-Ascoli theorem, we can prove the following lemma.

Lemma 2.7 *Let $\{q_k\}_{k \in \mathbb{N}}$ be the $2kT-$periodic solution of problem* (1) *which satisfies* (29) *for $k \in \mathbb{N}$. Then there exists a subsequence $\{q_{k_j}\}$ convergent to q in $C^1_{loc}(\mathbb{R}, \mathbb{R}^N)$.*

Proof. Arguing as in Theorem 2.1 in [11], we conclude from the fact

$$|q_k(t_2) - q_k(t_1)| \leq \int_{t_1}^{t_2} |\dot{q}_k(t)|dt \leq (t_2 - t_1)^{1/2} \left(\int_{t_1}^{t_2} |\dot{q}_k(t)|^2 dt \right)^{1/2}$$

that the sequence (q_k) is equicontinuous on every interval $[-lT, lT] \subset [-kT, kT]$. By (29) and Arzelà-Ascoli theorem, the sequence (q_k) has a uniformly convergent subsequence on each $[-lT, lT]$.

Let $(q^1_{k_m})$ be a subsequence of (q_k) that converges on $[-T, T]$. Then $(q^1_{k_m})$ is equicontinuous and uniformly bounded on $[-2T, 2T]$. So we can choose a subsequence $(q^2_{k_m})$ of $(q^1_{k_m})$ that converges uniformly on $[-2T, 2T]$. Repeat this procedure for all k and take the diagonal sequence $(q^m_{k_m})$. It is obvious that $(q^m_{k_m})_m$ is a subsequence of $(q^i_{k_m})$ for any $1 \le i \le m$. Hence, it converges uniformly to a function $q(t)$ on any bounded interval. In the following, for simplicity, we also denote the subsequence $(q^m_{k_m})$ by (q_k). The proof of Lemma 2.7 is complete. □

Lemma 2.8 *Let $q : \mathbb{R} \to \mathbb{R}^N$ be the function given in Lemma 2.7. Then q is the desired nontrivial homoclinic solution of (DS) such that $\dot{u}(t) \to 0$, as $t \to \pm\infty$.*

Proof. Firstly, we will show that q is a solution of (1). Let $\{q_{k_j}\}_{k\in\mathbb{N}}$ be defined in Lemma 2.7, then we have

$$\ddot{q}_{k_j}(t) + B\dot{q}_{k_j}(t) + V'(t, q_{k_j}(t)) \quad = \quad f_{k_j}(t) \tag{30}$$

for every $j \in \mathbb{N}$ and $t \in [-k_jT, k_jT]$. Take $a, b \in \mathbb{R}$ such that $a < b$. There exists $j_0 \in \mathbb{N}$ such that for all $j > j_0$ and $t \in [a, b] \subset [-k_jT, k_jT]$, we have

$$\ddot{q}_{k_j}(t) \quad = \quad -B\dot{q}_{k_j}(t) - V'(t, q_{k_j}(t)) + f_{k_j}(t). \tag{31}$$

Hence, $\ddot{q}_{k_j}(t)$ is continuous in $[a, b]$ and $\ddot{q}_{k_j}(t)$ is a classical derivative of $\dot{q}_{k_j}(t)$ in $[a, b]$ for every $j > j_0$. Moreover, since $\dot{q}_{k_j} \to \dot{q}$ uniformly on $[a, b]$ and

$$\ddot{q}_{k_j}(t) \quad = \quad -B\dot{q}_{k_j}(t) - V'(t, q_{k_j}(t)) + f_{k_j}(t) \tag{32}$$

we obtain

$$\ddot{q}(t) + B\dot{q}(t) + V'(t, q(t)) \quad = \quad f(t), \tag{33}$$

for every $t \in [a, b]$. Since a and b are arbitrary, we conclude that q satisfies (DS). □

