Nonlinear Dynamics and Systems Theory, **18** (1) (2018) 67–79

Weak Heteroclinic Solutions of Discrete Nonlinear Problems of Kirchhoff Type with Variable Exponents

A. Guiro [1], I. Ibrango [1] and S. Ouaro [2*]

[1] *Aboudramane Guiro, LAboratoire de Mathématiques et Informatique (LAMI), UFR, Sciences et Techniques, Université NAZI BONI, 01 BP 1091 Bobo-Dioulasso 01, Bobo Dioulasso, Burkina Faso*

[2] *Stanislas OUARO, LAboratoire de Mathématiques et Informatique (LAMI), UFR, Sciences Exactes et Appliquées, Université Ouaga I Pr Joseph KI-ZERBO, 03 BP 7021 Ouaga 03, Ouagadougou, Burkina Faso*

Received: November 21, 2016; Revised: December 17, 2017

Abstract: We prove the existence of weak heteroclinic solutions for discrete nonlinear problems of Kirchhoff type. The proof of the main result is based on a minimization method.

Keywords: *nonlinear difference equation, heteroclinic solution, anisotropic problems, Kirchhoff, critical points.*

Mathematics Subject Classification (2010): 47A75; 35B38; 35P30; 34L05; 34L30.

1 Introduction

In this paper, we study the following nonlinear discrete anisotropic problem

$$\begin{cases} -M(A(k-1,\Delta u(k-1)))\Delta(a(k-1,\Delta u(k-1))) \\ +\alpha(k)|u(k)|^{p(k)-2}u(k) = \delta(k)f(k,u(k)),\ k \in \mathbb{Z}^*, \\ u(0)=0, \quad \lim_{k\to-\infty} u(k) = -1, \quad \lim_{k\to+\infty} u(k) = 1, \end{cases} \tag{1}$$

where $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator, $\mathbb{Z}^* = \{k \in \mathbb{Z} : k \neq 0\}$ and $M, a, \alpha, \delta, f, p$ are functions to be defined later.

* Corresponding author: mailto:ouaro@yahoo.fr

Note that difference equations can be seen as a discrete counterpart of partial differential equations and are usually studied in connection with numerical analysis. In this way, the main operator in problem (1)

$$\Delta(a(k-1,\Delta u(k-1)))$$

can be seen as a discrete counterpart of the anisotropic operator

$$\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a\left(x,\frac{\partial}{\partial x_i}u\right).$$

The first study in this direction for constant exponents was done by Cabada et *al.* [2] and for variable exponent by Mihailescu et *al.* [8] (see also [6]). In [6], the authors studied the following problem

$$\begin{cases}-\Delta\big(a\big(k-1,\Delta u(k-1)\big)\big)+\alpha(k)g\big(k,u(k)\big)=\delta(k)f\big(k,u(k)\big), \quad k\in\mathbb{Z}^*,\\ u(0)=0,\ \lim\limits_{k\to-\infty}u(k)=-1,\ \lim\limits_{k\to+\infty}u(k)=1,\end{cases} \tag{2}$$

where

$$g(k,\xi)=|\xi-1|^{p(k)-2}(\xi-1)\chi_{\mathbb{Z}^+}(k)+|\xi+1|^{p(k)-2}(\xi+1)\chi_{\mathbb{Z}^-}(k).$$

The authors in [6] proved an existence result of weak heteroclinic solutions of problem (2).

In this paper, we consider the same boundary conditions as in [6], but the function

$$M(A(k-1,\Delta u(k-1)))$$

which appears in the left-hand side of problem (1) is more general than the one which appears in [6]. Indeed, if we take $M(t)=1$ in the problem (1), we obtain the probem studied by Guiro et *al* in [6].

To prove an existence result of problem (1), we define other new spaces and new associated norms and we adapt the classical minimization methods used for the study of anisotropic PDEs. The idea is to transfer the problem of the existence of solutions for (1) into the problem of the existence of a minimizer for some associated energy functional.

The study of heteroclinic connections for boundary value problems got a certain impulse in recent years, motivated by applications in various biological, physical and chemical models, such as phase-transition, physical processes in which the variable transits from an unstable equilibrium to a stable one, or front-propagation in reaction-diffusion equations. Indeed, heteroclinic solutions are often called transitional solutions (see [3,7]). Problem (1) involves variable exponents due to their use in image restoration (see [4]), in electrorheological and thermorheological fluids dynamic (see [5,9,10]).

The remaining part of this paper is organized as follows: Section 2 is devoted to mathematical preliminaries. The main existence result is stated and proved in Section 3.

2 Preliminaries and Assumptions

We use the notations

$$p^+=\sup_{k\in\mathbb{Z}}p(k), \qquad p^-=\inf_{k\in\mathbb{Z}}p(k)$$

and we set

$$\mathbb{Z}^+ := \{k \in \mathbb{Z} : k \geq 0\}; \qquad \mathbb{Z}^- := \{k \in \mathbb{Z} : k \leq 0\};$$
$$\mathbb{Z}^+_* := \{k \in \mathbb{Z} : k > 0\}; \qquad \mathbb{Z}^-_* := \{k \in \mathbb{Z} : k < 0\}.$$

