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Boundedness of the New Modified Hyperchaotic Pan System

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Abstract: In this paper, we attempt to investigate the ultimate bound and positively invariant set for the new modified hyperchaotic Pan system using a technique combining the generalized Lyapunov function theory and optimization. For this system, we derive a four-dimensional ellipsoidal ultimate bound and positively invariant set. Furthermore, the two-dimensional parabolic ultimate bound with respect to $x-z$ is established. Finally, a numerical example is provided to illustrate the main result.

Keywords: *Pan system; upper bounds.*

Mathematics Subject Classification (2010): 65P20, 65P30, 65P40.

1 Introduction

In the last four decades, chaos as a very interesting nonlinear phenomenon has been intensively studied. Hyperchaotic system is usually defined as a chaotic system with more than one positive Lyapunov exponent. It is even more complicated than chaotic systems and has more unstable manifolds. At the same time, due to its theoretical and practical applications in technological fields, such as secure communications, lasers, nonlinear circuits, control, synchronization, hyperchaos has recently become a central topic in the research of nonlinear sciences.

In particular, the ultimate boundedness is very important for the study of the qualitative behavior of a chaotic system. If one can show that a chaotic or a hyperchaotic system under consideration has a globally attractive set, one knows that the system cannot have the equilibrium points, periodic or quasi-periodic solutions, or other chaotic or hyperchaotic attractors existing outside the attractive set. This greatly simplifies the analysis

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of dynamics of the system of a chaotic or hyperchaotic system [7]. The boundedness of a chaotic system also plays an important role in chaos control and chaos synchronization.

Such an estimation is quite difficult to achieve technically, however, several works on this topic were realized for some $3D$ and $4D$ dynamical systems [2, 4–6, 8–15].

Furthermore, there are no unified methods for constructing the Lyapunov functions to study the boundedness of the chaotic systems. Therefore, it is necessary to study the boundedness of the hyperchaotic systems. In the present paper, we investigate the ultimate bound and positively invariant set for the new modified hyperchaotic Pan system using a technique combining the generalized Lyapunov function theory and optimization. First, we derive an ellipsoidal ultimate bound and positively invariant set. Then we obtain a two-dimensional parabolic ultimate bound with respect to $x - z$. Finally, a numerical example is provided to illustrate the main result.

2 The Ultimate Bound and Positively Invariant Set for the New Modified Hyperchaotic Pan System

- Consider the system

$$\dot{X} = f(X), \tag{1}$$

where $X \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $X = (x_1, x_2, ..., x_n)^T$, $t_0 \geq 0$ is the initial time, and $X(t, t_0, X_0)$ is a solution to system (1) satisfying $X(t_0, t_0, X_0) = X_0$, which for simplicity is denoted by $X(t)$. Assume $\Omega \in \mathbb{R}^n$ is a compact set.

- Define the distance between the solution $X(t, t_0, X_0)$ and the set Ω by $\rho(X(t, t_0, X_0), \Omega) = \inf_{Y \in \Omega} \|X(t, t_0, X_0) - Y\|$, and denote $\Omega_\varepsilon = \{X / \rho(X, \Omega) < \varepsilon\}$, Clearly, $\Omega \subset \Omega_\varepsilon$.

Definition 2.1 Suppose that there is a compact set $\Omega \subset \mathbb{R}^n$. If, for every $x_0 \in \mathbb{R}^n/\Omega$, $\lim_{t\to\infty} \rho(x(t), \Omega) = 0$, that is, for any $\varepsilon > 0$, there is a $T > t_0$, such that for $t \geq T$, $x(t, t_0, x_0) \subset \Omega_\varepsilon$, then the set Ω is called an ultimate bound for system (1). If, for any $x_0 \in \Omega$ and all $t \geq t_0$, $x(t, t_0, x_0) \subset \Omega$, then Ω is called the positively invariant set for system (1).

