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Monoaxial Attitude Stabilization of a Rigid Body under Vanishing Restoring Torque

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Abstract: The paper deals with the problem of monoaxial attitude stabilization of a rigid body. The possibility of implementing such a control system in which the restoring torque tends to zero as time increases is studied. With the aid of the Lyapunov direct method and the differential inequalities theory, conditions under which an equilibrium position of the body is stable with respect to all variables as well as with respect to a part of variables are derived. The results of a numerical modeling are presented to demonstrate the effectiveness of the proposed approaches.

Keywords: *rigid body; monoaxial attitude stabilization; dissipation; asymptotic stability; Lyapunov function; differential inequality.*

Mathematics Subject Classification (2010): 34H15, 70Q05, 93C10.

1 Introduction

In problems of a rigid body attitude control, restoring torques are usually the basis of control system functioning. However, attitude stabilization of a body is impossible without damping torques ensuring suppression of a body oscillations in a neighborhood of a stable equilibrium position. Therefore, the question how to create a damping torque and to design a specific damping mechanism is one of the main problems that should be solved for practical realization of attitude control systems [6, 7, 9, 14, 20, 24]. At the same time, due to limited resources of control systems based on jet propulsion, there arises a natural question on the possibility of implementing such a control system in which the restoring torque tends to zero as time increases.

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A more general formulation of the problem suggests that a mechanical system with dissipative and potential forces is given. Let the system admit an asymptotically stable equilibrium position. Consider the case of an evolution of the potential forces. We assume that the evolution consists of the appearance of a scalar positive time-varying multiplier at the vector of these forces. The issue of preservation of stability of the equilibrium position despite the evolution of potential forces is stated.

The stability problem in mechanical systems with a nonstationary parameter at potential forces was considered in many works, see, for example, [1, 3, 10, 13, 15, 22, 23, 26, 28] and the references cited therein. However, it should be noted that a few results were obtained for the case of vanishing potential forces.

In this contribution, the issue of monoaxial attitude stabilization of a rigid body is studied. It is assumed that the body is under the action of a time-invariant essentially nonlinear dissipative torque and a time-varying restoring torque that vanishes as time increases. Using the differential inequalities theory [11, 16–18] and approaches proposed in [1, 3, 23], conditions providing stability with respect to all variables as well as with respect to a part of variables of an equilibrium position of the body are derived.

2 Statement of the Problem

Consider a rigid body rotating about its mass center O with angular velocity $\boldsymbol{\omega}$. Assume that the axes $Oxyz$ are principal central axes of inertia of the body. Differential equations governing the attitude motion of the body under control torque $\mathbf{M}$ have the following form

$$\mathbf{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} = \mathbf{M}, \tag{1}$$

where $\mathbf{J} = \mathrm{diag}\{A, B, C\}$ is a body inertia tensor in the axes $Oxyz$.

Let unit vectors $\mathbf{s}$ and $\mathbf{r}$ be given, the vector $\mathbf{s}$ be constant in the inertial space and the vector $\mathbf{r}$ be constant in the body-fixed frame. Then the vector $\mathbf{s}$ rotates with respect to the coordinate system $Oxyz$ with angular velocity $-\boldsymbol{\omega}$. Hence,

$$\dot{\mathbf{s}} = -\boldsymbol{\omega} \times \mathbf{s}. \tag{2}$$

Thus, we will consider the differential system consisting of the Euler dynamic equations (1) and the Poisson kinematic equations (2).

Let the torque $\mathbf{M}$ be a sum of the dissipative component $\mathbf{M}_d$ and the restoring one $\mathbf{M}_r$: $\mathbf{M} = \mathbf{M}_d + \mathbf{M}_r$. We will assume that the dissipative torque is defined by the formula $\mathbf{M}_d = -\partial W(\boldsymbol{\omega})/\partial \boldsymbol{\omega}$, where $W(\boldsymbol{\omega})$ is a continuously differentiable for $\boldsymbol{\omega} \in \mathbb{R}^3$ positive definite homogeneous function of the order $\nu + 1$, $\nu > 1$. It should be noted that mechanical systems with essentially nonlinear dissipative forces were considered, for instance, in [19, 21]. In particular, such type forces arise when a body rotates in a viscous medium [19]. Moreover, it is worth mentioning that essentially nonlinear control laws are more robust with respect to the impact of delay and nonstationary perturbations than linear ones, see [2, 5].