In order to present the main result, for each $p(.) : \mathbb{Z} \to (0, +\infty)$ and $\beta \geq 1$, we introduce the following spaces:

$$\mathcal{L}^1 := \Big\{ u : \mathbb{Z} \to \mathbb{R}; \ \ \sum_{k\in\mathbb{Z}} |u(k)| < +\infty \Big\},$$

$$\mathcal{L}^\infty := \Big\{ u : \mathbb{Z} \to \mathbb{R}; \ \ \sup_{k\in\mathbb{Z}} |u(k)| < +\infty \Big\},$$

$$\mathcal{L}_0^{p(.)} := \Big\{ u : \mathbb{Z} \to \mathbb{R}; \ \ u(0) = 0 \text{ and } \ \rho_{p(.)}(u) := \sum_{k\in\mathbb{Z}} |u(k)|^{p(k)} < +\infty \Big\},$$

$$\mathcal{L}_{0,+}^{p(.)} := \Big\{ u : \mathbb{Z} \to \mathbb{R}; \ \ u(0) = 0 \text{ and } \ \rho_{p_+(.)}(u) := \sum_{k\in\mathbb{Z}^+} |u(k)|^{p(k)} < +\infty \Big\},$$

$$\mathcal{L}_{0,-}^{p(.)} := \Big\{ u : \mathbb{Z} \to \mathbb{R}; \ \ u(0) = 0 \text{ and } \ \rho_{p_-(.)}(u) := \sum_{k\in\mathbb{Z}^-} |u(k)|^{p(k)} < +\infty \Big\},$$

$$\mathcal{L}_{0,\alpha(.)}^{p(.)} := \Big\{ u : \mathbb{Z} \to \mathbb{R}; \ \ u(0) = 0 \text{ and } \ \rho_{\alpha(.),p(.)}(u) := \sum_{k\in\mathbb{Z}} \alpha(k)|u(k)|^{p(k)} < +\infty \Big\},$$

$$\mathcal{L}_{0,+,\alpha(.)}^{p(.)} := \Big\{ u : \mathbb{Z} \to \mathbb{R}; \ \ u(0) = 0 \text{ and } \ \rho_{\alpha(.),p_+(.)}(u) := \sum_{k\in\mathbb{Z}^+} \alpha(k)|u(k)|^{p(k)} < +\infty \Big\},$$

$$\mathcal{L}_{0,-,\alpha(.)}^{p(.)} := \Big\{ u : \mathbb{Z} \to \mathbb{R}; \ \ u(0) = 0 \text{ and } \ \rho_{\alpha(.),p_-(.)}(u) := \sum_{k\in\mathbb{Z}^-} \alpha(k)|u(k)|^{p(k)} < +\infty \Big\},$$

$$\begin{aligned} \mathcal{W}_{0,\alpha(.)}^{1,p(.)} := \Big\{ u : \mathbb{Z} \to \mathbb{R}; \ u(0) = 0, \ \rho_{1,\alpha(.),p(.)}(u) \ &:= \ \sum_{k\in\mathbb{Z}} \alpha(k)|u(k)|^{p(k)} \\ &+ \ \Big(\sum_{k\in\mathbb{Z}} |\Delta u(k)|^{p(k)} \Big)^{\beta} < +\infty \Big\}, \end{aligned}$$

$$\begin{aligned} \mathcal{W}_{0,+,\alpha(.)}^{1,p(.)} := \Big\{ u : \mathbb{Z} \to \mathbb{R}; \ u(0) = 0, \ \rho_{1,\alpha(.),p_+(.)}(u) \ &:= \ \sum_{k\in\mathbb{Z}^+} \alpha(k)|u(k)|^{p(k)} \\ &+ \ \Big(\sum_{k\in\mathbb{Z}^+} |\Delta u(k)|^{p(k)} \Big)^{\beta} < +\infty \Big\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{W}_{0,-,\alpha(.)}^{1,p(.)} := \Big\{ u : \mathbb{Z} \to \mathbb{R}; \ u(0) = 0, \ \rho_{1,\alpha(.),p_-(.)}(u) \ &:= \ \sum_{k\in\mathbb{Z}^-} \alpha(k)|u(k)|^{p(k)} \\ &+ \ \Big(\sum_{k\in\mathbb{Z}^-} |\Delta u(k)|^{p(k)} \Big)^{\beta} < +\infty \Big\}. \end{aligned}$$

For the data $a,\ f,\ \alpha$ and δ, we assume the following.

(H_1) : $\begin{cases} a(k,.) : \mathbb{R} \to \mathbb{R}, \quad k \in \mathbb{Z} \text{ and there exists a mapping } \ A(.,.) : \mathbb{Z} \times \mathbb{R} \to \mathbb{R} \text{ which} \\ \text{satisfies } \ a(k,\xi) = \frac{\partial}{\partial \xi} A(k,\xi) \ \text{ and } \ A(k,0) = 0, \quad \text{for all } k \in \mathbb{Z}. \end{cases}$

$(H_2) : |\xi|^{p(k)} \leq a(k,\xi)\xi \leq p(k)A(k,\xi), \quad \text{for all } k \in \mathbb{Z} \ \text{ and } \ \xi \in \mathbb{R}.$

(H_3): There exists a positive constant C_1 such that $|a(k,\xi)| \leq C_1(j(k) + |\xi|^{p(k)-1})$, for all

$k \in \mathbb{Z}$ and $\xi \in \mathbb{R}$, with $j \in \mathcal{L}^{q(.)}_{0,\alpha(.)}$, where $\frac{1}{p(k)} + \frac{1}{q(k)} = 1$.

