Nonlinear Dynamics and Systems Theory, **18** (1) (2018) 53–66

Some Gronwall Lemmas Using Picard Operator Theory: Application to Dynamical Systems

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Received: January 12, 2017; Revised: November 21, 2017

Abstract: In this paper we derive optimal explicit bounds for the solutions to integral inequalities. We rewrite the inequalities in terms of integral operators and we get the bound as a fixed point of the corresponding operator. As application, we study the stability of ceratin dynamical systems.

Keywords: *Gronwall lemma; dynamical system; fixed point; Picard operators.*

Mathematics Subject Classification (2010): 26D10, 26D15, 26D20, 34A40.

1 Introduction

Integral inequalities are a necessary tool in the study of various classes of equations. In 1919, Gronwall [10] introduced the famous Gronwall inequality in the study of the solutions of differential equations. Since then, many contributions have been made (see [1]-[3]). The applications of integral inequalities were developed in a remarkable way in the study of the existence, the uniqueness, the comparison, the stability and continuous dependence of the solution in respect to data. In the last few years, a series of generalizations of these inequalities appeared. The problem of stability can be solved by Lyapunov techniques for differential equations (see [12]- [14]), or in terms of nonlinear integral inequalities. These inequalities can be used in the analysis of various problems in the theory of nonlinear differential equations and control systems (see [3] and references therein). There is an extensive literature on the inequalities, for example, the Barbalats lemma is an integral inequality used in applied nonlinear control. The second Lyapunov method has long played an important role in the history of stability theory, and it will with no doubt continue to serve as an indispensable tool in future research

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papers (see [14]). V. I. Zubov studied the boundaries of the asymptotic stability domain in which he proved the theorem of the asymptotic stability domain. This result is now known as Zubov's theorem (see [17], [6]). The concepts of stability and boundedness of solutions have been studied extensively by Taro Yoshizawa (see [18], [19], [7]).

In a recent paper [11], I.A. Rus has formulated ten problems of interest in the theory of Gronwall lemmas. One of them concerns finding examples of Gronwall-type lemmas in which the upper bounds are fixed points of the corresponding operator A (**Problem 5**). The new inequalities, derived in this paper, are useful in many applications, in particular to the stability of dynamical systems. We propose new sufficient conditions to ensure the global uniform asymptotic stability of time-varying systems described by the following equation:

$$\dot{x} = f(t,x) + g(t,x), \tag{1}$$

where $f : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $g : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are piecewise continuous in t and locally Lipschitz in x on $\mathbb{R}_+ \times \mathbb{R}^n$, and the associated nominal system is given by:

$$\dot{x} = f(t,x). \tag{2}$$

For all $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}_+$, we will denote by $x(t;t_0,x_0)$, or simply by $x(t)$, the unique solution at time t_0 starting from the point x_0.

Unless otherwise stated, we assume throughout the paper that the functions encountered are sufficiently smooth. We often omit arguments of functions to simplify notation, $\|.\|$ stands for the Euclidean norm vectors. We recall now some standard concepts from stability and practical stability theory; any book on Lyapunov stability can be consulted for these; particularly good references are [4]: $\mathcal{K}$ is the class of functions $\mathbb{R}_+ \to \mathbb{R}_+$ which are zero at the origin, strictly increasing and continuous. $\mathcal{K}_\infty$ is the subset of $\mathcal{K}$ functions that are unbounded. $\mathcal{L}$ is the set of functions $\mathbb{R}_+ \to \mathbb{R}_+$ which are continuous, decreasing and converging to zero as their argument tends to $+\infty$. $\mathcal{KL}$ is the class of functions $\mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ which are class $\mathcal{K}$ on the first argument and class $\mathcal{L}$ on the second argument. A positive definite function $\mathbb{R}_+ \to \mathbb{R}_+$ is the one that is zero at the origin and positive otherwise. We define the closed ball $B_r := \Big\{x \in \mathbb{R}^n : \ \|x\| \leq r\Big\}$.

2 Abstract Gronwall Lemma

To present our problem we need some standard notations of Nonlinear Analysis. Let X be a nonempty set and $A : X \to X$ be an operator. We denote by $F_A = \{x \in X / Ax = x\}$ the fixed point set of the operator A. The symbol, $F_A = \{x^*_A\}$, has the following meaning: the operator A has a unique fixed point and we denote this unique fixed point by x^*_A. In general, throughout this paper we follow the notation and terminology from I.A. Rus [15] and [16].

Definition 2.1 (I.A. Rus [15]). Let $(X, \to)$ be an L-space. An operator $f : X \to X$ is, by definition, a Picard operator if:
i) $F_f = \{x^*\}$.
ii) $(f^n(x))_{n\in\mathbb{N}} \to x^*$ as $n \to \infty$, for all $x \in X$.