The restoring torque $\mathbf{M}_r$ should be chosen such that the torque $\mathbf{M}$ ensures monoaxial stabilization of a rigid body [29]: the system of equations (1), (2) should admit the asymptotically stable equilibrium position

$$\boldsymbol{\omega} = \mathbf{0}, \qquad \mathbf{s} = \mathbf{r}. \tag{3}$$

From the results of [25, 29] it follows that the torque $\mathbf{M}_r$ can be determined by the formula

$$\mathbf{M}_r = -a\|\mathbf{s}-\mathbf{r}\|^{\mu-1}\mathbf{s}\times\mathbf{r},$$

where $\mu \geq 1$, $a > 0$, and $\|\cdot\|$ denotes the Euclidean norm of a vector.

Next, consider the case where the restoring torque evolves with time, and the evolution is expressed in the appearance of a scalar multiplier $h(t)$ at the vector of the torque. Thus, system (1) can be rewritten as follows

$$\mathbf{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times\mathbf{J}\boldsymbol{\omega} = -\frac{\partial W(\boldsymbol{\omega})}{\partial \boldsymbol{\omega}} - h(t)a\|\mathbf{s}-\mathbf{r}\|^{\mu-1}\mathbf{s}\times\mathbf{r}. \tag{4}$$

Assume that $h(t)$ is a positive and continuously differentiable for $t \geq 0$ function, and $h(t) \to 0$ as $t \to +\infty$. Hence, the restoring torque vanishes as time increases. We will look for conditions under which the equilibrium position (3) of system (2), (4) is stable with respect to all or a part of variables.

3 Main Results

First, according to the approach proposed in [23], construct a Lyapunov function in the form

$$V_1 = \frac{1}{2}\boldsymbol{\omega}^\top\mathbf{J}\boldsymbol{\omega} + \frac{ah(t)}{\mu+1}\|\mathbf{s}-\mathbf{r}\|^{\mu+1}.$$

Differentiating the function with respect to system (2), (4), we obtain

$$\dot{V}_1 = -(\nu+1)W(\boldsymbol{\omega}) + \frac{a\dot{h}(t)}{\mu+1}\|\mathbf{s}-\mathbf{r}\|^{\mu+1} \leq \varphi(t)V_1,$$

where $\varphi(t) = \max\left\{0;\, \dot{h}(t)/h(t)\right\}$.

Thus, on the basis of the theory of differential inequalities, see [11, 16], we arrive at the following theorem.

Theorem 3.1 *If there exists a constant $L > 0$ such that $\int_0^t \varphi(\tau)d\tau \leq L$ for $t \geq 0$, then the equilibrium position* (3) *of system* (2), (4) *is stable with respect to $\boldsymbol{\omega}$.*

Corollary 3.1 *If $\dot{h}(t) \leq 0$ for $t \geq 0$, then the equilibrium position* (3) *of system* (2), (4) *is stable with respect to $\boldsymbol{\omega}$.*

Next, we will show that with the aid of more precise estimates of the derivative of V_1, conditions of the asymptotic stability with respect to $\boldsymbol{\omega}$ of the equilibrium position (3) can be derived.

Let $\dot{h}(t) \leq 0$ for $t \geq 0$. Denote

$$z_1 = \frac{1}{2}\boldsymbol{\omega}^\top\mathbf{J}\boldsymbol{\omega}, \quad z_2 = \frac{ah(t)}{\mu+1}\|\mathbf{s}-\mathbf{r}\|^{\mu+1}.$$

Then $V_1 = z_1 + z_2$.

Choose a positive number Δ. We obtain

$$\dot{V}_1 \leq -cz_1^{\nu+1} - \psi(t)z_2^{\nu+1}$$

for $t \geq 0$, $z_1 \geq 0$, $0 \leq z_2 \leq \Delta$, where

$$c = (\nu + 1) \min_{\|\boldsymbol{\omega}\|=1} \frac{W(\boldsymbol{\omega})}{(\boldsymbol{\omega}^\top \mathbf{J}\boldsymbol{\omega}/2)^{\frac{\nu+1}{2}}} > 0, \qquad \psi(t) = -\frac{\dot{h}(t)}{h(t)} \Delta^{\frac{1-\nu}{2}}.$$