(H_4) : $\big(a_i(x,\xi) - a_i(x,\eta)\big)(\xi - \eta) > 0$ for all $k \in \mathbb{Z}$ and $\xi, \eta \in \mathbb{R}$ such that $\xi \neq \eta$.

(H_5) : $f : \mathbb{Z} \times \mathbb{R} \longrightarrow \mathbb{R}$ and there exists a constant $C_2 > 0$ such that

$$|f(k,t)| \leq C_2\big(1 + |t-1|^{p(k)-1}\big)\chi_{\mathbb{Z}^+}(k) + C_2\big(1 + |t+1|^{p(k)-1}\big)\chi_{\mathbb{Z}^-_*}(k),$$

for all $k \in \mathbb{Z},\ t \in \mathbb{R}$, where $\chi_A(k) = 1$ if $k \in A$ and $\chi_A(k) = 0$ if $k \notin A$.

Assumption (H_5) implies that

$$\begin{cases} |f(k,t+1)| \leq C_2(1 + |t|^{p(k)-1}) & \text{if } \ k \geq 0, \\ |f(k,t-1)| \leq C_2(1 + |t|^{p(k)-1}) & \text{if } \ k < 0. \end{cases} \tag{3}$$

So by denoting

$$F(k,t) = \int_0^t f(k,s)ds \quad \text{for } \ k \in \mathbb{Z},\ t \in \mathbb{R},$$

we deduce that there exists a positive constant $C_3 > 1$ such that

$$\begin{cases} |F(k,t+1)| \leq C_3(1 + |t|^{p(k)}) & \text{if } \ k \geq 0, \\ |F(k,t-1)| \leq C_3(1 + |t|^{p(k)}) & \text{if } \ k < 0. \end{cases} \tag{4}$$

(H_6) : $\begin{cases} \alpha : \mathbb{Z} \to \mathbb{R} \text{ and } \delta : \mathbb{Z} \to \mathbb{R} \text{ are such that } \alpha(k) \geq \alpha_0 > 0 \ \text{ for all } k \in \mathbb{Z}, \\ 0 < \delta(k) \leq \bar{\delta} = \sup\limits_{k \in \mathbb{Z}} |\delta(k)| < +\infty \text{ and } \delta \in \mathcal{L}^1. \end{cases}$

$(H_7) : \alpha_0 > \bar{\delta} p^+ C_3.$

This condition means that the parameter $\alpha(.)$ should be bigger than the parameter $\bar{\delta}$ and is called the competition phenomenon between $\alpha(.)$ and $\delta(.)$.

We also assume that

$(H_8) : p : \mathbb{Z} \longrightarrow (1, +\infty)$ with $1 < p^- \leq p^+ < +\infty$.

$(H_9) : M : (0,+\infty) \longrightarrow (0,+\infty)$ is continuous, nondecreasing and there exist three positive real numbers $B_1,\ B_2,\ \beta$ with $B_1 \leq B_2$, and $\beta \geq 1$ such that

$$B_1 t^{\beta-1} \leq M(t) \leq B_2 t^{\beta-1}, \quad \text{for all} \quad t > 0.$$

Example 2.1 We can give the following functions which satisfy assumptions (H_1) – (H_4):

- $A(k,\xi) = \dfrac{1}{p(k)}|\xi|^{p(k)}$, where $a(k,\xi) = |\xi|^{p(k)-2}\xi$, $\forall k \in \mathbb{Z}$ and $\xi \in \mathbb{R}$.
- $A(k,\xi) = \dfrac{1}{p(k)}\Big(\big(1+|\xi|^2\big)^{p(k)/2} - 1\Big)$, where $a(k,\xi) = \big(1+|\xi|^2\big)^{(p(k)-2)/2}\xi$, $\forall\, k \in \mathbb{Z}$ and $\xi \in \mathbb{R}$.

We introduce on $\mathcal{L}^{p(.)}_{0,+}$ and $\mathcal{L}^{p(.)}_{0,+,\alpha(.)}$ the Luxemburg norms

$$||u||_{p+(.)} := \inf\Big\{\lambda > 0; \quad \sum_{k\in\mathbb{Z}^+}\Big|\frac{u(k)}{\lambda}\Big|^{p(k)} \leq 1\Big\},$$

$$||u||_{\alpha(.),p+(.)} := \inf\Big\{\lambda > 0; \quad \sum_{k\in\mathbb{Z}^+}\alpha(k)\Big|\frac{u(k)}{\lambda}\Big|^{p(k)} \leq 1\Big\}$$

and we define, on the space $\mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$, the norm

$$||u||_{1,\alpha(.),p+(.)} := \inf\Big\{\lambda > 0; \ \sum_{k\in\mathbb{Z}^+}\alpha(k)\Big|\frac{u(k)}{\lambda}\Big|^{p(k)} + \Big(\sum_{k\in\mathbb{Z}^+}\Big|\frac{\Delta u(k)}{\lambda}\Big|^{p(k)}\Big)^{\beta} \leq 1\Big\}.$$

We replace $\mathbb{Z}^+$ by $\mathbb{Z}^-$ to get the norms on $\mathcal{L}^{p(.)}_{0,-}$, $\mathcal{L}^{p(.)}_{0,-,\alpha(.)}$ and $\mathcal{W}^{1,p(.)}_{0,-,\alpha(.)}$.

