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Refinements of Some Pachpatte and Bihari Inequalities on Time Scales

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Abstract: In this study, some generalizations and refinements of some inequalities of Pachpatte and Bellman-Bihari types are established on arbitrary time domains using the time scale theory. The obtained results unify continuous and discrete inequalities and extend some results known in the literature. The paper ends up with two illustrative examples to highlight the utility of our results.

Keywords: *dynamic equations; time scale, Gronwall-Bellman inequality; Bellman-Bihari inequality.*

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1 Introduction

The Gronwall-Bellman and Bellman-Bihari integral inequalities play important roles in the study of qualitative and quantitative properties of differential equations [1–6]. Similarly, discrete Gronwall and Bihari inequalities have been developed for the analysis of difference equations [7]. New classes of differential and integral equations have been studied using Gronwall-Bellman-Pachpatte inequalities [5, 8, 9]. Recently, the time scaly theory, which was introduced in [10], gives a promising direction that unifies continuous

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and discrete analysis in a consistent way. Using this theory, many works (see, for instance, [11–16]) have investigated new Patchpatte-type and Gronwall-Bihari inequalities for dynamic equations defined on arbitrary time scales.

The aim of this paper is to extend some results on Patchpatte-type and Gronwall-Bihari inequalities for dynamic equations defined on time scales. On the one hand, some Pachpatte-type inequalities, containing in the right-hand side two nonlinear integral terms involving Lipshitz kind functions, are studied. Using elementary analytic methods, we investigate extensions of some continuous and discrete inequalities appearing in [5,8,9,17,18] to an arbitrary time scale and refine some Pachpatte-type inequalities given in [14, 19–23]. On the other hand, some Bellman-Bihari inequalities on time scale, including two nonlinear integral terms using class $\mathcal{S}$ or $\mathcal{T}$ functions, are introduced. Some similar inequalities have been studied for the continuous-time case in [3,4,6]. However, there are very few results for Bellman-Bihari inequalities on arbitrary time scales involving class S functions (see [24–26]). These inequalities can be applied to analyze qualitative and quantitative properties of integro-differential equations on time scales.

The rest of this paper is as follows. In Section 2, some basics on the time scale theory are recalled. In Section 3, some generalizations of Pachpatte-type inequalities on arbitrary time scale are presented. In Section 4, some new Bellman-Bihari inequalities on time scales are given. In the last section, some illustrative examples are presented to highlight the utility of our results.

2 Preliminaries on Time Scale

Let us consider the time scale $\mathbb{T}$ which is an arbitrary non-empty closed subset of $\mathbb{R}$. For $t \in \mathbb{T}$, we can define

- the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T}; s > t\}$,
- the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T}; s < t\}$,
- the graininess function $\mu : \mathbb{T} \to \mathbb{R}_+$ by $\mu(t) := \sigma(t) - t$.

An element $t \in \mathbb{T}$ is said to be right-dense if $\sigma(t) = t$, left-dense if $\rho(t) = t$, left-scattered if $\rho(t) < t$. If $\mathbb{T}$ has a left-scattered maximal element m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$ or else $\mathbb{T}^\kappa = \mathbb{T}$. For a function $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, we define the delta derivative $f^\Delta(t)$ at t (provided it exists) such that

$$f^\Delta(t) := \lim_{s \to t, s \neq \sigma(t)} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}.$$

Clearly, it becomes the usual derivative when $\mathbb{T} = \mathbb{R}$, i.e. $f^\Delta(t) = f'(t)$ and the usual forward difference operator $f^\Delta(t) = \Delta f(t)$, if $\mathbb{T} = \mathbb{Z}$. A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\Delta(t) = f(t)$ hods for all $t \in \mathbb{T}^\kappa$. The Cauchy integral of f is defined by

$$\int_a^t f(s)\Delta s = F(t) - F(a) \quad \text{for all } a, t \in \mathbb{T}.$$

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous (denoted by $f \in \mathcal{C}_{rd} := \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R})$), provided f is continuous at every right-dense points in $\mathbb{T}$ and $\lim_{s \to t^-} f(s)$ exists and is finite at every left-dense point $t \in \mathbb{T}$. A function $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is said to be rd-continuous if h defined by $h(t) = f(t, x(t))$ is rd-continuous for any continuous function $x : \mathbb{T} \to \mathbb{R}^n$. A rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ is said to be regressive (denoted by $f \in \mathcal{R} := \mathcal{R}(\mathbb{T}, \mathbb{R})$) if $\quad 1 + \mu(t)f(t) \neq 0 \quad$ for all $t \in \mathbb{T}^\kappa$. $\mathcal{R}^+ := \{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0 \; for\ all\ t \in \mathbb{T}\}$ is the set of all positively regressive elements of $\mathcal{R}$.

Theorem 2.1 *[27, 28] Let $t_0 \in \mathbb{T}$, $p \in \mathcal{R}$. The first order linear dynamic equation*

$$x^\Delta(t) = p(t)x, \;\; x(t_0) = 1$$

has a unique solution on $\mathbb{T}$ called the exponential function, denoted by $e_p(t, t_0)$.