Hence, the differential inequality

$$\dot{V}_1 \leq -\tilde{\varphi}(t) V_1^{\frac{\nu+1}{2}} \tag{5}$$

holds in a neighborhood of the equilibrium position (3) and for all $t \geq 0$. Here

$$\tilde{\varphi}(t) = \min_{u_1 \geq 0,\ u_2 \geq 0,\ u_1+u_2=1} \left(c\, u_1^{\nu+1} + \psi(t)\, u_2^{\nu+1} \right).$$

It can be shown that

$$\tilde{\varphi}(t) = \frac{c\psi(t)}{\left(c^{\frac{2}{\nu-1}} + \psi^{\frac{2}{\nu-1}}(t) \right)^{\frac{\nu-1}{2}}}.$$

Assume that for a solution $(\boldsymbol{\omega}^\top(t), \mathbf{s}^\top(t))^\top$ of (2), (4) the condition

$$\frac{ah(t)}{\mu+1} \|\mathbf{s}(t) - \mathbf{r}\|^{\mu+1} \leq \Delta$$

is fulfilled on an interval $[t_0, t_1]$, where $0 \leq t_0 < t_1$. Then, integrating differential inequality (5), we obtain

$$\begin{gathered} \frac{1}{2}\boldsymbol{\omega}^\top(t)\mathbf{J}\boldsymbol{\omega}(t) + \frac{ah(t)}{\mu+1}\|\mathbf{s}(t) - \mathbf{r}\|^{\mu+1} = \hat{V}_1(t) \\ \leq \hat{V}_1(t_0) \left(1 + \frac{\nu-1}{2} \hat{V}_1^{\frac{\nu-1}{2}}(t_0) \int_{t_0}^{t} \tilde{\varphi}(\tau) d\tau \right)^{-\frac{2}{\nu-1}} \end{gathered} \tag{6}$$

for $t \in [t_0, t_1]$. Here $\hat{V}_1(t) = V_1(t, \boldsymbol{\omega}(t), \mathbf{s}(t))$.

Thus, we arrive at the following theorem.

Theorem 3.2 *If* $\dot{h}(t) \leq 0$ *for* $t \geq 0$ *and*

$$\int_0^t \tilde{\varphi}(\tau) d\tau \to +\infty \quad \text{as} \quad t \to +\infty, \tag{7}$$

then the equilibrium position (3) *of system* (2), (4) *is asymptotically stable with respect to* $\boldsymbol{\omega}$.

Example 3.1 Let the nonstationary multiplier $h(t)$ in system (4) be defined by the formula $h(t) = e^{-\beta t}$, where $\beta = \text{const} > 0$. Then, for any $\beta > 0$ and any $\Delta > 0$, we obtain $\tilde{\varphi}(t) \equiv \text{const} > 0$. Hence, the equilibrium position (3) of system (2), (4) is asymptotically stable with respect to $\boldsymbol{\omega}$.

Remark 3.1 Function $\tilde{\varphi}(t)$ depends on the chosen number Δ. To guarantee that the equilibrium position is asymptotically stable with respect to $\boldsymbol{\omega}$, it is sufficient to find at least one value of Δ for which condition (7) is fulfilled.

Remark 3.2 It is easy to verify that, the smaller the value of Δ, the more precise estimate (6). However, decreasing the value of Δ, we narrow the domain of initial conditions of solutions of system (2), (4) for which the estimate can be applied.

Remark 3.3 The use of estimate (6) does not permit us to obtain conditions of stability with respect to $\mathbf{s}$.

Really, for any $\Delta > 0$, the inequality $\tilde{\varphi}(t) \leq \psi(t)$ holds for $t \geq 0$. Hence,

$$\frac{1}{h(t)}\left(1+\frac{\nu-1}{2}\hat{V}_1^{\frac{\nu-1}{2}}(t_0)\int_{t_0}^{t}\tilde{\varphi}(\tau)d\tau\right)^{-\frac{2}{\nu-1}}$$

$$\geq \frac{1}{h(t)}\left(1-\frac{\nu-1}{2}\hat{V}_1^{\frac{\nu-1}{2}}(t_0)\Delta^{\frac{1-\nu}{2}}\int_{t_0}^{t}\frac{\dot{h}(\tau)}{h(\tau)}d\tau\right)^{-\frac{2}{\nu-1}}$$

$$= \frac{1}{h(t)}\left(1-\frac{\nu-1}{2}\hat{V}_1^{\frac{\nu-1}{2}}(t_0)\Delta^{\frac{1-\nu}{2}}\log\frac{h(t)}{h(t_0)}\right)^{-\frac{2}{\nu-1}} \to +\infty \quad \text{as} \quad t \to +\infty.$$

Finally in this section, we consider one more approach to a Lyapunov function construction for system (2), (4) which permits us to find stability conditions not only with respect to $\boldsymbol{\omega}$, but also with respect to all variables.