Remark 2.1 We have the following:

$$\mathcal{L}^{p(.)}_{0,+,\alpha(.)} \supset \mathcal{L}^{p(.)}_{0,\alpha(.)}, \quad \mathcal{L}^{p(.)}_{0,-,\alpha(.)} \supset \mathcal{L}^{p(.)}_{0,\alpha(.)}, \quad \mathcal{W}^{1,p(.)}_{0,+,\alpha(.)} \supset \mathcal{W}^{1,p(.)}_{0,\alpha(.)} \ \text{ and } \ \mathcal{W}^{1,p(.)}_{0,-,\alpha(.)} \supset \mathcal{W}^{1,p(.)}_{0,\alpha(.)}.$$

Indeed, $\alpha(k)|u(k)|^{p(k)}$ *is nonnegative for all* $k \in \mathbb{Z}$*. Therefore, if* $\sum_{k\in\mathbb{Z}}\alpha(k)|u(k)|^{p(k)} < +\infty$*, then* $\sum_{k\in\mathbb{Z}^+}\alpha(k)|u(k)|^{p(k)} < +\infty$.

In the sequel, we will use the following result.

Proposition 2.1 *([6], Proposition 2.5). If* $u \in \mathcal{L}^{p(.)}_{0,+,\alpha(.)}$ *and* $p^+ < \infty$*, then the following properties hold:*

1. $||u||_{\alpha(.),p+(.)} < 1 \implies ||u||^{p+}_{\alpha(.),p+(.)} \leq \rho_{\alpha(.),p+(.)}(u) \leq ||u||^{p^-}_{\alpha(.),p+(.)}$;
2. $||u||_{\alpha(.),p+(.)} > 1 \implies ||u||^{p-}_{\alpha(.),p+(.)} \leq \rho_{\alpha(.),p+(.)}(u) \leq ||u||^{p^+}_{\alpha(.),p+(.)}$;
3. $||u||_{\alpha(.),p+(.)} < 1\ (= 1; > 1) \iff \rho_{\alpha(.),p+(.)}(u) < 1\ (= 1; > 1)$;
4. $||u||_{\alpha(.),p+(.)} \longrightarrow 0 \iff \rho_{\alpha(.),p+(.)}(u) \longrightarrow 0$.

Lemma 2.1 *([6], Lemma 2.8)(discrete Hölder type inequality). Let* $u \in \mathcal{L}^{p(.)}_{0,+,\alpha(.)}$ *and* $v \in \mathcal{L}^{q(.)}_{0,+,\alpha(.)}$ *with* $\frac{1}{p(k)} + \frac{1}{q(k)}$ *for any* k *in* $\mathbb{Z}$*. Then*

$$\sum_{k\in\mathbb{Z}^+}|uv| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)||u||_{\alpha(.),p+(.)}||v||_{\alpha(.),q+(.)}. \tag{5}$$

As in [6], we have the following results.

Proposition 2.2

1. $\rho_{1,\alpha(.),p_+(.)}(u+v) \leq 2^{\beta p^+-1}\big(\rho_{1,\alpha(.),p_+(.)}(u) + \rho_{1,\alpha(.),p_+(.)}(v)\big), \quad \forall\ u,\ v \in \mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$.

2. *Let* $u \in \mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$. *Then:*

   i) if $\lambda > 1$, *we have*

$$\lambda^{p^-}\rho_{1,\alpha(.),p_+(.)}(u) \leq \rho_{1,\alpha(.),p_+(.)}(\lambda u) \leq \lambda^{\beta p^+}\rho_{1,\alpha(.),p_+(.)}(u); \tag{6}$$

   ii) if $0 < \lambda < 1$, *we have*

$$\lambda^{\beta p^+}\rho_{1,\alpha(.),p_+(.)}(u) \leq \rho_{1,\alpha(.),p_+(.)}(\lambda u) \leq \lambda^{p^-}\rho_{1,\alpha(.),p_+(.)}(u). \tag{7}$$

Theorem 2.1 *Let* $u \in \mathcal{W}^{1,p(.)}_{0,+,\alpha(.)} \setminus \{0\}$. *Then*

$$||u||_{1,\alpha(.),p_+(.)} = a \quad \text{if and only if} \quad \rho_{1,\alpha(.),p_+(.)}(u/a) = 1.$$

Proposition 2.3 *If* $u \in \mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$ *and* $p^+ < \infty$, *then the following properties hold:*

1. $||u||_{1,\alpha(.),p_+(.)} < 1 \implies ||u||^{\beta p^+}_{1,\alpha(.),p_+(.)} \leq \rho_{1,\alpha(.),p_+(.)}(u) \leq ||u||^{p^-}_{1,\alpha(.),p_+(.)}$;
2. $||u||_{1,\alpha(.),p_+(.)} > 1 \implies ||u||^{p-}_{1,\alpha(.),p_+(.)} \leq \rho_{1,\alpha(.),p_+(.)}(u) \leq ||u||^{\beta p^+}_{1,\alpha(.),p_+(.)}$;
3. $||u||_{1,\alpha(.),p_+(.)} < 1\ (=1; >1) \iff \rho_{1,\alpha(.),p_+(.)}(u) < 1\ (=1; >1)$;
4. $||u||_{1,\alpha(.),p_+(.)} \longrightarrow 0 \iff \rho_{1,\alpha(.),p_+(.)}(u) \longrightarrow 0$.