To derive our main results, one must recall the Gronwall's inequality on time scale.

Theorem 2.2 *([29, Theorem 5.4]). Let $t_0 \in \mathbb{T}$, $x,f \in \mathcal{C}_{rd}$ and $p \in \mathcal{R}^+$. Then*

$$x^\Delta(t) \leq p(t)x(t) + f(t) \qquad \text{for all } t \in \mathbb{T}^\kappa$$

implies

$$x(t) \leq x(t_0)e_p(t, t_0) + \int_{t_0}^{t} e_p(t, \sigma(\tau))f(\tau)\Delta\tau \qquad \text{for all } t \in \mathbb{T}.$$

Lemma 2.1 *([27, Theorem 1.117]) Let $t_0 \in \mathbb{T}^\kappa$ and assume that $L : \mathbb{T} \times \mathbb{T}^\kappa \to \mathbb{R}$ is continuous at (t, t), where $t \in \mathbb{T}^\kappa$ with $t > t_0$. Also assume that $L^{\Delta_t}(t, .)$ is rd–continuous on $[t_0, \sigma(t)]$.*
Suppose that for each $\varepsilon > 0$ there exists a neighborhood U of t, independent of $\tau \in [t_0, \sigma(t)]$, such that

$$\left|L(\sigma(t), \tau) - L(s, \tau) - L^{\Delta_t}(t, \tau)(\sigma(t) - s)\right| \leq \varepsilon\,|\sigma(t) - s| \qquad \text{for all } s \in U,$$

where L^{Δ_t} denotes the derivative of L with respect to the first variable. Then

$$g(t) := \int_{t_0}^{t} L(t, \tau)\Delta\tau$$

implies

$$g^\Delta(t) = L(\sigma(t), t) + \int_{t_0}^{t} L^{\Delta_t}(t, \tau)\Delta\tau.$$

Lemma 2.2 *([24, Lemma 2.1]) Let $a, b \in \mathbb{T}$, and a delta differentiable function $r : [a, b]_{\mathbb{T}} \to]0, \infty[$ with $r^\Delta(t) \geq 0$ on $[a, b] \cap \mathbb{T}^\kappa$. Define*

$$G(x) = \int_{x_0}^{x} \frac{ds}{g(s)}, x > 0, x_0 > 0, \tag{1}$$

where $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ is positive and nondecreasing on $]0, \infty[$. Then, for each $t \in [a, b]_{\mathbb{T}}$ one has

$$G(r(t)) \leq G(r(a)) + \int_{a}^{t} \frac{r^\Delta(\tau)}{g(r(\tau))}\Delta\tau.$$

Lemma 2.3 *([30, Lemma 2.1]) Assume that $a \geq 0$, $p \geq q \geq 0$ and $p \neq 0$, then*

$$a^{\frac{q}{p}} \leq \frac{q}{p}K^{\frac{q-p}{p}}a + \frac{p-q}{p}K^{\frac{q}{p}}, \text{ for any } K > 0.$$

Definition 2.1

- A nondecreasing continuous function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{F}$ (see [3, Section 3]) if it satisfies the following conditions:
  $\star$ $g(x)$ is positive for $x \geq 0$;
  $\star\star$ $(1/y)g(x) \leq g(x/y)$, for $x \geq 0$ and $y > 0$ (or $(1/y)g(x) \leq g(x/y)$, for $x > 0$ and $y \geq 1$ (see [1, Section 5] as an equivalent characterization)).

- A nondecreasing continuous function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{S}$ (see [24]) if it satisfies the following conditions:
  $\star$ $g(x)$ is positive for $x > 0$;
  $\star\star$ $(1/y)g(x) \leq g(x/y)$ for $x \geq 0$ and $y \geq 1$.

- A strictly increasing continuous function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{T}$ if it satisfies the following conditions:
  $\star$ $g(x)$ is positive for $x > 0$;
  $\star\star$ $(1/y)g(x) \geq g(x/y)$ for $x \geq 0$ and $y \geq 1$.

Remark 2.1

- Any function of class $\mathcal{F}$ is of class $\mathcal{S}$. The converse is not true. For example, $f(x) = x^{\alpha},\ x \in \mathbb{R}_+,\ \alpha \in [0, 1]$, is of class $\mathcal{S}$ but is not of class $\mathcal{F}$.

- In [25], the authors introduce the class $\mathcal{F}$ as similar to class $\mathcal{S}$, without distinguishing slight difference between these two classes. In [26] class $\mathcal{S}$ functions are designed by the class $\mathcal{S}^*$.

3 Pachpatte-type Inequalities

In this section, we derive some new results on Pachpatte-type inequalities, which can be used in the analysis of differential equations on arbitrary time scales. We suppose that $t \geq t_0,\ t \in \mathbb{T}^{\kappa}$.