Let

$$V_2 = \frac{1}{2}\boldsymbol{\omega}^\top \mathbf{J}\boldsymbol{\omega} + \frac{ah(t)}{\mu+1}\|\mathbf{s}-\mathbf{r}\|^{\mu+1} + \gamma h^{\sigma}(t)\|\mathbf{s}\times\mathbf{r}\|^{\beta-1}\boldsymbol{\omega}^\top \mathbf{J}(\mathbf{s}\times\mathbf{r}),$$

where $\gamma > 0$, $\beta \geq 1$, $\sigma > 0$. Then there exist positive numbers α_1, α_2, α_3 such that

$$\alpha_1\|\boldsymbol{\omega}\|^2 + \frac{ah(t)}{\mu+1}\|\mathbf{s}-\mathbf{r}\|^{\mu+1} - \alpha_3\gamma h^{\sigma}(t)\|\boldsymbol{\omega}\|\|\mathbf{s}-\mathbf{r}\|^{\beta} \leq V_2$$

$$\leq \alpha_2\|\boldsymbol{\omega}\|^2 + \frac{ah(t)}{\mu+1}\|\mathbf{s}-\mathbf{r}\|^{\mu+1} + \alpha_3\gamma h^{\sigma}(t)\|\boldsymbol{\omega}\|\|\mathbf{s}-\mathbf{r}\|^{\beta}.$$

Differentiating function V_2 with respect to system (2), (4), we obtain

$$\dot{V}_2 = -(\nu+1)W(\boldsymbol{\omega}) + \frac{a\dot{h}(t)}{\mu+1}\|\mathbf{s}-\mathbf{r}\|^{\mu+1} + \sigma\gamma h^{\sigma-1}(t)\dot{h}(t)\|\mathbf{s}\times\mathbf{r}\|^{\beta-1}\boldsymbol{\omega}^\top \mathbf{J}(\mathbf{s}\times\mathbf{r})$$

$$+\gamma h^{\sigma}(t)\|\mathbf{s}\times\mathbf{r}\|^{\beta-1}(\mathbf{s}\times\mathbf{r})^\top\left(-\boldsymbol{\omega}\times\mathbf{J}\boldsymbol{\omega} - \frac{\partial W(\boldsymbol{\omega})}{\partial\boldsymbol{\omega}} - h(t)a\|\mathbf{s}-\mathbf{r}\|^{\mu-1}(\mathbf{s}\times\mathbf{r})\right)$$

$$+\gamma h^{\sigma}(t)\boldsymbol{\omega}^\top \mathbf{J}\,\frac{\partial\left(\|\mathbf{s}\times\mathbf{r}\|^{\beta-1}(\mathbf{s}\times\mathbf{r})\right)}{\partial\mathbf{s}}\left(-\boldsymbol{\omega}\times\mathbf{s}\right).$$

Assume that $\dot{h}(t) \leq 0$ for $t \geq 0$. It is easy to verify that one can choose positive constants α_4, α_5, α_6 and δ such that the inequality

$$\dot{V}_2 \leq -\alpha_4\left(\|\boldsymbol{\omega}\|^{\nu+1} + \gamma h^{\sigma+1}(t)\|\mathbf{s}-\mathbf{r}\|^{\beta+\mu}\right) + \alpha_5\gamma h^{\sigma-1}(t)\dot{h}(t)\|\mathbf{s}-\mathbf{r}\|^{\beta}\|\boldsymbol{\omega}\|$$

$$+\alpha_6\gamma h^{\sigma}(t)\left(\|\mathbf{s}-\mathbf{r}\|^{\beta}\|\boldsymbol{\omega}\|^2 + \|\mathbf{s}-\mathbf{r}\|^{\beta}\|\boldsymbol{\omega}\|^{\nu} + \|\mathbf{s}-\mathbf{r}\|^{\beta-1}\|\boldsymbol{\omega}\|^2\right)$$

holds for $t \geq 0$, $\boldsymbol{\omega} \in \mathbb{R}^3$, $\|\mathbf{s}-\mathbf{r}\| < \delta$.