In terms of the Picard operators, a classical result in metric fixed point theory has the following form ([13], [9]).

Proposition 2.1 *(Contraction principle). Let (X, d) be a complete metric space and let $f : X \to X$ be an a-contraction, i.e., $a \in]0, 1[$ and $d(f(x), f(y)) \leq a.d(x, y)$, for each $x, y \in X$. Then f is a Picard operator.*

Proposition 2.2 *(Abstract Gronwall lemma). Let $(X, \to, \leq)$ be an ordered L-space and $A : X \to X$ be an operator. We suppose that:*
i) A is a Picard operator ($F_A = \{x_A^*\}$).
ii) A is an increasing operator.
Then we have:
a) $x \in X$, $x \leq A(x) \Rightarrow x \leq x_A^*$.
b) $x \in X$, $x \geq A(x) \Rightarrow x \geq x_A^*$.

3 Main Results

In this section we point out some Gronwall-type inequalities using some results concerning Picard operator theory.

The following result is well known from the book of A.N. Filatov (see [8]), here we will give a new proof of it using the theory of operators.

Theorem 3.1 *Let $x \in \mathcal{C}([a, b], \mathbb{R}_+)$ be such that*

$$x(t) \leq \delta_2(t - a) + \delta_1 \int_a^t x(s)ds + \delta_3, \quad \forall t \in [a, b], \tag{3}$$

where $\delta_1 > 0$, δ_2 and δ_3 are real numbers, then

$$x(t) \leq \left(\frac{\delta_2}{\delta_1} + \delta_3\right) \exp \delta_1(t - a) - \frac{\delta_2}{\delta_1}, \quad \forall t \in [a, b]. \tag{4}$$

Proof. Let $(X, \to, \leq) = (\mathcal{C}[a, b], \xrightarrow{\|.\|_\tau}, \leq)$, where $\|.\|_\tau$ is the Bielecki norm on $\mathcal{C}[a, b]$, i.e., τ is a positive real number and

$$\|x\|_\tau = \max_{a \leq t \leq b} (|x(t)| \exp(-\tau(t - a))).$$

We consider on $X = \mathcal{C}[a, b]$ the operator $A : X \to X$ defined by

$$A(x)(t) = \delta_2(t - a) + \delta_1 \int_a^t x(s)ds + \delta_3, \quad t \in [a, b].$$

Suppose that x is a fixed point of A, then $A(x) = x$ or, equivalently,

$$x(t) = \delta_2(t - a) + \delta_1 \int_a^t x(s)ds + \delta_3, \quad t \in [a, b].$$

By differentiation, we get

$$x'(t) = \delta_1 x(t) + \delta_2,$$

which is an ordinary differential equation (added to an initial condition, this ODE admits a unique solution according to the Cauchy-Lipschitz theorem). Since $x(a) = \delta_3$, it comes out that

$$x(t) = \left(\frac{\delta_2}{\delta_1} + \delta_3\right) \exp \delta_1(t - a) - \frac{\delta_2}{\delta_1}.$$

Conversly, we can easily verify that $A(x) = x$ and using the fact that A admits a unique fixed point, we get

$$x_A^*(t) = \left(\frac{\delta_2}{\delta_1} + \delta_3\right) \exp \delta_1(t-a) - \frac{\delta_2}{\delta_1}, \quad t \in [a,b].$$

One can easily check that A is an increasing operator: let $x,\, y \in \mathcal{C}[a,b]$, if $x \leq y$, then $A(x) \leq A(y)$. The last point is to show that A is a contraction with respect to $\|.\|_\tau$. We have

$$\begin{aligned} |A(x)(t) - A(y)(t)|e^{-\tau(t-a)} &\leq \delta_1 e^{-\tau(t-a)} \int_a^t |x(s) - y(s)| ds \\ &\leq \|x-y\|_\tau \delta_1 e^{-\tau(t-a)} \int_a^t e^{\tau(s-a)} ds \\ &\leq \|x-y\|_\tau \frac{\delta_1}{\tau} \left[1 - e^{-\tau(t-a)}\right] \\ &\leq \|x-y\|_\tau \frac{\delta_1}{\tau} \left[1 - e^{-\tau(b-a)}\right]. \end{aligned}$$

Then $\|A(x) - A(y)\|_\tau \leq \|x-y\|_\tau \dfrac{\delta_1}{\tau}\left[1 - e^{-\tau(b-a)}\right]$ and A is a contraction with τ suitably chosen. Finally, the proof follows from Proposition 2.2. □

Remark 3.1 If $\delta_2 \geq 0$, then there is a direct proof for this well known Gronwall-type lemma.