Theorem 3.1 *Assume that $u, f \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}_+)$ and $S : \mathbb{T} \times \mathbb{R}_+ \to \mathbb{R}_+$ is a rd-continuous function which satisfies*

$$0 \leq S(t, x) - S(t, y) \leq R(t, y)(x - y) \tag{2}$$

for $t \in \mathbb{T}$, $x \geq y \geq 0$ and

$$S^{\Delta_t}(t, 0) \geq 0,\ R^{\Delta_t}(t, 0) \geq 0, \tag{3}$$

for $t \in \mathbb{T}^{\kappa}$, where $R : \mathbb{T} \times \mathbb{R}_+ \to \mathbb{R}_+^*$ is a rd-continuous function. If $L(t, s)$ is defined as in Lemma 2.1 so that $L(t, s) \geq 0$ and $L^{\Delta_t}(t, s) \geq 0$ for $t, s \in \mathbb{T}$ with $s \leq t$, then

$$u(t) \leq c + \int_{t_0}^{t} f(\eta) \left(S(\eta, u(\eta)) + \int_{t_0}^{\eta} L(\eta, \tau) S(\tau, u(\tau)) \Delta\tau \right) \Delta\eta, \tag{4}$$

with $c \geq 0$, for all $t \in \mathbb{T}^{\kappa}$, implies

$$u(t) \leq c + \int_{t_0}^{t} f(\eta) \left(\Big[S(t_0, 0) + R(t_0, 0)c \Big] e_{A_1^*}(\eta, t_0) + \int_{t_0}^{\eta} e_{A_1^*}(\eta, \sigma(\tau)) B_1^*(\tau) \Delta\tau \right) \Delta\eta, \tag{5}$$

for all $t \in \mathbb{T}^\kappa$ *with*

$$A_1^*(t) = R(\sigma(t),0)f(t) + \frac{R^{\Delta_t}(t,0)}{R(t,0)} + L(\sigma(t),t) + \int_{t_0}^t L^{\Delta_t}(t,\tau)\Delta\tau, \quad t \in \mathbb{T}^\kappa,$$

and

$$B_1^*(t) = S^{\Delta_t}(t,0) + L(\sigma(t),t)S(t,0) + \int_{t_0}^t L^{\Delta_t}(t,\tau)S(\tau,0)\Delta\tau, \quad t \in \mathbb{T}^\kappa.$$

Proof. Let us set function $z(t)$ by the right-hand side of (4).
The Delta-derivative of z satisfies the following inequality

$$z^\Delta(t) \leq f(t)v(t)$$

with

$$v(t) = S(t,0) + R(t,0)z(t) + \int_{t_0}^t L(t,\tau)[S(\tau,0) + R(\tau,0)z(\tau)]\Delta\tau.$$

Using Lemma 2.1, one can easily obtain

$$\begin{aligned} v^\Delta(t) \quad &= \quad S^{\Delta_t}(t,0) + R(\sigma(t),0)z^\Delta(t) + R^{\Delta_t}(t,0)z(t) + \\ &\quad L(\sigma(t),t)\Big[S(t,0) + R(t,0)z(t)\Big] + \int_{t_0}^t L^{\Delta_t}(t,\tau)\Big[S(\tau,0) + R(\tau,0)z(\tau)\Big]\Delta\tau. \end{aligned}$$

It is easy to see that $v(t)$ is nonnegative nondecreasing function. Further, one gets

$$v^\Delta(t) \leq A_1^*(t)v(t) + B_1^*(t).$$

Theorem 2.2 yields the following inequality

$$v(t) \leq \Big[S(t_0,0) + R(t_0,0)c\Big]e_{A_1^*}(t,t_0) + \int_{t_0}^t e_{A_1^*}(t,\sigma(\tau))B_1^*(\tau)\Delta\tau.$$

Hence, one can deduce inequality (5).

For special forms of function S one can see that the proposed Pachpatte-type inequality is a generalization of some existing results.

Remark 3.1 Assume that $S(t,u(t)) = u(t)$.

- Let $t_0 = 0$. If $\mathbb{T} = \mathbb{R}$, Theorem 3.1 implies Theorem 2.1(a_1) of [9]. If $\mathbb{T} = \mathbb{N}_0$, Theorem 3.1 reduces to Theorem 2.5(c_1) in [9]. If $\mathbb{T} = \mathbb{N}_0$ and $L(t,s) = l(s)$ with $l(.)$ being a nonnegative function, then Theorem 3.1 validates Theorem 1.4.1 of [8].
- For an arbitrary time scale $\mathbb{T}$, one can easily obtain [21, Theorem 3.1]. Moreover, when $\mathbb{T} = \mathbb{T}^\kappa$, Theorem 3.1 includes [14, Lemma] and [19, Corrolary 4.9], if $L(t,s) = l(s)$ with $l(.)$ being a nonnegative rd–continuous function.
- The inequality given in Theorem 3.1, when $\mathbb{T} = \mathbb{T}^\kappa$, solves the integral approximation of Theorem 3.2 in [22] satisfied by an rd-continuous nonnegative function $u(.)$ designated to bound the solution of the considered nonlinear integro-differential equation. In this case, $L(t,.) = f(.)$ is Lebesgue Δ-integrable function ($f \in L^1_{\mathbb{T}} := L^1_{\mathbb{T}}(\mathbb{T},\mathbb{R}_+)$, for more information about the Lebesgue Δ-integration see [31]), we obtain $u(t) \leq Mc$, for all $t \geq t_0$ with $M = \frac{1}{2}(1 + \exp(2\|f\|_{L^1_{\mathbb{T}}}))$.