With the aid of the substitution $\xi = h^{\frac{1}{\mu+1}} \|\mathbf{s} - \mathbf{r}\|$, we arrive at the estimates

$$\alpha_1 \|\boldsymbol{\omega}\|^2 + \frac{a}{\mu+1} \xi^{\mu+1} - \alpha_3 \gamma h^{\sigma - \frac{\beta}{\mu+1}}(t) \|\boldsymbol{\omega}\| \xi^{\beta} \le V_2$$

$$\le \alpha_2 \|\boldsymbol{\omega}\|^2 + \frac{a}{\mu+1} \xi^{\mu+1} + \alpha_3 \gamma h^{\sigma - \frac{\beta}{\mu+1}}(t) \|\boldsymbol{\omega}\| \xi^{\beta},$$

$$\dot{V}_2 \le -\alpha_4 \left(\|\boldsymbol{\omega}\|^{\nu+1} + \gamma h^{\sigma - \frac{\beta-1}{\mu+1}}(t) \xi^{\beta+\mu} \right) + \alpha_5 \gamma h^{\sigma - 1 - \frac{\beta}{\mu+1}}(t) \dot{h}(t) \|\boldsymbol{\omega}\| \xi^{\beta}$$

$$+ \alpha_6 \gamma h^{\sigma - \frac{\beta}{\mu+1}}(t) \left(\|\boldsymbol{\omega}\|^2 + \|\boldsymbol{\omega}\|^{\nu} \right) \xi^{\beta} + \alpha_6 \gamma h^{\sigma - \frac{\beta-1}{\mu+1}}(t) \|\boldsymbol{\omega}\|^2 \xi^{\beta-1}.$$

Hence, if $\beta \ge \mu\nu$, $\sigma \ge \beta/\mu$, γ is sufficiently small, $\|\mathbf{s} - \mathbf{r}\| < \delta$, and

$$|\dot{h}(t)| \le L h^{1 + \frac{\beta - \sigma\mu}{\beta+\mu}}(t) \quad \text{for} \quad t \ge 0, \tag{8}$$

where $L = \text{const} > 0$, then

$$\frac{1}{2} \left(\alpha_1 \|\boldsymbol{\omega}\|^2 + \frac{a}{\mu+1} \xi^{\mu+1} \right) \le V_2 \le 2 \left(\alpha_2 \|\boldsymbol{\omega}\|^2 + \frac{a}{\mu+1} \xi^{\mu+1} \right), \tag{9}$$

$$\dot{V}_2 \le -\frac{1}{2} \alpha_4 h^{\sigma - \frac{\beta-1}{\mu+1}}(t) \left(\|\boldsymbol{\omega}\|^{\nu+1} + \gamma \xi^{\beta+\mu} \right) \le -\alpha_7 h^{\sigma - \frac{\beta-1}{\mu+1}}(t) V_2^{\frac{\beta+\mu}{\mu+1}}. \tag{10}$$

Here α_7 is a positive constant.

Using estimates (9) and (10), we obtain that if there exist numbers β and σ such that $\beta \ge \mu\nu$, $\sigma \ge \beta/\mu$, inequality (8) is valid, and

$$h^{\frac{\beta-1}{\mu+1}}(t) \int_0^t h^{\sigma - \frac{\beta-1}{\mu+1}}(\tau)\, d\tau \to +\infty \quad \text{as} \quad t \to +\infty, \tag{11}$$

then the equilibrium position (3) of system (2), (4) is asymptotically stable with respect to all variables.

Denote $\theta = \sigma - \beta/\mu$. Then conditions (8) and (11) can be rewritten as follows

$$|\dot{h}(t)| \le L h^{1 - \frac{\theta\mu}{\beta+\mu}}(t) \quad \text{for} \quad t \ge 0,$$

$$h^{\frac{\beta-1}{\mu+1}}(t) \int_0^t h^{\theta + \frac{\beta+\mu}{\mu(\mu+1)}}(\tau)\, d\tau \to +\infty \quad \text{as} \quad t \to +\infty.$$

It is easy to see that, to derive less conservative stability conditions, we should take $\beta = \mu\nu$. As a result, we obtain the following theorem.