Theorem 3.2 *Let $x \in \mathcal{C}([a,b], \mathbb{R}_+)$ be such that*

$$x(t) \leq \delta_2(t-a) + \delta_1 \int_a^t x(s) ds + \varphi(t), \quad \forall t \in [a,b], \tag{5}$$

where $\delta_1 > 0$, δ_2, δ_3 are real numbers and φ is a continuous function on $[a,b]$, then

$$x(t) \leq \frac{\delta_2}{\delta_1} \exp \delta_1(t-a) + \delta_1 \int_a^t \varphi(s) \exp \delta_1(t-s) ds + \varphi(a) - \frac{\delta_2}{\delta_1}, \quad \forall t \in [a,b]. \tag{6}$$

Proof. We use the same notations as in the last proof. Let the operator A be defined by

$$A(x)(t) = \delta_2(t-a) + \delta_1 \int_a^t x(s) ds + \varphi(t), \quad t \in [a,b].$$

Suppose that x is a fixed point of A, then $A(x) = x$ or, equivalently,

$$x(t) = \delta_2(t-a) + \delta_1 \int_a^t x(s) ds + \varphi(t), \quad t \in [a,b].$$

By differentiation, we get

$$x'(t) = \delta_1 x(t) + \delta_2 + \varphi'(t),$$

which is an ordinary differential equation. Since $x(a) = \varphi(a)$, it comes out that

$$x(t) = \frac{\delta_2}{\delta_1} \exp \delta_1(t-a) + \delta_1 \int_a^t \varphi(s) \exp \delta_1(t-s) ds + \varphi(a) - \frac{\delta_2}{\delta_1}.$$

Conversely, we can easily verify that $A(x) = x$ and using the fact that A admits a unique fixed point, we get

$$x_A^*(t) = \frac{\delta_2}{\delta_1} \exp \delta_1 (t-a) + \delta_1 \int_a^t \varphi(s) \exp \delta_1 (t-s) ds + \varphi(a) - \frac{\delta_2}{\delta_1}, \quad \forall t \in [a,b].$$

One can easily check that A is an increasing operator: let $x,\ y \in \mathcal{C}[a,b]$, if $x \leq y$, then $A(x) \leq A(y)$. On the other hand, by the same calculation as in the previous theorem, one can easily check that A is a contraction with respect to $\|.\|_\tau$, with τ suitably chosen. Finally, the proof follows from Proposition 2.2. □

Remark 3.2 If the function φ is a constant, then we get the particular case of Theorem 3.1.

Theorem 3.3 *Let $x \in \mathcal{C}([a,b], \mathbb{R}_+)$ be such that*

$$x(t) \leq \alpha(t) + \beta(t) \int_a^t x(s) ds, \quad \forall t \in [a,b], \tag{7}$$

where α is continuous and β is a continuous function on $[a,b]$, then

$$x(t) \leq \alpha(t) + \beta(t) \int_a^t \alpha(s) \exp\left(\int_s^t \beta(u) du\right) ds, \quad \forall t \in [a,b]. \tag{8}$$

Proof. Using the same notations, let the operator A be defined by

$$A(x)(t) = \alpha(t) + \beta(t) \int_a^t x(s) ds, \quad t \in [a,b].$$

Suppose that x is a fixed point of A, then $A(x) = x$ or, equivalently,

$$x(t) = \alpha(t) + \beta(t) \int_a^t x(s) ds, \quad t \in [a,b].$$

By differentiation, we get

$$\beta(t) x'(t) = \left[\beta'(t) + \beta^2(t)\right] x(t) + \alpha'(t)\beta(t) - \beta'(t)\alpha(t),$$

which is an ordinary differential equation. Since $x(a) = \alpha(a)$, it comes out that

$$x(t) = \alpha(t) + \beta(t) \int_a^t \alpha(s) \exp\left(\int_s^t \beta(u) du\right) ds.$$

Conversely, we can easily verify that $A(x) = x$ and using the fact that A admits a unique fixed point, we get

$$x_A^*(t) = \alpha(t) + \beta(t) \int_a^t \alpha(s) \exp\left(\int_s^t \beta(u) du\right) ds, \quad \forall t \in [a,b].$$

One can easily check that A is an increasing operator: let $x,\ y \in \mathcal{C}[a,b]$, if $x \leq y$, then $A(x) \leq A(y)$. The last point is to show that A is a contraction with respect to $\|.\|_\tau$. We have

$$\begin{aligned} |A(x)(t) - A(y)(t)|e^{-\tau(t-a)} &\leq \beta(t)e^{-\tau(t-a)} \int_a^t |x(s) - y(s)|ds \\ &\leq \|x - y\|_\tau \beta(t)e^{-\tau(t-a)} \int_a^t e^{\tau(s-a)} ds \\ &\leq \|x - y\|_\tau \frac{\beta(t)}{\tau} \left[1 - e^{-\tau(t-a)}\right] \\ &\leq \|x - y\|_\tau \frac{\|\beta\|_\infty}{\tau} \left[1 - e^{-\tau(b-a)}\right]. \end{aligned}$$