Consider the new modified hyperchaotic Pan system [1] :

$$\begin{cases} x' = ay - ax, \\ y' = cx - xz + u, \\ z' = xy - bz, \\ u' = -dy, \end{cases} \tag{2}$$

where a, b, c, d are real parameters. System (2) displays a typical hyperchaotic attractor when $(a, b, c, d) = \left(10, \frac{8}{3}, 28, 10\right)$, the corresponding three-dimensional phase diagrams in $(x - y - z)$, $(x - z - u)$ spaces are shown in Fig. 1.

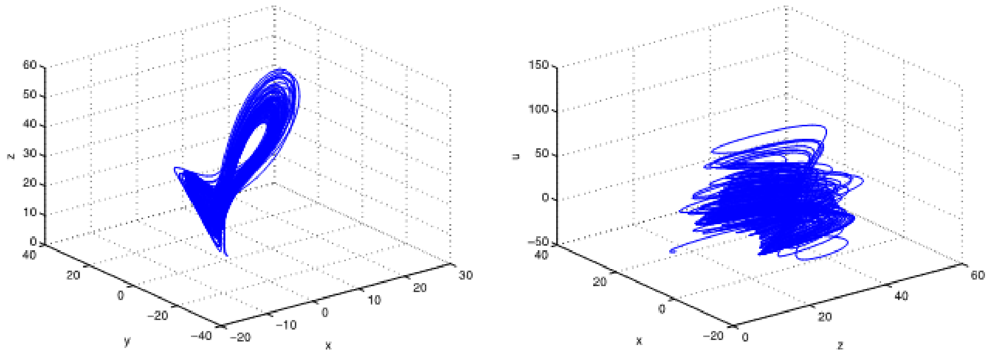

Figure 1: Hyperchaotic attractor of the new modified hyperchaotic Pan system (2) with $(a,b,c,d)=\left(10,\frac{8}{3},28,10\right)$ and the initial value $(x_0,y_0,z_0,w_0)=(1,1,1,1)$.

Some basic dynamical properties of the new modified hyperchaotic Pan system (2) were studied in [1]. But many properties of the system (2) remain to be uncovered. In the following, we will discuss the boundedness of the new modified hyperchaotic Pan system (2).

Theorem 2.1 *Denote*

$$\Omega=\left\{(x,y,z,u)/x^2+dy^2+d\left(z-c-\frac{a}{d}\right)^2+u^2\leq R^2\right\}, \tag{3}$$

where

$$R^2=\begin{cases}\dfrac{b^2(dc+a)^2}{4ad(b-a)}, & si\ b\geq 2a\\ \dfrac{(cd+a)^2}{d}, & si\ b<2a.\end{cases} \tag{4}$$

If $a>0$, $b>0$, $c>0$ and $d>0$, then all orbits of system (2), including hyperchaotic attractors, are trapped into a bounded region, and so the hyperellipsoid Ω is an ultimate bound and positively invariant set for system (2).

Proof. Define the following Lyapunov function

$$V=x^2+dy^2+d\left(z-c-\frac{a}{d}\right)^2+u^2. \tag{5}$$

Then, its time derivative along the orbits of system (2) is

$$\frac{1}{2}\dot{V}=-ax^2-dbz^2+b(cd+a)z=0. \tag{6}$$

That is to say, for $a>0$, $b>0$, $d>0$, the surface, defined by

$$\Gamma=\left\{(x,y,z,u)/ax^2+db\left(z-\frac{cd+a}{2d}\right)^2=\frac{b(cd+a)^2}{4d},\right\} \tag{7}$$

is an ellipsoid in $4D$ space for certain values of a, b, c and d. Outside Γ, we have $\dot{V} < 0$, while inside Γ, we have $\dot{V} > 0$. Since the function $V = x^2 + dy^2 + d\left(z - c - \frac{a}{d}\right)^2 + u^2$ is continuous on the closed set Γ, V can reach its maximum on the surface Γ. Denote the maximum value of V by R^2, that is $R^2 = \max V_{(x,y,z,u)\in\Gamma}$. Next, we use the Lagrange multiplier method to obtain the optimal value of V on Γ. Define