- Suppose that $L(t,.) = g(.)$, where g is a nonnegative rd–continuous function, if $\mathbb{T}$ is an arbitrary time scale, with Theorem 3.1 one can obtain the inequality proved in [26, Theorem 1].

As an extension of Theorem 3.1, one can derive an integral inequality involving positive real powers. In the following, it is supposed that $p \neq 0, p, q, r$ are real constants such that $0 \leq q, r \leq p$.

Proposition 3.1 *Assume that all conditions of Theorem 3.1 are satisfied except for inequalities (3). Then*

$$u^p(t) \quad \leq \quad c + \int_{t_0}^{t} f(\eta)\left(u^q(\eta) + \int_{t_0}^{\eta} L(\eta,\tau)S\big(\tau, u^r(\tau)\big)\Delta\tau\right)\Delta\eta, \tag{6}$$

for all $t \in \mathbb{T}^\kappa$, *implies*

$$u(t) \leq \left(c + \int_{t_0}^{t} f(\eta)\left[\left(\frac{q}{p}K^{\frac{q-p}{p}}c + \frac{p-q}{p}K^{\frac{q}{p}}\right)e_{A_2^*}(\eta, t_0) + \int_{t_0}^{\eta} e_{A_2^*}(\eta, \sigma(\tau))B_2^*(\tau)\Delta\tau\right]\Delta\eta\right)^{\frac{1}{p}},$$

for all $t \in \mathbb{T}^\kappa$, *for any* $K > 0$ *with*

$$\begin{aligned} A_2^*(t) \quad = \quad & \frac{q}{p}K^{\frac{q-p}{p}}f(t) + \frac{r}{q}K^{\frac{r-q}{p}}\Bigg(L(\sigma(t),t)R\Big(t, \frac{p-r}{p}K^{\frac{r}{p}}\Big) \\ & + \int_{t_0}^{t} L^{\Delta_t}(t,\tau)R\Big(\tau, \frac{p-r}{p}K^{\frac{r}{p}}\Big)\Delta\tau\Bigg),\ t \in \mathbb{T}^\kappa \end{aligned}$$

and

$$B_2^*(t) = L(\sigma(t),t)S\Big(t, \frac{p-r}{p}K^{\frac{r}{p}}\Big) + \int_{t_0}^{t} L^{\Delta_t}(t,\tau)S\Big(\tau, \frac{p-r}{p}K^{\frac{r}{p}}\Big)\Delta\tau, t \in \mathbb{T}^\kappa.$$

Proof. The proof is similar to the proof of Theorem 3.1. Hence, the details are omitted.

One can highlight that the last result generalizes some existing works as follows.

Remark 3.2

- The result of Proposition 3.1 holds for any arbitrary time scales. Setting $S(t, u(t)) = u(t)$, we see that the obtained inequality is as seen in Theorem 3.1 in [20].
- Proposition 3.1 can be viewed as a generalization of some results on some particular time scales. For example, letting $S(t, u(t)) = u(t)$ and $\mathbb{T} = \mathbb{R}$. If $K = 1$, one can easily derive Theorem 3.1 in [17].

One can extend the result of Theorem 3.1, changing the nonnegative constant c on the right-side of (4) by an increasing positive function.

Proposition 3.2 *Assume that all conditions of Theorem 3.1 are satisfied except for inequalities (3). Let* $a(.)$ *be a nondecreasing function in* $\mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}_+^*)$. *Suppose that*

$$S^{\Delta_t}(t,0)a(t) \geq S(t,0)a^{\Delta}(t), \quad R^{\Delta_t}(t,0) \geq 0 \qquad \textit{for all } \mathbb{T}^\kappa.$$

Then,

$$u(t) \leq a(t) + \int_{t_0}^{t} f(\eta) \left(S(\eta, u(\eta)) + \int_{t_0}^{\eta} L(\eta, \tau) S(\tau, u(\tau)) \Delta\tau \right) \Delta\eta, \qquad (7)$$

for all $t \in \mathbb{T}^\kappa$, *implies*

$$u(t) \leq a(t) \left(1 + \int_{t_0}^{t} f(\eta) \left[\Big[R(t_0, 0) + M(t_0, 0) \Big] e_{A_1^*}(\eta, t_0) + \int_{t_0}^{\eta} e_{A_1^*}(\eta, \sigma(\tau)) B_3^*(\tau) \Delta\tau \right] \Delta\eta \right),$$

for all $t \in \mathbb{T}^\kappa$, *with*

$$B_3^*(t) = M^{\Delta_t}(t, 0) + L(\sigma(t), t) M(t, 0) + \int_{t_0}^{t} L^{\Delta_t}(t, \tau) M(\tau, 0) \Delta\tau, \quad t \in \mathbb{T}^\kappa,$$

and

$$M(t, z) = \frac{1}{a(t)} S(t, a(t)z), \quad z \geq 0, \quad t \in \mathbb{T}.$$