3 Proof of Theorem 1.1.

We have shown that q satisfies (1). It remains to prove that q is nontrivial and homoclinic to 0. First, we show that q is nontrivial. Obviously, this will be the case if $f \not\equiv 0$. Consider the function $\varphi : [0, +\infty[\to [0, +\infty[$ defined by

$$\varphi(s) = \begin{cases} \displaystyle\max_{t\in\mathbb{R}, 0<|x|\le s} \frac{W'(t, x).v(x)}{|x|^2}, & s > 0, \\ 0, & s = 0. \end{cases}$$

Then by (V_2), (V_3), (8) and (9) φ is a continuous, nondecreasing function and $\varphi(s) \ge 0$ for $s \ge 0$. The definition of φ implies that

$$\int_{-kT}^{kT} W'(t, q_k(t)).v(q_k(t))dt \le \varphi(\|q_k\|_{L^\infty_{2kT}})\|q_k\|^2_{E_k} \tag{34}$$

for every $n \in \mathbb{N}$. Since $I'_k(q_k).v(q_k) = 0$, we have

$$\int_{-kT}^{kT} W'(t, q_k(t)).v(q_k(t))dt =$$

$$\int_{-kT}^{kT} |\dot{q}_k(t)|^2 dt - \int_{-kT}^{kT} B\dot{q}_k(t).v(q_k(t))dt + \int_{-kT}^{kT} K'(t,q_k(t)).v(q_k(t))dt. \tag{35}$$

From (34), (35), (V_1) and (V_6), we obtain

$$\begin{aligned}\varphi(\|q_k\|_{L^\infty_{2kT}})\|q_k\|^2_{E_k} &\geq \int_{-kT}^{kT}|\dot{q}_k(t)|^2dt - \int_{-kT}^{kT}B\dot{q}_k(t).v(q_k(t)))dt + \int_{-kT}^{kT}K'(t,q_k(t))v(q_k(t))dt \\ &\geq (\min\{1,b_1\} - b\|B\|)\,\|q_k\|^2_{E_k}.\end{aligned}$$

Since $\|q_k\|_{E_k} > 0$, it follows that

$$\varphi(\|q_k\|_{L^\infty_{2kT}}) \geq (\min\{1,b_1\} - b\|B\|) > 0.$$

If $\|q_k\|_{L^\infty_{2kT}} \to 0$ as $k \to \infty$, we have $\varphi(0) \geq (\min\{1,b_1\} - b\|B\|) > 0$, which is a contradiction. Passing to a subsequence of (q_k) if necessary, we see that there is a constant $C_1 > 0$ such that

$$\|q_k\|_{L^\infty_{2kT}} \geq C_1 \tag{36}$$

for every $k \in \mathbb{N}$. Moreover, for all $j \in \mathbb{N}$, $t \mapsto q_k^j(t) = q_k(t+jT)$ is also a $2kT$-periodic solution of system (3). Hence, if the maximum of $|q_k|$ occurs in $\theta_k \in [-kT,kT]$ then the maximum of $|q_k^j|$ occurs in $\tau_k^j = \theta_k - jT$. Then there exists a $j_k \in \mathbb{Z}$ such that $\tau_k^{j_k} \in [-T,T]$. Consequently,

$$\|q_k^{j_k}\|_{L^\infty([-kT,kT],\mathbb{R}^N)} = \max_{t\in[-T,T]} |q_k^{j_k}(t)|.$$

Suppose the contrary to our claim, that $q \equiv 0$. Then

$$\|q_k^{j_k}\|_{L^\infty([-kT,kT],\mathbb{R}^N)} = \max_{t\in[-T,T]} |q_k^{j_k}(t)| \to 0,$$

which contradicts (36).