Then $\|A(x) - A(y)\|_\tau \leq \|x - y\|_\tau \frac{\|\beta\|_\infty}{\tau} \left[1 - e^{-\tau(b-a)}\right]$ and A is a contraction with τ suitably chosen. Finally, the proof follows from Proposition 2.2. □

Remark 3.3 If $\alpha(t) = \delta_3$ and $\beta(t) = \delta_1$, then we get the particular case of Theorem 3.1.

The following result is well known from the book of Filatov and Scharova (1976), here we will give a new proof of it using the theory of operators.

Theorem 3.4 *Let $x \in \mathcal{C}([a, b], \mathbb{R}_+)$ be such that*

$$x(t) \leq \alpha(t) + \beta(t) \int_a^t k(s)x(s)ds, \quad \forall t \in [a, b], \tag{9}$$

where α is continuous, β and k are continuous functions on $[a, b]$, then

$$x(t) \leq \alpha(t) + \beta(t) \int_a^t \alpha(s)k(s) \exp\left(\int_s^t \beta(u)k(u)du\right) ds, \quad \forall t \in [a, b]. \tag{10}$$

Proof. Using the same notations, let the operator A be defined by

$$A(x)(t) = \alpha(t) + \beta(t) \int_a^t k(s)x(s)ds, \quad t \in [a, b].$$

Suppose that x is a fixed point of A, then $A(x) = x$ or, equivalently,

$$x(t) = \alpha(t) + \beta(t) \int_a^t k(s)x(s)ds, \quad t \in [a, b].$$

By differentiation, we get

$$x'(t) = \left[\beta(t)k(t) + \frac{\beta'(t)}{\beta(t)}\right] x(t) + \alpha'(t) - \alpha(t)\frac{\beta'(t)}{\beta(t)},$$

which is an ordinary differential equation. The solutions of the homogenous equation are

$$x(t) = \lambda\beta(t) \exp\left(\int_a^t \beta(s)k(s)ds\right).$$

A particular solution can be obtained using the method of variation of the constant, then the solutions of our ODE are

$$x(t) = \lambda\beta(t)\exp\left(\int_a^t \beta(s)k(s)ds\right) + \alpha(t) + \beta(t)\int_a^t \alpha(s)k(s)\exp\left(\int_s^t \beta(u)k(u)du\right)ds.$$

Since $x(a) = \alpha(a)$, it comes out that

$$x(t) = \alpha(t) + \beta(t)\int_a^t \alpha(s)k(s)\exp\left(\int_s^t \beta(u)k(u)du\right)ds.$$

Conversly, we can easily verify that $A(x) = x$ and using the fact that A admits a unique fixed point, we get

$$x_A^*(t) = \alpha(t) + \beta(t)\int_a^t \alpha(s)k(s)\exp\left(\int_s^t \beta(u)k(u)du\right)ds, \quad \forall t \in [a,b].$$

One can easily check that A is an increasing operator: let $x,\ y \in \mathcal{C}[a,b]$, if $x \le y$, then $A(x) \le A(y)$. On the other hand, A is a contraction with respect to $\|.\|_\tau$. We have

$$\begin{aligned} |A(x)(t) - A(y)(t)|e^{-\tau(t-a)} &\le \beta(t)e^{-\tau(t-a)}\int_a^t k(s)|x(s)-y(s)|ds \\ &\le \|x-y\|_\tau \beta(t)e^{-\tau(t-a)}\int_a^t k(s)e^{\tau(s-a)}ds \\ &\le \|x-y\|_\tau \frac{\beta(t)}{\tau}\|k\|_\infty\left[1-e^{-\tau(t-a)}\right] \\ &\le \|x-y\|_\tau \frac{\|\beta\|_\infty\|k\|_\infty}{\tau}\left[1-e^{-\tau(b-a)}\right]. \end{aligned}$$

Then $\|A(x) - A(y)\|_\tau \le \|x-y\|_\tau \dfrac{\|\beta\|_\infty\|k\|_\infty}{\tau}\left[1-e^{-\tau(b-a)}\right]$ and A is a contraction with τ suitably chosen. Finally, the proof follows from Proposition 2.2. □