$$F = x^2 + dy^2 + d\left(z - c - \frac{a}{d}\right)^2 + u^2 + \lambda\left[ax^2 + db\left(z - \frac{cd+a}{2d}\right)^2 - \frac{b\,(cd+a)^2}{4d},\right] \tag{8}$$

and let

$$\begin{cases} \dfrac{\partial F(x,y,z,u)}{\partial x} = 2x + 2\lambda ax = 0, \\ \dfrac{\partial F(x,y,z,u)}{\partial y} = 2dy = 0, \\ \dfrac{\partial F(x,y,z,u)}{\partial z} = 2d\left(z - c - \dfrac{a}{d}\right) + 2\lambda db\left(z - \dfrac{cd+a}{2d}\right) = 0, \\ \dfrac{\partial F(x,y,z,u)}{\partial u} = 2u = 0, \\ \dfrac{\partial F(x,y,z,u)}{\partial \lambda} = ax^2 + db\left(z - \dfrac{cd+a}{2d}\right)^2 - \dfrac{b\,(cd+a)^2}{4d} = 0. \end{cases} . \tag{9}$$

Thus,

(i) When $\lambda \neq \dfrac{-1}{a}$, we have $(x,y,z,u) = (0,0,0,0)$ or $(x,y,z,u) = \left(0, 0, \dfrac{cd+a}{d}, 0\right)$ and $R^2 = \dfrac{(cd+a)^2}{d}$ or $R^2 = 0$ correspondingly.

(ii) When $\lambda = \dfrac{-1}{a}$, and $b \geq 2a$, we have $(x,y,z,u) = \left(\pm\dfrac{b\,(cd+a)\sqrt{b-2a}}{2\sqrt{ad}\,(a-b)}, 0, \dfrac{(cd+a)(2a-b)}{2d\,(a-b)}, 0\right)$ and $R^2 = \dfrac{b^2\,(dc+a)^2}{4ad\,(b-a)}$. Summarizing (i)–(ii) above, we have

$$R^2 = \begin{cases} \dfrac{b^2\,(dc+a)^2}{4ad\,(b-a)}, & \text{if } b \geq 2a, \\ \dfrac{(cd+a)^2}{d}, & \text{if } b < 2a. \end{cases} \tag{10}$$

For the set Ω, as shown in (3), we have $\Gamma \subset \Omega$. Next, we will show

$$\lim_{t\to\infty} \rho\,(X(t), \Omega) = 0, \tag{11}$$

using the reduction to absurdity, where $X(t) = (x(t), y(t), z(t), u(t))$. If (11) does not hold, we can conclude that the orbits of system (2) are outside Ω permanently, thus $\dot{V} < 0$. Therefore,$V(X(t))$ monotonously decreases outside Ω, which leads to the following result $\lim\limits_{t\to\infty} V(X(t)) = v^* > l$. Let $s = \inf\limits_{X\in D}\left(-\dot{V}(X(t))\right)$ where $D = \{X(t)/V^* \leq V(X(t)) \leq V(X(t_0))\}$, while t_0 is the initial time. Consequently, we have that s, V^* are positive constants, and $\dfrac{dV(X(t))}{dt} \leq -s$. As $t \to \infty$, we have $0 \leq V(X(t)) \leq$

$V(X(t_0)) - s(t - t_0) \to -\infty$ this is inconsistent. Therefore (11) actually holds, that is to say, Ω is the ultimate bound of system (2). Finally, to see that Ω is also the positively invariant set, reason as follows. Suppose V attains its maximum value on surface Γ at point $P_0(\widehat{x}_0, \widehat{y}_0, \widehat{z}_0, \widehat{u}_0)$. Since $\Gamma \subset \Omega$, for any point $X(t)$ on Ω and $X(t) \neq P_0$, we have $\dot{V}(X) < 0$, thus, any orbit $X(t)$ ($X(t) \neq P_0$) of system (2) will go into Ω. When $X(t) = P_0$, by the continuation theorem [3], $X(t)$ will also go into Ω. Summarizing the above, we conclude that Ω is the positively invariant set of system (2).