Second, we now prove that $q(t) \to 0$ as $t \to \pm\infty$. We have, from (29)

$$\int_{-kT}^{kT}(|q_k(t)|^2 + |\dot{q}_k(t)|^2)dt \leq \|q_k\|^2_{E_k} \leq M_1^2.$$

Obviously, for each $i \in \mathbb{N}$ there is $k_i \in \mathbb{N}$ such that for all $k \geq k_i$

$$\int_{-iT}^{iT}(|q_k(t)|^2 + |\dot{q}_k(t)|^2)dt \leq \|q_k\|^2_{E_k} \leq M_1^2.$$

Letting $k \to +\infty$, we obtain

$$\int_{-iT}^{iT}(|q(t)|^2 + |\dot{q}(t)|^2)dt \leq M_1^2.$$

As $i \to +\infty$, we have

$$\int_{-\infty}^{+\infty}(|q(t)|^2 + |\dot{q}(t)|^2)dt \leq M_1^2.$$

Hence we get

$$\int_{|t|\geq r}(|q(t)|^2+|\dot{q}(t)|^2)dt \to 0 \text{ as } r\to+\infty. \tag{37}$$

By Corollary 2.2 in [16], we have

$$|q(t)|^2 \leq \int_{t-1}^{t+1}(|q(s)|^2+|\dot{q}(s)|^2)ds \tag{38}$$

for every $t\in\mathbb{R}$. Then, by (37) and (38) we conclude that

$$q(t)\to 0\, as\; |t|\to\infty.$$

Finally, we have to show that $\dot{q}(t)\to 0$ as $t\to\pm\infty$. From Corollary 2.2 in [16] we have

$$|\dot{q}(t)|^2 \leq \int_{t-1}^{t+1}(|q(s)|^2+|\dot{q}(s)|^2)ds+\int_{t-1}^{t+1}|\ddot{q}(s)|^2ds,$$

for every $t\in\mathbb{R}$. Since $q\in W^{1,2}(\mathbb{R},\mathbb{R}^N)$, we get

$$\int_{t-1}^{t+1}(|q(s)|^2+|\dot{q}(s)|^2)ds \to 0 \text{ as } |t|\to\infty.$$

Hence, it suffices to prove that

$$\int_{t-1}^{t+1}|\ddot{q}(s)|^2ds \to 0 \text{ as } |t|\to\infty. \tag{39}$$

Since q is a solution of (DS), we obtain

$$\begin{aligned}
\int_{t-1}^{t+1}|\ddot{q}(s)|^2ds &\leq \|B\|^2\int_{t-1}^{t+1}|\dot{q}(s)|^2ds+\int_{t-1}^{t+1}|V'(t,q(s))|^2ds+\int_{t-1}^{t+1}|f(s)|^2ds\\
&+ 2\|B\|\left(\int_{t-1}^{t+1}|\dot{q}(s)|^2ds\right)^{\frac{1}{2}}\left(\int_{t-1}^{t+1}|V'(s,q(s))|^2ds\right)^{\frac{1}{2}}\\
&+ 2\|B\|\left(\int_{t-1}^{t+1}|\dot{q}(s)|^2ds\right)^{\frac{1}{2}}\left(\int_{t-1}^{t+1}|f(s)|^2ds\right)^{\frac{1}{2}}\\
&+ 2\left(\int_{t-1}^{t+1}|f(s)|^2\right)^{\frac{1}{2}}\left(\int_{t-1}^{t+1}|V'(s,q(s))|^2ds\right)^{\frac{1}{2}}ds.
\end{aligned}$$

By (V_5), we get

$$\int_{t-1}^{t+1}|f(s)|^2ds\to 0,\;\; as\; |t|\to\infty. \tag{40}$$

Since $\int_{t-1}^{t+1}|\dot{q}(s)|^2ds\to 0$ as $|t|\to\infty$, $q(t)\to 0$ as $|t|\to\infty$ and $V'(t,q)\to 0$ as $|q|\to 0$ uniformly in $t\in\mathbb{R}$, then (39) follows. The proof of Theorem 1.1 is complete. □

Acknowledgment

I wish to thank the anonymous referee for his/her suggestions and interesting remarks.

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