Proof. Setting $w(t) = \frac{u(t)}{a(t)}$, one can reformulate (7) as

$$w(t) \leq 1 + \int_{t_0}^{t} f(\eta) \left(M(\eta, w(\eta)) + \int_{t_0}^{\eta} L(\eta, \tau) M(\tau, w(\tau)) \Delta\tau \right) \Delta\eta.$$

Clearly, M verifies relation (2), i.e.

$$M(t, x) - M(t, y) \leq R_1(t, y)(x - y), \quad x \geq y \geq 0, \quad t \in \mathbb{T},$$

where $R_1(t, y) = R(t, a(t)y)$. From our hypothesis we see that S and R_1 verify relation (3). Using Theorem 3.1, it yields

$$w(t) \leq 1 + \int_{t_0}^{t} f(\eta) \left[\Big[R(t_0, 0) + M(t_0, 0) \Big] e_{A_1^*}(\eta, t_0) + \int_{t_0}^{\eta} e_{A_1^*}(\eta, \sigma(\tau)) B_3^*(\tau) \Delta\tau \right] \Delta\eta.$$

This concludes the proof.

This proposition generalizes some well known inequalities.

Remark 3.3 Assume that $S(t, u(t)) = u(t)$.

- Take a special case in references [8, 21, 26] with $a(.) \neq 0$.
  For an arbitrary time scale $\mathbb{T}$, inequalities in Proposition 3.2 reduce to [21, Theorem 3.2]. If $L(t, s) = g(s)$ with $g \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}_+)$, then Proposition 3.2 includes [26, Theorem 2 (a)]. For $\mathbb{T} = \mathbb{N}_0$ and $t_0 = 0$, the result [8, Theorem 1.4.2] is a particular case of Proposition 3.2 where $L(t, s) = c(s)$ with c(.) being a nonnegative function defined on $\mathbb{N}_0$.

- If $\mathbb{T} = \mathbb{R}_+$ and $t_0 = 0$, then Proposition 3.2 generalizes [5, Theorem 1.7.4] when $L(t, s) = g(s)$ with $g(.)$ being a nonnegative continuous function on $\mathbb{R}_+$.

4 Bihari-type Inequalities

In this part, some new Gronwall-Bellman-Bihari type inequalities, containing in the right-hand side two nonlinear integral terms involving class $\mathcal{S}$ or $\mathcal{T}$ functions, are introduced. These inequalities can be applied to analyze qualitative and quantitative properties of integro-differential equations on time scales. In this section it is assumed that $t \geq t_0,\ t \in \mathbb{T}$.

Let us begin with the following inequality which will be used in the proof of the next results.

Theorem 4.1 *Let us consider $u, f \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}_+)$ and c is a positive constant. Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function which is nondecreasing positive on $]0, +\infty[$, $L : \mathbb{T} \times \mathbb{T} \to \mathbb{R}_+$ be a rd-continuous function and G be given by (1). If*

$$u(t) \leq c + \int_{t_0}^{t} f(\eta) \left(g(u(\eta)) + \int_{t_0}^{\eta} L(\eta, \tau) g(u(\tau)) \Delta\tau \right) \Delta\eta, \tag{8}$$

for $t \in \mathbb{T}$, then for all $t \in \mathbb{T}$ satisfying

$$G(c) + \int_{t_0}^{t} f(\eta) \left[1 + \int_{t_0}^{\eta} L(\eta, \tau) \Delta\tau \right] \Delta\eta \in Dom(G^{-1}),$$

we have

$$u(t) \leq G^{-1} \left(G(c) + \int_{t_0}^{t} f(\eta) \left[1 + \int_{t_0}^{\eta} L(\eta, \tau) \Delta\tau \right] \Delta\eta \right),$$

where G^{-1} is the inverse function of G.

Proof. Let us define function $z(t)$ by the right-hand side of (8). Then, we have $z(t_0) = c$ and

$$u(t) \leq z(t).$$

As g is a nondecreasing function, the Delta-derivative of $z(t)$ satisfies the following inequality

$$z^{\Delta}(t) \quad \leq \quad f(t) g(z(t)) \left[1 + \int_{t_0}^{t} L(t, \tau) \Delta\tau \right].$$

Dividing both sides by $g(z(t))$, one can get

$$\frac{z^{\Delta}(t)}{g(z(t))} \quad \leq \quad f(t) \left[1 + \int_{t_0}^{t} L(t, \tau) \Delta\tau \right].$$

Since $z(t)$ is nondecreasing, from Lemma 2.2, one can obtain

$$G(z(t)) \leq G(c) + \int_{t_0}^{t} f(\eta) \left[1 + \int_{t_0}^{\eta} L(\eta, \tau) \Delta\tau \right] \Delta\eta.$$

Further

$$z(t) \leq G^{-1} \left(G(c) + \int_{t_0}^{t} f(\eta) \left[1 + \int_{t_0}^{\eta} L(\eta, \tau) \Delta\tau \right] \Delta\eta \right).$$

This concludes the proof.