3 Existence of Weak Heteroclinic Solutions

In this section we investigate the existence of weak heteroclinic solutions of problem (1) in the following sense.

Definition 3.1 A weak heteroclinic solution of problem (1) is a function $u \in \mathcal{W}^{1,p(.)}_{0,\alpha(.)}$ such that

$$\begin{cases} M\Big(\displaystyle\sum_{k\in\mathbb{Z}} A\big(k-1,\Delta u(k-1)\big)\Big)\displaystyle\sum_{k\in\mathbb{Z}} a(k-1,\Delta u(k-1))\Delta v(k-1) \\ \\ +\displaystyle\sum_{k\in\mathbb{Z}}\alpha(k)|u(k)|^{p(k)-2}u(k)v(k) = \displaystyle\sum_{k\in\mathbb{Z}}\delta(k)f\big(k,u(k)\big)v(k), \end{cases} \tag{8}$$

for any $v \in \mathcal{W}^{1,p(.)}_{0,\alpha(.)}$, with $u(0) = 0$, $\lim\limits_{k\to-\infty} u(k) = -1$ and $\lim\limits_{k\to+\infty} u(k) = 1$.

The main result is the following.

Theorem 3.1 *Assume that assumptions* (H_1)-(H_9) *hold true. Then, there exists at least one weak heteroclinic solution of problem (1).*

Proof. We first consider the following problem

$$\begin{cases} -M(A(k-1,\Delta u(k-1)))\Delta(a(k-1,\Delta u(k-1))) \\ +\alpha(k)|u(k)|^{p(k)-2}u(k)=\delta(k)f\big(k,u(k)+1\big),\ k\in\mathbb{Z}^{+}_{*}, \\ u(0)=0, \quad \lim\limits_{k\to+\infty}u(k)=0. \end{cases} \tag{9}$$

Definition 3.2 A weak solution of problem (9) is a function $u\in\mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$ such that

$$\begin{cases} M\Big(\sum\limits_{k=1}^{+\infty}A\big(k-1,\Delta u(k-1)\big)\Big)\sum\limits_{k=1}^{+\infty}a(k-1,\Delta u(k-1))\Delta v(k-1) \\ +\sum\limits_{k=1}^{+\infty}\alpha(k)|u(k)|^{p(k)-2}u(k)v(k)=\sum\limits_{k=1}^{+\infty}\delta(k)f\big(k,u(k)+1\big)v(k), \end{cases} \tag{10}$$

for any $v\in\mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$.

Theorem 3.2 *Assume that hypotheses* (H_1)*-*(H_9) *hold. Then, there exists at least one weak solution of problem (9).*

To prove Theorem 3.2, we consider the energy functional corresponding to problem (9) defined by $J:\mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}\longrightarrow\mathbb{R}$ such that

$$J(u)=\widehat{M}\Big(\sum_{k=1}^{+\infty}A\big(k-1,\Delta u(k-1)\big)\Big)+\sum_{k=1}^{+\infty}\frac{\alpha(k)}{p(k)}|u(k)|^{p(k)}-\sum_{k=1}^{+\infty}\delta(k)F\big(k,u(k)+1\big), \tag{11}$$

where $\widehat{M}(t)=\displaystyle\int_0^t M(s)\,ds$ and we present some basic properties of the functional J.

Proposition 3.1 *The functional* J *is well defined on* $\mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$ *and is of class* $C^1\big(\mathcal{W}^{1,p(.)}_{0,+,\alpha(.)},\mathbb{R}\big)$ *with the derivative given by*

$$\begin{cases} \langle J'(u),v\rangle=M\Big(\sum\limits_{k=1}^{+\infty}A\big(k-1,\Delta u(k-1)\big)\Big)\sum\limits_{k=1}^{+\infty}a(k-1,\Delta u(k-1))\Delta v(k-1) \\ \qquad +\sum\limits_{k=1}^{+\infty}\alpha(k)|u(k)|^{p(k)-2}u(k)v(k)-\sum\limits_{k=1}^{+\infty}\delta(k)f\big(k,u(k)+1\big)v(k), \end{cases} \tag{12}$$

for all $u,v\in\mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$.

Indeed, we denote

$$I(u)=\widehat{M}\Big(\sum_{k=1}^{+\infty}A\big(k-1,\Delta u(k-1)\big)\Big),\qquad L(u)=\sum_{k=1}^{+\infty}\frac{\alpha(k)}{p(k)}|u(k)|^{p(k)}$$

and

$$\Lambda(u)=\sum_{k=1}^{+\infty}\delta(k)F\big(k,u(k)+1\big).$$

We have, by using (H_9), that

$$\begin{aligned}|I(u)| &= \Big|\int_0^{\sum_{k=1}^{+\infty}A\left(k-1,\Delta u(k-1)\right)} M(t)dt\Big|\\
&\le B_2\Big|\int_0^{\sum_{k=1}^{+\infty}A\left(k-1,\Delta u(k-1)\right)} t^{\beta-1}dt\Big|\\
&\le \frac{B_2}{\beta}\Big(\sum_{k=1}^{+\infty}\big|A(k-1,\Delta u(k-1))\big|\Big)^{\beta}.\end{aligned}$$