Theorem 3.5 *Let $x(t)$ be continuous and nonnegative on $[0,h]$ and satisfy*

$$x(t) \le a(t) + \int_0^t (a_1(s)x(s) + b(s))\,ds, \tag{11}$$

where $a_1(t)$ and $b(t)$ are nonnegative integrable functions. Then, on $[0,h]$

$$x(t) \le a(t) + \int_0^t (a_1(s)a(s) + b(s))\exp\left(\int_s^t a_1(\xi)d\xi\right)ds. \tag{12}$$

Proof. Using the same notations, let the operator A be defined by

$$A(x)(t) = a(t) + \int_0^t (a_1(s)x(s) + b(s))\,ds, \quad t \in [0,h].$$

We note that $F_A = \{x_A^*\}$, where

$$x_A^*(t) = a(t) + \int_0^t (a_1(s)a(s) + b(s))\exp\left(\int_s^t a_1(\xi)d\xi\right)ds, \quad \forall t \in [0,h].$$

One can easily check that A is an increasing operator: let x, $y \in \mathcal{C}[a,b]$, if $x \leq y$, then $A(x) \leq A(y)$. On the other hand, A is a contraction with respect to $\|.\|_\tau$, with τ suitably chosen. Finally, the proof follows from Proposition 2.2. □

For the following results we will use another norm, the rest of data are the same.

Theorem 3.6 *Let $x(t)$ be bounded continuous in $J = [\alpha, \infty)$, and suppose*

$$x(t) \leq ae^{-\gamma(t-\alpha)} + \int_\alpha^\infty be^{-\gamma|t-s|}x(s)ds, \quad t \in J, \tag{13}$$

where $a \geq 0$, $b \geq 0$, and $\gamma > 0$ are constants and $b < \frac{\gamma}{2}$. Then

$$x(t) \leq \frac{a}{b}(\gamma - \delta)e^{-\delta(t-\alpha)}, \quad t \in J, \tag{14}$$

where $\delta = \sqrt{\gamma^2 - 2b\gamma}$.

Proof. Let $(X, \to, \leq) = (\mathcal{C}(J), \xrightarrow{\|.\|}, \leq)$, where $\mathcal{C}(J)$ is the Banach space of functions x which are bounded and continuous in $J = [\alpha, \infty)$ with norm $\|x\| = \sup_{t\in J}|x(t)|$. Using the same notations, let the operator A be defined by

$$A(x)(t) = ae^{-\gamma(t-\alpha)} + \int_\alpha^\infty be^{-\gamma|t-s|}x(s)ds, \quad t \in J,$$

Suppose that x is a fixed point of A, then $A(x) = x$ or, equivalently,

$$\begin{aligned} x(t) &= ae^{-\gamma(t-\alpha)} + \int_\alpha^\infty be^{-\gamma|t-s|}x(s)ds \\ &= ae^{-\gamma(t-\alpha)} + be^{-\gamma t}\int_\alpha^t e^{\gamma s}x(s)ds + be^{\gamma t}\int_t^\infty e^{-\gamma s}x(s)ds. \end{aligned}$$

By differentiation, we get

$$x'(t) = -2a\gamma e^{-\gamma(t-\alpha)} + \gamma x(t) - 2b\gamma e^{-\gamma t}\int_\alpha^t e^{\gamma s}x(s)ds,$$

we derive once again, it comes out that

$$x''(t) = (\gamma^2 - 2b\gamma)x(t),$$

which is an ordinary differential equation. Using $x(\alpha)$ and $x'(\alpha)$, we get

$$x(t) = \frac{a}{b}(\gamma - \delta)e^{-\delta(t-\alpha)}.$$

Conversly, we can easily verify that $A(x) = x$ and using the fact that A admits a unique fixed point, we arrive at

$$x_A^*(t) = \frac{a}{b}(\gamma - \delta)e^{-\delta(t-\alpha)}, \quad t \in J.$$

One can easily check that A is an increasing operator: let $x,\ y \in \mathcal{C}[a,b]$, if $x \le y$, then $A(x) \le A(y)$. If $x, y \in \mathcal{C}(J)$ and $\|x-y\| = L$, it is easy to see that

$$|A(x)(t) - A(y)(t)| \le \int_\alpha^t bLe^{-\gamma(t-s)}ds + \int_t^\infty bLe^{\gamma(t-s)}ds \le \frac{2b}{\gamma}L = \frac{2b}{\gamma}\|x-y\|,$$

whence we conclude that $A(x) \in \mathcal{C}(J)$ and A is a contraction. Finally, the proof follows from Proposition 2.2. □

Theorem 3.7 *Let $x(t)$ be a continuous function for $\alpha \le t \le \beta$, and suppose*

$$x(t) \le ae^{-\gamma(\beta-t)} + \int_\alpha^\beta be^{-\gamma|t-s|}x(s)ds, \quad \alpha \le t \le \beta, \tag{15}$$

where $a \ge 0$, $b \ge 0$, and $\gamma > 0$ are constants and $b < \dfrac{\gamma}{2}$. Then

$$x(t) \le \frac{a}{b}(\gamma - \delta)e^{-\delta(\beta-t)}, \quad \alpha \le t \le \beta, \tag{16}$$

where $\delta = \sqrt{\gamma^2 - 2b\gamma}$.