In the following, some results which can be considered as some extensions of Theorem 4.1 are investigated. The next corollary allows us to get a relaxed integral bound of an unknown function using the image of a continuous increasing function.

Corollary 4.1 *Let us consider $u, f \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}_+)$ and c is a positive constant. Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous strictly increasing function with $g([0, +\infty[) = [0, +\infty[$ and $L : \mathbb{T} \times \mathbb{T} \to \mathbb{R}_+$ be a rd-continuous function. Define*

$$F(y) = \int_{y_0}^{y} \frac{1}{g^{-1}(s)} ds, \ y > 0, \ y_0 > 0, \tag{9}$$

where g^{-1} is the inverse function of g. If

$$g(u(t)) \leq c + \int_{t_0}^{t} f(\eta) \left(u(\eta) + \int_{t_0}^{\eta} L(\eta, \tau) u(\tau) \Delta\tau \right) \Delta\eta, \tag{10}$$

for $t \in \mathbb{T}$, then for all $t \in \mathbb{T}$ satisfying

$$F(c) + \int_{t_0}^{t} f(\eta) \left(1 + \int_{t_0}^{\eta} L(\eta, \tau) \Delta\tau \right) \Delta\eta \in Dom(F^{-1}),$$

and

$$F^{-1} \left[F(c) + \int_{t_0}^{t} f(\eta) \left(1 + \int_{t_0}^{\eta} L(\eta, \tau) \Delta\tau \right) \Delta\eta \right] \in Dom(g^{-1}),$$

we get

$$u(t) \quad \leq \quad g^{-1} \left(F^{-1} \left[F(c) + \int_{t_0}^{t} f(\eta) \left(1 + \int_{t_0}^{\eta} L(\eta, \tau) \Delta\tau \right) \Delta\eta \right] \right),$$

where F^{-1} is the inverse function of F.

Proof. Let us define function $z(t)$ by the right-hand side of (10). Using the properties of g, one can get

$$z(t) \quad \leq \quad c + \int_{t_0}^{t} f(\eta) \left(g^{-1}(z(\eta)) + \int_{t_0}^{\eta} L(\eta, \tau) g^{-1}(z(\tau)) \Delta\tau \right) \Delta\eta.$$

Applying Theorem 4.1, one can obtain

$$z(t) \quad \leq \quad F^{-1} \left[F(c) + \int_{t_0}^{t} f(\eta) \left(1 + \int_{t_0}^{\eta} L(\eta, \tau) \Delta\tau \right) \Delta\eta \right].$$

Inspired by the concept of inequality in [26, Theorem 7], one can derive a Bihari type bound of an integral inequality in the next corollary using functions of class $\mathcal{S}$ (introduced in Section 2).

Corollary 4.2 *Assume that $u, f \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}_+)$, $a : \mathbb{T} \to \mathbb{R}_+$ is a rd-continuous non-decreasing function. Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be of class $\mathcal{S}$, $L : \mathbb{T} \times \mathbb{T} \to \mathbb{R}_+$ be a rd-continuous function and G be defined by (1). If*

$$u(t) \leq a(t) + \int_{t_0}^{t} f(\eta) \left(g(u(\eta)) + \int_{t_0}^{\eta} L(\eta, \tau) g(u(\tau)) \Delta\tau \right) \Delta\eta, \tag{11}$$

for $t \in \mathbb{T}$, then for all $t \in \mathbb{T}$ satisfying

$$G(1) + \int_{t_0}^{t} f(\eta) \left[1 + \int_{t_0}^{\eta} L(\eta, \tau) \Delta\tau \right] \Delta\eta \in Dom(G^{-1}),$$

we obtain

$$u(t) \le \max(a(t),1) G^{-1}\left(G(1)+\int_{t_0}^{t} f(\eta)\left[1+\int_{t_0}^{\eta} L(\eta,\tau)\Delta\tau\right]\Delta\eta\right).$$

Proof. Define function b by $b(t)=\max(a(t),1)$. Then, from (11) we get

$$\frac{u(t)}{b(t)} \quad \le \quad 1+\int_{t_0}^{t}\frac{f(\eta)}{b(\eta)}\left(g(u(\eta))+\int_{t_0}^{\eta} L(\eta,\tau) g(u(\tau))\Delta\tau\right)\Delta\eta.$$

Set $w(t):=\frac{u(t)}{b(t)}$. As $g\in\mathcal{S}$, we deduce the following inequality

$$w(t) \quad \le \quad 1+\int_{t_0}^{t} f(\eta)\left(g(w(\eta))+\int_{t_0}^{\eta} L(\eta,\tau) g(w(\tau))\Delta\tau\right)\Delta\eta.$$

A suitable application of Theorem 4.1 gives

$$w(t)\le G^{-1}\left(G(1)+\int_{t_0}^{t} f(\eta)\left[1+\int_{t_0}^{\eta} L(\eta,\tau)\Delta\tau\right]\Delta\eta\right).$$

It is equivalent to the desired inequality, in view of the fact that $u(t)=w(t)b(t)$.

By a similar reasoning as in Corollary 4.1, an integral approximation of an unknown function using its image by a function of class $\mathcal{T}$ is derived in the next corollary.