Corollary 2.1 *For $a > 0$, $b > 0$, $c > 0$ and $d > 0$, the solution of the system (2) is bounded by the conditions*

$$\begin{cases} |x| \le R, \\ |y| \le \dfrac{R}{\sqrt{d}}, \\ \dfrac{cd+a}{d} - \dfrac{R}{\sqrt{d}} \le z \le \dfrac{R}{\sqrt{d}} + \dfrac{cd+a}{d}, \\ |u| \le R. \end{cases} \tag{12}$$

Proof. Direct consequence of the previous theorem.

3 The Estimation of the Two-Dimensional Parabolic Ultimate Bound with Respect to x–z

Theorem 3.1 *When $b < 2a$, the system (2) has the following two-dimensional parabolic ultimate bound*

$$z \ge \frac{x^2}{2a}. \tag{13}$$

Proof. Define

$$V(t) = \frac{1}{2a}x^2(t) - z(t).$$

Then, its time derivative along the orbits of system (2) is

$$\dot{V} = \frac{1}{a}x\dot{x} - \dot{z} = -x^2 + bz.$$

Thus,

$$\dot{V} + bV = -x^2 + bz + \frac{b}{2a}x^2 - bz = \left(\frac{b}{2a} - 1\right)x^2.$$

When $b < 2a$, we have

$$\dot{V} + bV \le 0.$$

For any initial value $V(t_0) = V_0$, according to the comparison theorem, we have

$$V(t) \le V_0 e^{-b(t-t_0)} \to 0 \, (t \to \infty)$$

Thus

$$\lim_{t\to\infty} V(t) = \lim_{t\to\infty}\left[\frac{1}{2a}x^2(t) - z(t)\right] \le 0.$$

So, we get that system orbits satisfy the parabolic ultimate bound

$$z \ge \frac{x^2}{2a}.$$

This completes the proof.

4 Example

Consider the system (2), when $a = 10$, $b = \frac{8}{3}$, $c = 28$ and $d = 10$.

We have

$$V(x, y, z, u) = x^2 + 10y^2 + 10(z - 29)^2 + u^2,$$

$$\Gamma = \left\{ (x, y, z, u) / 10x^2 + \frac{80}{3}\left(z - \frac{29}{2}\right)^2 = \frac{20}{3} \times 29^2 \right\}$$

and

$$R^2 = \max V_{(x,y,z,u)\in\Gamma} = \frac{(cd + a)^2}{d} = 10 \times 29^2.$$

Therefore, the estimate of ultimate bound for system (2) is

$$\Omega = \left\{ (x, y, z, u) / x^2 + 10y^2 + 10(z - 29)^2 + u^2 \leq 10 \times 29^2 \right\}.$$

Consequently, we have

$$\begin{cases} |x| \leq 29 \times \sqrt{10}, \\ |y| \leq 29, \\ 0 \leq z \leq 58, \\ |u| \leq 29 \times \sqrt{10}. \end{cases}.$$

It is obvious that the orbits of system (2) locate in the section where $z \geq 0$.

5 Conclusion

In this paper, we have investigated the ultimate bound and positively invariant set for the new modified hyperchaotic Pan system. We have first derived a four-dimensional ellipsoidal ultimate bound and positively invariant set. Then, we have obtained a two-dimensional parabolic bound with respect to $x - z$, which shows that, in the four-dimensional space, the orbits of the system are located inside the parabolic cylinder $z \geq \frac{x^2}{2a}$, accordingly, we have also got $z \geq 0$. Finally, a numerical example is provided to illustrate the main result.

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