Theorem 3.3 *If $\dot{h}(t) \le 0$ for $t \ge 0$, and there exist positive numbers θ and L such that*

$$|\dot{h}(t)| \le L h^{1 - \frac{\theta}{\nu+1}}(t) \quad \textit{for} \quad t \ge 0, \tag{12}$$

$$h^{\frac{\mu\nu-1}{\mu+1}}(t) \int_0^t h^{\theta + \frac{\nu+1}{\mu+1}}(\tau)\, d\tau \to +\infty \quad \textit{as} \quad t \to +\infty,$$

then the equilibrium position (3) *of system* (2), (4) *is asymptotically stable with respect to all variables.*

Corollary 3.2 *If $\dot{h}(t) \leq 0$ for $t \geq 0$, there exist positive numbers θ and L such that condition* (12) *is valid, and*

$$h^{\frac{\mu\nu-1}{\mu+1}}(t)\left(1+\int_0^t h^{\theta+\frac{\nu+1}{\mu+1}}(\tau)\,d\tau\right) \geq \rho \quad for \;\; t \geq 0,$$

where $\rho = \mathrm{const} > 0$, then the equilibrium position (3) *of system* (2), (4) *is stable with respect to all variables and asymptotically stable with respect to $\boldsymbol{\omega}$.*

Example 3.2 Let the nonstationary multiplier $h(t)$ in system (4) be defined by the formula $h(t) = (t+1)^{\alpha}$, where $\alpha < 0$. In this case Theorem 3.3 and Corollary 3.2 provide less conservative stability conditions for $\theta = 0$.

We obtain that if $\alpha > -1/\nu$, then the equilibrium position (3) of system (2), (4) is asymptotically stable with respect to all variables, whereas if $\alpha = -1/\nu$, then the equilibrium position is stable with respect to all variables and asymptotically stable with respect to $\boldsymbol{\omega}$.

Remark 3.4 Recently, attention was paid to the problems of synchronization in various nonlinear systems such as dumbbell satellites [8], coupled systems [27], dissimilar and uncoupled rotating systems [12]. As the stability properties are important in studying oscillations in such systems, it seems that the results obtained in this paper may be extended to the mentioned classes of nonlinear systems.

4 Results of a Numerical Simulation

In this section, we demonstrate the previous theoretical results by means of a numerical simulation. Consider the monoaxial attitude stabilization of a rigid body with the inertia tensor $\mathbf{J} = \mathrm{diag}\{1.0,\, 1.2,\, 0.8\}$ in the equilibrium position (3). Denote the unit vectors of the body-fixed frame $Oxyz$ by $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ and the direction cosines of the unit vector $\mathbf{r}$ in the body-fixed frame $Oxyz$ by $\gamma_1, \gamma_2, \gamma_3$. Let $\mathbf{r}$ be chosen as $\mathbf{r} = \frac{1}{\sqrt{3}}\mathbf{r}_1 + \frac{1}{\sqrt{3}}\mathbf{r}_2 + \frac{1}{\sqrt{3}}\mathbf{r}_3$. So, in the equilibrium position (3) the direction cosines $\gamma_1, \gamma_2, \gamma_3$ are equal to $1/\sqrt{3}$.

Assume that a positive definite homogeneous dissipative function W is defined by the formula

$$W = \frac{3}{8}\left(\omega_x^{8/3} + \omega_y^{8/3} + \omega_z^{8/3}\right).$$

Here ω_x, ω_y, ω_z are components of the vector $\boldsymbol{\omega}$. In this case $\nu = 5/3$, and the dissipative torque is $\mathbf{M}_d = -\left(\omega_x^{5/3},\, \omega_y^{5/3},\, \omega_z^{5/3}\right)^{\top}$.

Choose the restoring torque as a linear function of $\mathbf{s}$ ($\mu = 1$). Such approach is commonly used for satellite attitude stabilization, see [25, 29]. In particular, in [4], it was applied to the problem of monoaxial satellite stabilization in the orbital frame. Let

$$\mathbf{M}_r = -\frac{h(t)}{5\sqrt{3}}\,\mathbf{s} \times \mathbf{r},$$

where $h(t) = (t+0.1)^{\alpha}$, $\alpha = \mathrm{const} < 0$. We will consider two values of the parameter α: 1) $\alpha = -1/5$ and 2) $\alpha = -12/5$.

In the first case, in accordance with Theorem 3.3, the equilibrium position (3) of system (2), (4) is asymptotically stable with respect to all variables $\gamma_1, \gamma_2, \gamma_3, \omega_x, \omega_y, \omega_z$ (see Figs. 1 and 2).