Corollary 4.3 *Let u, a, f, L be as defined in Corollary 4.2 and F be as given in (9). Suppose that $g\in\mathcal{T}$ with $g([0,+\infty[)=[0,+\infty[$. If*

$$g(u(t))\le a(t)+\int_{t_0}^{t} f(\eta)\left(u(\eta)+\int_{t_0}^{\eta} L(\eta,\tau)u(\tau)\Delta\tau\right)\Delta\eta, \tag{12}$$

then for all $t\in\mathbb{T}$ satisfying

$$F(1)+\int_{t_0}^{t} f(\eta)\left(1+\int_{t_0}^{\eta} L(\eta,\tau)\Delta s\right)\Delta\eta \in Dom(F^{-1}),$$

and

$$F^{-1}\left[F(1)+\int_{t_0}^{t} f(\eta)\left(1+\int_{t_0}^{\eta} L(\eta,\tau)\Delta\tau\right)\Delta\eta\right]\in Dom(g^{-1}),$$

we have

$$u(t)\le \max(a(t),1) g^{-1}\left(F^{-1}\left[F(1)+\int_{a}^{t} f(\eta)\left(1+\int_{a}^{\eta} L(\eta,\tau)\Delta\tau\right)\Delta\eta\right]\right).$$

Proof. Take $b(t)=\max(a(t),1)$, then (12) can be written as

$$\frac{g(u(t))}{b(t)}\le 1+\int_{t_0}^{t}\frac{f(\eta)}{b(\eta)}\left(u(\eta)+\int_{t_0}^{\eta} L(\eta,\tau)u(\tau)\Delta s\right)\Delta\eta. \tag{13}$$

Set $w(t)=\frac{u(t)}{b(t)}$ on $\mathbb{T}$. Taking into account the fact that g is a nondecreasing function of class $\mathcal{T}$ and $g([0,+\infty[)=[0,+\infty[$, from (13), one can get

$$g(w(t))\le 1+\int_{t_0}^{t} f(\eta)\left(w(\eta)+\int_{t_0}^{\eta} L(\eta,\tau)w(\tau)\Delta\tau\right)\Delta\eta.$$

Applying Corollary 4.1 the requested inequality is obtained.

5 Illustrative Examples

In this section, we apply some inequalities obtained in the previous sections to investigate certain properties of the solutions of dynamic equations on arbitrary time scales.

Example 5.1 Using a straightforward extension of Theorem 3.1, let us discuss the boundedness behavior of the solution of the nonlinear dynamic equation defined as:

$$\begin{cases} \big(x^p(t)\big)^{\Delta} = \mathrm{P}\left(t, S\big(t, x^q(\sigma(t))\big), \displaystyle\int_{t_0}^{t} \mathrm{H}\Big(t, s, S\big(s, x^r(s)\big)\Big)\Delta s\right), \quad t_0, t \in \mathbb{T}^{\kappa}, \\ x^p(t_0) = c, \end{cases}$$

where $t \geq t_0$, $c \neq 0$ a real constant, $\mathrm{P} : \mathbb{T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\mathrm{H} : \mathbb{T} \times \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ are rd-continuous functions and p, q and r are real constants such that $p \neq 0$, $0 \leq q, r \leq p$. We shall assume that the proposed initial value problem has a unique solution $x(t)$. We also consider that functions P and H satisfy

$$|\mathrm{P}(t, U, V)| \leq f(t)(|U| + |V|), \quad t \in \mathbb{T}, \quad U, V \in \mathbb{R}, \tag{14}$$

$$|\mathrm{H}(t, s, U)| \leq L(t, s)|U|, \quad t, s \in \mathbb{T}, \quad U \in \mathbb{R}, \tag{15}$$

where f and L are as mentioned in Theorem 3.1. Let us assume that functions $S, R : \mathbb{T} \times \mathbb{R} \to \mathbb{R}_+$ satisfy the following properties

$$S(t, y) \leq S(t, x), \quad t \in \mathbb{T}, \quad y \leq x, \ x, y \in \mathbb{R}, \tag{16}$$

$$S(t, x) - S(t, y) \leq R(t, y)(x - y), \quad t \in \mathbb{T}, \quad 0 \leq y \leq x. \tag{17}$$

Suppose that there exists $K > 0$ such that

$$R^{\Delta_t}\Big(t, \frac{p-q}{p} K^{\frac{q}{p}}\Big) \geq 0, \ S^{\Delta_t}\Big(t, \frac{p-q}{p} K^{\frac{q}{p}}\Big) \geq 0, \tag{18}$$

for all $t \in \mathbb{T}^{\kappa}$ and $-\frac{q}{p} K^{\frac{q-p}{p}} R\Big(., \frac{p-q}{p} K^{\frac{q}{p}}\Big) f(.) \in \mathcal{R}^+$.