According to $(H_1),(H_3)$ and the *discrete Hölder type inequality*, we write

$$\begin{aligned}\sum_{k=1}^{+\infty}\big|A(k-1,\Delta u(k-1))\big| &\le \sum_{k=1}^{+\infty}\int_0^{\Delta u(k-1)}|a(k-1,t)|dt\\
&\le C_1\sum_{k=1}^{+\infty}\Big(j(k-1)+\frac{1}{p(k-1)}|\Delta u(k-1)|^{p(k-1)-1}\Big)\Delta u(k-1)\\
&\le C_1\sum_{k=1}^{+\infty}j(k-1)|\Delta u(k-1)|+\frac{C_1}{p^-}\sum_{k=1}^{+\infty}|\Delta u(k-1)|^{p(k-1)}\\
&\le C_1\Big(\frac{1}{q^-}+\frac{1}{p^-}\Big)||j||_{q_+(.)}||\Delta u||_{p_+(.)}+\frac{C_1}{p^-}||\Delta u||_{p_+(.)}\\
&< +\infty\end{aligned}$$

and we deduce that $|I(u)|<+\infty$. We have

$$|L(u)|=\Big|\sum_{k=1}^{+\infty}\frac{\alpha(k)}{p(k)}|u(k)|^{p(k)}\Big|\le\frac{1}{p^-}\Big|\sum_{k=1}^{+\infty}\alpha(k)|u(k)|^{p(k)}\Big|<+\infty$$

and

$$\begin{aligned}|\Lambda(u)| &= \Big|\sum_{k=1}^{+\infty}\delta(k)F\big(k,u(k)+1\big)\Big|\\
&\le\sum_{k=1}^{+\infty}|\delta(k)||F\big(k,u(k)+1\big)|\\
&\le\sum_{k=1}^{+\infty}C_3|\delta(k)|\big(1+|u(k)|^{p(k)}\big)\\
&\le C_3\sum_{k=1}^{+\infty}|\delta(k)|+C_3\bar{\delta}\sum_{k=1}^{+\infty}|u(k)|^{p(k)}\\
&<+\infty.\end{aligned}$$

Hence, J is well-defined. Clearly, the functionals I, L and Λ are in $C^1\big(\mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}, \mathbb{R}\big)$.

In what follows we prove (12). Let $u, v \in \mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$. Since

$$\begin{cases} \lim\limits_{\lambda\to 0^+} \dfrac{I(u+\lambda v)-I(u)}{\lambda} \\ = \lim\limits_{\lambda\to 0^+} \dfrac{\widehat{M}\Big(\sum\limits_{k=1}^{+\infty} A\big(k-1,\Delta u(k-1)+\lambda\Delta v(k-1)\big)\Big)-\widehat{M}\Big(\sum\limits_{k=1}^{+\infty} A\big(k-1,\Delta u(k-1)\big)\Big)}{\lambda} \\ = M\Big(\sum\limits_{k=1}^{+\infty} A\big(k-1,\Delta u(k-1)\big)\Big)\sum\limits_{k=1}^{+\infty} a\big(k-1,\Delta u(k-1)\big)\Delta v(k-1), \end{cases}$$

$$\begin{aligned} \lim_{\lambda\to 0^+} \frac{L(u+\lambda v)-L(u)}{\lambda} &= \lim_{\lambda\to 0^+} \sum_{k=1}^{+\infty} \frac{|u(k)+\lambda v(k)|^{p(k)}-|u(k)|^{p(k)}}{p(k)\lambda} \\ &= \sum_{k=1}^{+\infty} \lim_{\lambda\to 0^+} \frac{|u(k)+\lambda v(k)|^{p(k)}-|u(k)|^{p(k)}}{p(k)\lambda} \\ &= \sum_{k=1}^{+\infty} |u(k)|^{p(k)-2}u(k)v(k) \end{aligned}$$

and

$$\begin{aligned} \lim_{\lambda\to 0^+} \frac{\Lambda(u+\lambda v)-\Lambda(u)}{\lambda} &= \lim_{\lambda\to 0^+} \sum_{k=1}^{+\infty} \delta(k)\frac{F\big(k,u(k)+\lambda v(k)+1\big)-F\big(k,u(k)+1\big)}{\lambda} \\ &= \sum_{k=1}^{+\infty} \delta(k) \lim_{\lambda\to 0^+} \frac{F\big(k,u(k)+\lambda v(k)+1\big)-F\big(k,u(k)+1\big)}{\lambda} \\ &= \sum_{k=1}^{+\infty} \delta(k) f\big(k,u(k)+1\big)v(k), \end{aligned}$$

we obtain the relation (12). □

Proposition 3.2 *The functional J is weakly lower semi-continuous.*

Indeed, by (H_1), (H_4) and (H_9) we have that J is convex. Thus, it is enough to show that J is lower semi-continuous. For this, we fix $u \in \mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$ and $\epsilon > 0$. Since J is convex, we deduce that for any $v \in \mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$,

$$\begin{aligned} J(v) &\geq J(u) + \langle J'(u), v-u\rangle \\ &\geq J(u) + R(u,v) + S(u,v) + T(u,v), \end{aligned}$$

with

$$R(u,v) = M\Big(\sum_{k=1}^{+\infty} A\big(k-1,\Delta u(k-1)\big)\Big)\sum_{k=1}^{+\infty} a(k-1,\Delta u(k-1))\big(\Delta v(k-1)-\Delta u(k-1)\big),$$

$$S(u,v) = \sum_{k=1}^{+\infty} |u(k)|^{p(k)-2}u(k)\big(v(k)-u(k)\big)$$

and

$$T(u,v) = \sum_{k=1}^{+\infty} \delta(k)f\big(k,u(k)+1\big)\big(u(k)-v(k)\big).$$