Proof. Since the proof of this result follows by the similar arguments as in the last theorem, we omit the details. □

Remark 3.4 We use the condition $b < \dfrac{\gamma}{2}$ to prove that the operator A is a contraction but we can omit this condition and use the Gronwall lemma to prove the last proposition.

4 Application to Stability of Dynamical Systems

We consider the following system:

$$\dot{x} = f(t,x), \quad x(t_0) = x_0, \tag{17}$$

where $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$ and $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in t and locally Lipschitz in x. We begin by giving the definition of uniform boundedness and uniform stability (see [14], [18], [19], [7]).

Definition 4.1 (uniform boundedness) A solution of (17) is said to be globally uniformly bounded if for every $\alpha > 0$ there exists $c = c(\alpha)$ such that, for all $t_0 \ge 0$,

$$\|x_0\| \le \alpha \Rightarrow \|x(t)\| \le c(\alpha), \quad \forall\ t \ge t_0.$$

Definition 4.2 (uniform stability)
(i) The origin $x = 0$ is uniformly stable if for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$, such that for all $t_0 \ge 0$,

$$\|x_0\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall\ t \ge t_0.$$

(ii) The origin $x = 0$ is globally uniformly stable if it is uniformly stable and the solutions of system (17) are globally uniformly bounded.

We recall in the following definition the notion of practical stability (see [5]).

Definition 4.3 (practical stability) The system (17) is said to be $(PS1)$ uniformly practically stable if, given (λ, A) with $0<\lambda<A$, we have

$$\|x_0\|<\lambda \Rightarrow \|x(t)\|<A, \quad t \geq t_0, \quad \forall t_0 \in \mathbb{R}_+.$$

$(PS2)$ quasi-uniformly asymptotically stable (in the large) if $\forall \varepsilon>0$, $\alpha>0$, $t_0 \in \mathbb{R}_+$, there exists a positive number $T=T(\varepsilon, \alpha)$ such that

$$\|x_0\| \leq \alpha \Rightarrow \|x(t)\|<\varepsilon, \quad t \geq t_0+T.$$

$(PS3)$ uniformly practically asymptotically stable if $(PS1)$ and $(PS2)$ hold at the same time.

As application to stability, let us consider the nonlinear dynamical system:

$$\dot{x}=A(t) x+g(t, x), \tag{18}$$

where $t \geq 0$, $x(t) \in \mathbb{R}^n$, the matrix $A(.)$ is continuous and bounded, $g: \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous in (t, x), locally Lipschitz in x such that $g(t, 0)=0$. We suppose that $x=0$ is globally uniformly asymptotically stable equilibrium point for the nominal system $\dot{x}=A(t) x$, this is equivalent to saying that

$$\|\Phi(t, t_0)\| \leq k \exp -\gamma(t-t_0), \quad \forall t \geq t_0, k>0, \gamma>0, \tag{19}$$

where $\Phi(t, t_0)$ is the state transition matrix associated to $A(t)$. The solution of this system with the initial condition (t_0, x_0) is given by:

$$x(t)=\Phi(t, t_0) x(t_0)+\int_{t_0}^t \Phi(t, s) g(s, x(s)) d s. \tag{20}$$

We have

$$\|x(t)\| \leq k \exp -\gamma(t-t_0)\|x(t_0)\|+\int_{t_0}^t k e^{-\gamma(t-s)}\|g(s, x(s))\| d s. \tag{21}$$

It follows that

$$e^{\gamma t}\|x(t)\| \leq k e^{\gamma t_0}\|x(t_0)\|+\int_{t_0}^t k e^{\gamma s}\|g(s, x(s))\| d s. \tag{22}$$

We will impose a restriction on g to study the practical stability.

If we suppose that for all (t, x),

$$\|g(t, x)\| \leq \rho(t),$$

with ρ being a nonnegative continuous function which tends to zero as $t \rightarrow \infty$, then (22) becomes

$$e^{\gamma t}\|x(t)\| \leq k e^{\gamma t_0}\|x(t_0)\|+\int_{t_0}^t k e^{\gamma s} \rho(s) d s.$$

The assumption on ρ means that: $\forall \varepsilon>0, \quad \exists T>0/ \quad \forall t \geq t_0+T, \quad \rho(t)<\varepsilon$. We have also $\exists \beta/ \quad \forall t \in[t_0, t_0+T], \quad \rho(t) \leq \beta$.