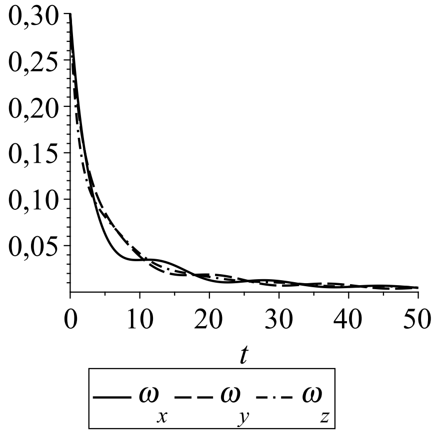

Figure 1: Angular velocity for $\alpha = -1/5$.

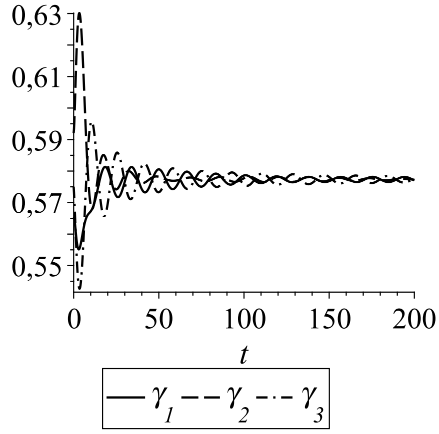

Figure 2: Direction cosines for $\alpha = -1/5$.

In the second case, in accordance with Theorem 3.2, the equilibrium position is asymptotically stable with respect to $\omega_x, \omega_y, \omega_z$ (see Fig. 3). At the same time, Fig. 4 demonstrates that there is no asymptotic stability with respect to $\gamma_1, \gamma_2, \gamma_3$.

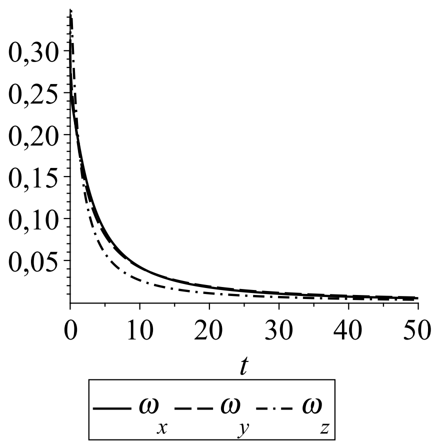

Figure 3: Angular velocity for $\alpha = -12/5$.

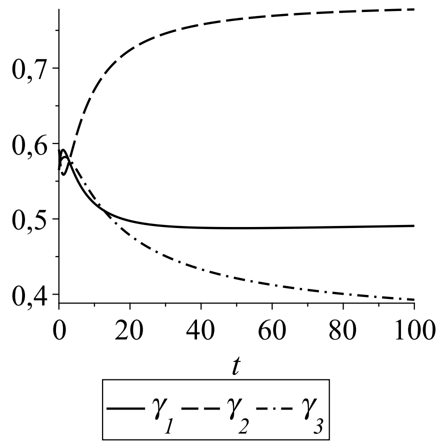

Figure 4: Direction cosines for $\alpha = -12/5$.

In both cases one and the same set of initial conditions was taken. The initial values of "aircraft" angles $\varphi(0) = 0.8$, $\psi(0) = 1.0$, $\theta(0) = -0.6$ result in the following initial values of direction cosines: $\gamma_1(0) = 0.5646424737$, $\gamma_2(0) = 0.5920595303$, $\gamma_3(0) = 0.5750168603$. The initial values of angular velocity projections are $\omega_x(0) = \omega_y(0) = \omega_z(0) = 0.3$.

5 Conclusion

The method of differential inequalities is a powerful tool for the stability analysis of nonlinear systems. In the present paper, the method is used for the investigation of the problem of monoaxial attitude stabilization of a rigid body. The possibility of implementing such a control system in which the restoring torque tends to zero as time increases is studied. The practical use of the investigation is connected with the challenge of propellant economy in control systems. With the aid of the Lyapunov direct method and the differential inequalities theory, stability conditions of an equilibrium position of the body are derived. It should be noted that Theorem 3.1 provides conditions of stability with respect to the angular velocity, in Theorem 3.2 conditions of the asymptotic stability with respect to the angular velocity are given, whereas, under the conditions of Theorem 3.3, we can guarantee the asymptotic stability with respect to all variables.

An interesting direction for further research is the application of the proposed approaches to the problem of three-axial stabilization of a rigid body.

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