Using the *discrete Hölder type inequality*, there exists three nonnegative constants C_4, C_5 and C_6 such that

$$\begin{aligned} R(u,v) &\geq -M\Big(\sum_{k=1}^{+\infty} A\big(k-1,\Delta u(k-1)\big)\Big)\sum_{k=1}^{+\infty} |a(k-1,\Delta u(k-1))||\Delta v(k-1)-\Delta u(k-1)| \\ &\geq -C_4'||\Delta u-\Delta v||_{\alpha(.),p_+(.)} \\ &\geq -C_4||u-v||_{1,\alpha(.),p_+(.)}, \end{aligned} \tag{13}$$

$$T(u,v) \geq -C_5||u-v||_{1,\alpha(.),p_+(.)} \tag{14}$$

and

$$\begin{aligned} S(u,v) &\geq -\sum_{k=1}^{+\infty} |u(k)|^{p(k)-1}|v(k)-u(k)| \\ &\geq -\Big(\frac{1}{p^-}+\frac{1}{q^-}\Big)\big|\big||u|^{p(.)-1}\big|\big|_{\alpha(.),q_+(.)}||u-v||_{\alpha(.),p_+(.)} \\ &\geq -C_6||u-v||_{1,\alpha(.),p_+(.)}. \end{aligned} \tag{15}$$

Then, combining (13), (14) and (15), we get

$$J(v) \geq J(u) - K||u-v||_{1,\alpha(.),p_+(.)}, \tag{16}$$

with $K = C_4 + C_5 + C_6$. Finally, for all $v \in \mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$ with $||v-u||_{1,\alpha(.),p_+(.)} < \tau = \dfrac{\epsilon}{K}$, we get

$$J(v) \geq J(u) - \epsilon.$$

Then J is lower semi-continuous and by [1], Corollary III.8, J is weakly lower semi-continuous. □

Proposition 3.3 *The functional J is coercive and bounded from below.*

Indeed, according to $(H_2),(H_5)-(H_9)$, we have

$$\begin{aligned}
J(u) &= \widehat{M}\Big(\sum_{k=1}^{+\infty} A\big(k-1,\Delta u(k-1)\big)\Big) + \sum_{k=1}^{+\infty}\frac{\alpha(k)}{p(k)}|u(k)|^{p(k)} - \sum_{k=1}^{+\infty}\delta(k)F\big(k,u(k)+1\big)\\
&\geq \frac{B_1}{\beta}\Big(\sum_{k=1}^{+\infty} A\big(k-1,\Delta u(k-1)\big)\Big)^{\beta} + \frac{1}{p^+}\sum_{k=1}^{+\infty}\alpha(k)|u(k)|^{p(k)}\\
&-C_3\sum_{k=1}^{+\infty}\delta(k)|u(k)|^{p(k)} - C_7\\
&\geq \frac{B_1}{\beta}\Big(\sum_{k=1}^{+\infty}\frac{1}{p(k-1)}\big|\Delta u(k-1)\big|^{p(k-1)}\Big)^{\beta} + \frac{1}{p^+}\sum_{k=1}^{+\infty}\alpha(k)|u(k)|^{p(k)}\\
&-\frac{C_3\bar{\delta}}{\alpha_0}\sum_{k=1}^{+\infty}\alpha(k)|u(k)|^{p(k)}\\
&\geq \frac{B_1}{\beta(p^+)^{\beta}}\Big(\sum_{k=1}^{+\infty}\big|\Delta u(k-1)\big|^{p(k-1)}\Big)^{\beta} + \Big(\frac{1}{p^+} - \frac{C_3\bar{\delta}}{\alpha_0}\Big)\sum_{k=1}^{+\infty}\alpha(k)|u(k)|^{p(k)} - C_7\\
&\geq \min\Big\{\frac{B_1}{\beta(p^+)^{\beta}};\frac{1}{p^+}-\frac{C_3\bar{\delta}}{\alpha_0}\Big\}\rho_{1,\alpha(.),p_+(.)}(u) - C_7.
\end{aligned}$$

To prove the coerciveness of the functional J, we may assume that $||u||_{1,\alpha(.),p_+(.)} > 1$ and, using Proposition 2.3, we deduce from the above inequality that

$$J(u) \geq \min\Big\{\frac{B_1}{\beta(p^+)^{\beta}};\frac{1}{p^+}-\frac{C_3\bar{\delta}}{\alpha_0}\Big\}||u||^{p^-}_{1,\alpha(.),p_+(.)} - C_7.$$

Thus, by assumption (H_7),

$$J(u) \longrightarrow +\infty \ \text{ as } \ ||u||_{1,\alpha(.),p_+(.)} \longrightarrow +\infty,$$

namely J is coercive. Besides, for $||u||_{1,\alpha(.),p_+(.)} \leq 1$, we have

$$\begin{aligned}
J(u) \qquad &\geq \min\Big\{\frac{B_1}{\beta(p^+)^{\beta}};\frac{1}{p^+}-\frac{C_3\bar{\delta}}{\alpha_0}\Big\}\rho_{1,\alpha(.),p_+(.)}(u) - C_7.\\
&\geq -C_7 > -\infty.
\end{aligned}$$

Thus J is bounded from below. □

Since J is weakly lower semi-continuous, bounded from below and coercive on $\mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$, using the relation between critical points of J and problem (9), we deduce that J has a minimizer which is a weak solution of (9).