Then, $\forall t \geq t_0 + T$,

$$\begin{aligned} e^{\gamma t}\|x(t)\| &\leq ke^{\gamma t_0}\|x(t_0)\| + \int_{t_0}^{t_0+T} ke^{\gamma s}\rho(s)ds + \int_{t_0+T}^{t} ke^{\gamma s}\rho(s)ds \\ &\leq ke^{\gamma t_0}\|x(t_0)\| + k\beta\int_{t_0}^{t_0+T} e^{\gamma s}ds + k\varepsilon\int_{t_0+T}^{t} e^{\gamma s}ds \\ &\leq ke^{\gamma t_0}\|x(t_0)\| + \frac{k\beta}{\gamma}\left[e^{\gamma(t_0+T)} - e^{\gamma t_0}\right] + \frac{k\varepsilon}{\gamma}\left[e^{\gamma t} - e^{\gamma(t_0+T)}\right], \end{aligned}$$

or equivalently, $\forall t \geq t_0 + T$

$$\|x(t)\| \leq ke^{-\gamma(t-t_0)}\|x(t_0)\| + \frac{k\beta}{\gamma}e^{-\gamma(t-t_0)}\left[e^{\gamma T} - 1\right] + \frac{k\varepsilon}{\gamma}.$$

We see that the function : $t \mapsto ke^{-\gamma(t-t_0)}\|x(t_0)\| + \frac{k\beta}{\gamma}e^{-\gamma(t-t_0)}\left[e^{\gamma T} - 1\right]$ vanishes, then

$$\|x(t)\| \leq M\varepsilon, \quad \forall t \geq t_0 + T',$$

for a certain $T' > T > 0$, this shows the practical stability of the system.

Another approach is to study the asymptotic behavior of the system in a small neighborhood of the origin. For the rest of our presentation, we need the following definitions which are related to stability.

Definition 4.4 (uniform stability of B_r)
(i) B_r is uniformly stable if for all $\epsilon > r$, there exists $\delta = \delta(\epsilon) > 0$ such that for all $t_0 \geq 0$,

$$\|x_0\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall\ t \geq t_0.$$

(ii) B_r is globally uniformly stable if it is uniformly stable and the solutions of system (4.1) are globally uniformly bounded.

Definition 4.5 (uniform attractivity) The origin $x = 0$ is globally uniformly attractive if for all $\epsilon > 0$ and $c > 0$, there exists $T(\epsilon, c) > 0$, such that for all $t_0 \geq 0$,

$$\|x(t)\| < \epsilon, \quad \forall\ t \geq t_0 + T(\epsilon, c), \quad \|x_0\| < c.$$

Definition 4.6 (Class $\mathcal{K}$ function) A continuous function $\alpha : [0, a) \to [0, +\infty)$ is said to belong to class $\mathcal{K}$, if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = +\infty$ and $\alpha(r) \to +\infty$ as $r \to +\infty$.

Definition 4.7 (Class $\mathcal{KL}$ function) A continuous function $\beta : [0, a) \times [0, +\infty) \to [0, +\infty)$ is said to belong to class $\mathcal{KL}$, if for each fixed point s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to +\infty$.

The following proposition provides a characterization of global uniform attractivity and global uniform stability.

Proposition 4.1 *If there exists a class $\mathcal{KL}$ function β, a class $\mathcal{K}_\infty$ α, a constant $r > 0$ such that, given any initial state x_0, the solution satisfies*

$$\|x(t)\| \leq \beta(\|x_0\|, t) + r, \quad \forall t \geq 0,$$

then B_r is globally uniformly attractive and globally uniformly stable.

Note that, if the class $\mathcal{KL}$-function β in the above relation is of the form $\beta(r,s) = kre^{-\lambda t}$, with $\lambda, k > 0$ we say that the ball B_r is globally uniformly exponentially stable. It is also worth noting that if, in the above definitions, we take $r = 0$, then one deals with the standard concept of GUAS and GUES of the origin (see [4] for more details). Moreover, in the rest of this paper, we study the asymptotic behavior of a small ball centered at the origin for $0 \leq \| x(t) \| - r$, so that if $r = 0$, we find the classical definition of the uniform asymptotic stability of the origin viewed as an equilibrium point.

Other applications to stability will be done in the following example by considering the system (18), we keep the same assumptions.

Example 4.1 1) Suppose that condition (22) holds and for all (t, x),

$$\|g(t,x)\| \leq \eta(t)\|x\|,$$

with η being an integrable function, then (22) becomes

$$e^{\gamma t}\|x(t)\| \leq ke^{\gamma t_0}\|x(t_0)\| + \int_{t_0}^{t} k\eta(s)e^{\gamma s}\|x(s)\|ds.$$

Let $u(t) = e^{\gamma t}\|x(t)\|$, then the last inequality becomes

$$u(t) \leq ku(t_0) + \int_{t_0}^{t} k\eta(s)u(s)ds,$$

using Theorem 3.4 we get

$$u(t) \leq ku(t_0) + \int_{t_0}^{t} k^2 u(t_0)\eta(s) \left(\exp \int_{s}^{t} k\eta(u)du \right) ds,$$

then

$$u(t) \leq kMu(t_0), \quad \text{where} \quad M = 1 + k\|\eta\|_1 e^{k\|\eta\|_1}.$$

One can obtain an estimation on the trajectories as follows, for all $t \geq t_0$,

$$\|x(t)\| \leq kM\|x(t_0)\|e^{-\gamma(t-t_0)}.$$

Then the origin is a globally uniformly exponentially stable equilibrium point for the system.

2) If we suppose that for all (t, x),

$$\|g(t,x)\| \leq \eta(t)\|x\| + \eta',$$

with η being an integrable function and $\eta' > 0$, then (22) becomes

$$e^{\gamma t}\|x(t)\| \leq ke^{\gamma t_0}\|x(t_0)\| + \int_{t_0}^{t} ke^{\gamma s}\{\eta(s)\|x(s)\| + \eta'\}ds.$$

Let $u(t) = e^{\gamma t}\|x(t)\|$, then the last inequality becomes

$$u(t) \leq ku(t_0) + \int_{t_0}^{t} \{k\eta(s)u(s) + k\eta' e^{\gamma s}\}ds,$$

using Theorem 3.5 we get

$$u(t) \le ku(t_0) + \int_{t_0}^{t} \{k^2 u(t_0)\eta(s) + k\eta' e^{\gamma s}\} \left(\exp \int_s^t k\eta(u)du\right) ds,$$

then $u(t) \le kMu(t_0) + re^{\gamma t}$, where $M = 1 + k\|\eta\|_1 e^{k\|\eta\|_1}$ and $r = \frac{k\eta'}{\gamma} e^{k\|\eta\|_1}$.

One can obtain an estimation on the trajectories as follows, for all $t \ge t_0$,

$$\|x(t)\| \le kM\|x(t_0)\|e^{-\gamma(t-t_0)} + r.$$

Then B_r is globally uniformly exponentially stable.

In the following example $g(t, 0)$ is not necessarily zero, in such a situation $x = 0$ is no longer an equilibrium point.

3) We suppose that for all (t, x),

$$\|g(t, x)\| \le \eta(t)\|x\| + \eta'(t),$$

with η being integrable and η' being a piecewise continuous function, then (22) becomes

$$e^{\gamma t}\|x(t)\| \le ke^{\gamma t_0}\|x(t_0)\| + \int_{t_0}^{t} ke^{\gamma s}\{\eta(s)\|x(s)\| + \eta'(s)\}ds.$$

Let $u(t) = e^{\gamma t}\|x(t)\|$, then the last inequality becomes

$$u(t) \le ku(t_0) + \int_{t_0}^{t} \{k\eta(s)u(s) + k\eta'(s)e^{\gamma s}\}ds,$$

using Theorem 3.5 we get

$$u(t) \le ku(t_0) + \int_{t_0}^{t} \{k^2 u(t_0)\eta(s) + k\eta'(s)e^{\gamma s}\} \left(\exp \int_s^t k\eta(u)du\right) ds,$$

then $u(t) \le kMu(t_0) + \varepsilon(t)$, where $M = 1 + k\|\eta\|_1 e^{k\|\eta\|_1}$ and $\varepsilon(t) = ke^{k\|\eta\|_1} \int_{t_0}^{t} \eta'(s)e^{\gamma s}ds$.

Finally, we get for all $t \ge t_0$,

$$\|x(t)\| \le kM\|x(t_0)\|e^{-\gamma(t-t_0)} + \varepsilon(t)e^{-\gamma t}.$$

If we suppose that the function : $t \mapsto \varepsilon(t)e^{-\gamma t}$ vanishes, we obtain that the system (18) is uniformly practically asymptotically stable.

5 Conclusion

In this paper we have reduced the study of various integral inequalities to fixed point problems. We have also derived some general Gronwall-type results and have given examples of such results in the particular case of the Banach space $\mathcal{C}(J)$ using two different norms.

Acknowledgment

The authors wish to thank the reviewers for their valuable and careful comments.

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