We will show that every weak solution u of (9) is such that $u(k) \to 0$ as $k \to +\infty$. Let u be a weak solution of problem (9). Since $u \in \mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$, we have $\sum_{k=1}^{+\infty}|u(k)|^{p(k)} < +\infty$. Denote

$$S_1 = \{k \in \mathbb{Z}^+_*; |u(k)| < 1\} \text{ and } S_2 = \{k \in \mathbb{Z}^+_*; |u(k)| \geq 1\}.$$

Since $u \in \mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$, S_2 is necessary a finite set and $|u(k)| < +\infty$ for any $k \in S_2$. As S_2 is a finite set, then $\sum_{k \in S_2} |u(k)|^{p^+} < +\infty$.

On the other hand, we have $\sum_{k \in S_1} |u(k)|^{p^+} \leq \sum_{k \in S_1} |u(k)|^{p(k)} \leq \sum_{k=1}^{+\infty} |u(k)|^{p(k)} < +\infty$.

Therefore,

$$\sum_{k=1}^{+\infty} |u(k)|^{p^+} = \sum_{k \in S_1} |u(k)|^{p^+} + \sum_{k \in S_2} |u(k)|^{p^+} < +\infty.$$

Thus, $\lim_{k \to +\infty} |u(k)| = 0$, which completes the proof of Theorem 3.2. □

To end the proof of Theorem 3.1, let us consider the following problem

$$\begin{cases} -M(A(k-1, \Delta u(k-1)))\Delta(a(k-1, \Delta u(k-1))) \\ \\ +\alpha(k)|u(k)|^{p(k)-2}u(k) = \delta(k) f\big(k, u(k) - 1\big), \ k \in \mathbb{Z}^-_*, \\ \\ u(0) = 0, \quad \lim_{k \to -\infty} u(k) = 0. \end{cases} \tag{17}$$

Definition 3.3 A weak solution of problem (17) is a function $u \in \mathcal{W}^{1,p(.)}_{0,-,\alpha(.)}$ such that

$$\begin{cases} M\Big(\sum_{k=-\infty}^{0} A\big(k-1, \Delta u(k-1)\big)\Big) \sum_{k=-\infty}^{0} a(k-1, \Delta u(k-1))\Delta v(k-1) \\ \\ + \sum_{k=-\infty}^{0} \alpha(k)|u(k)|^{p(k)-2}u(k)v(k) = \sum_{k=-\infty}^{0} \delta(k) f\big(k, u(k) - 1\big)v(k), \end{cases} \tag{18}$$

for any $v \in \mathcal{W}^{1,p(.)}_{0,-,\alpha(.)}$.

By mimicking the proof of Theorem 3.2, we prove the following result.

Theorem 3.3 *Assume that assumptions* (H_1)*-*(H_9) *hold true. Then, there exists at least one weak solution of (17).*

Now, we end the proof of Theorem 3.1. For this, we define $v_1 = u_1 + 1$, where u_1 is a weak solution of problem (9) and $v_2 = u_2 - 1$, where u_2 is a weak solution of problem (17). Therefore, according to (H_5), we deduce that

$$u = v_1 \chi_{\mathbb{Z}^+} + v_2 \chi_{\mathbb{Z}^-} \tag{19}$$

is a weak heteroclinic solution of problem (1). □

Acknowledgment

The authors express their deepest thanks to the editor and anonymous referee for their comments and suggestions on the paper.

References

[1] Brezis, H. *Analyse Fonctionnelle: Theorie et Applications.* Paris, Masson, 1983.

[2] Cabada, A., Li, C. and Tersian, S. On homoclinic solutions of a semilinear p-Laplacian difference equation with periodic coefficients. *Adv. Differ. Equ.* **2010**, Art. ID 195376, 17 pp.

[3] Cabada, A. and Tersian, S. Existence of heteroclinic solutions for discrete p-Laplacian problems with a parameter. *Nonlinear Anal.* RWA **12** (2011) 2429–2434.

[4] Chen, Y., Levine, S. and Rao, M. Variable exponent, linear growth functionals in image processing. *SIAM J. Appl. Math.* **66** (4) (2006) 1383–1406.

[5] Diening, L. *Theoretical and Numerical Results for Electrorheological Fluids.* Ph.D. thesis, University of Freiburg, Germany, 2002.

[6] Guiro, A., Koné, B. and Ouaro, S. Weak heteroclinic solutions and competition phenomena to anisotropic difference equations with variable exponents. *Opuscula Math.* **34** (4) (2014) 733–745.

[7] Marcelli, C. and Papalini, F. Heteroclinic connections for fully non-linear non-autonomous second-order differential equations. *J. Diff. Equa.* **241** (2007) 160–183.

[8] Mihailescu, M., Radulescu, V. and Tersian, S. Homoclinic solutions of difference equations with variable exponents. *Topol. Methods Nonlinear Anal.* **38** (2) (2011) 277–289.

[9] Rajagopal, K. R. and Ruzicka, M. Mathematical Modeling of Electrorheological materials. *Contin. Mech. Thermodyn.* **13** (2001) 59–78.

[10] Ruzicka, M. *Electrorheological Fluids: Modelling and Mathematical Theory.* Lecture Notes in Mathematics, 1748, Springer-Verlag, Berlin, 2002.