Clearly, the solution $x(t)$ of the system under consideration satisfies the following integral equation

$$x^p(t) = c + \int_{t_0}^{t} \mathrm{P}\left(\eta, S\big(\eta, x^q(\sigma(\eta))\big), \int_{t_0}^{\eta} \mathrm{H}\Big(\eta, \tau, S\big(\tau, x^r(\tau)\big)\Big)\Delta\tau\right)\Delta\eta. \tag{19}$$

It follows from relations (14)-(19) that

$$|x(t)|^p \leq |c| + \int_{t_0}^{t} f(\eta)\left(S\big(\eta, |x(\sigma(\eta))|^q\big) + \int_{t_0}^{\eta} L(\eta, \tau) S\big(\tau, |x(\tau)|^r\big)\Delta\tau\right)\Delta\eta.$$

Then, following a similar approach as in Theorem 3.1, one can easily obtain

$$\begin{aligned} |x(t)| \leq & \left(|c| + \int_{t_0}^{t} f_{\mu}(\eta)\left[\left(\frac{q}{p} K^{\frac{q-p}{p}} R\Big(t_0, \frac{p-q}{p} K^{\frac{q}{p}}\Big)|c|\right.\right.\right. \\ & \left.\left.\left. + S\Big(t_0, \frac{p-q}{p} K^{\frac{q}{p}}\Big)\right) e_{A_4^*}(\eta, t_0) + \int_{t_0}^{\eta} e_{A_4^*}(\eta, \sigma(\tau)) B_4^*(\tau)\Delta\tau\right]\Delta\eta\right)^{\frac{1}{p}}, \end{aligned}$$

where

$$f_\mu(t) = \frac{f(t)}{1-\mu(t)\frac{q}{p}K^{\frac{q-p}{p}}R\Big(t,\frac{p-q}{p}K^{\frac{q}{p}}\Big)f(t)},$$

$$A_4^*(t) = \frac{q}{p}K^{\frac{q-p}{p}}R\Big(\sigma(t),\frac{p-q}{p}K^{\frac{q}{p}}\Big)f_\mu(t)+\frac{R^{\Delta_t}\Big(t,\frac{p-q}{p}K^{\frac{q}{p}}\Big)}{R\Big(t,\frac{p-q}{p}K^{\frac{q}{p}}\Big)}$$
$$+\frac{r}{q}K^{\frac{r-q}{p}}\left(L(\sigma(t),t)\frac{R\Big(t,\frac{p-r}{p}K^{\frac{r}{p}}\Big)}{R\Big(t,\frac{p-q}{p}K^{\frac{q}{p}}\Big)}+\int_{t_0}^{t}L^{\Delta_t}(t,\tau)\frac{R\Big(\tau,\frac{p-r}{p}K^{\frac{r}{p}}\Big)}{R\Big(\tau,\frac{p-q}{p}K^{\frac{q}{p}}\Big)}\Delta\tau\right)$$

and

$$B_4^*(t) = S^{\Delta_t}\Big(t,\frac{p-q}{p}K^{\frac{q}{p}}\Big)+S\Big(t,\frac{p-r}{p}K^{\frac{r}{p}}\Big)L(\sigma(t),t)+\int_{t_0}^{t}L^{\Delta_t}(t,\tau)S\Big(\tau,\frac{p-r}{p}K^{\frac{r}{p}}\Big)\Delta\tau.$$

In the following example, applying Theorem 4.1, an integral approximation of the solution of a dynamical system is presented below.

Example 5.2 Let us consider the following initial value problem on an arbitrary time scale

$$\begin{cases} x^{\Delta}(t) = f(t)\left(g(x(t))+\displaystyle\int_{t_0}^{t}L(t,\tau)g(x(\tau))\Delta\tau\right) & t_0, t\in\mathbb{T},\\ x(t_0)=c, \end{cases}$$

where $t\ge t_0$, $c\neq 0$ a real constant, f, L are as defined in Theorem 4.1 and $g:\mathbb{R}\to\mathbb{R}_+$ a continuous function and nondecreasing positive on $\mathbb{R}^*$. Assume that $x(t)$ is the unique solution of the system under investigation, then it can be expressed as

$$x(t) = c+\int_{t_0}^{t}f(\eta)\left(g(x(\eta))+\int_{t_0}^{t}L(\eta,\tau)g(x(\tau))\Delta\tau\right)\Delta\eta.$$

Further,

$$|x(t)|\le |c|+\int_{t_0}^{t}f(\eta)\left(g(|u(\eta)|)+\int_{t_0}^{t}L(\eta,\tau)g(|x(\tau)|)\Delta\tau\right)\Delta\eta.$$

Applying Theorem 4.1, one can obtain

$$|x(t)|\le G^{-1}\left(G(|c|)+\int_{t_0}^{t}f(\eta)\left(1+\int_{t_0}^{\eta}L(\eta,\tau)\Delta\tau\right)\Delta\eta\right).$$

where G, as given by (1), is such that

$$G(|c|)+\int_{t_0}^{t}f(\eta)\left(1+\int_{t_0}^{\eta}L(\eta,\tau)\Delta\tau\right)\Delta\eta\in Dom(G^{-1}).$$

6 Conclusion

In this work, some new inequalities of Pachpatte and Bellman-Bihari types were derived on arbitrary time scales. As discussed in the paper, they can be thought of as generalizations and refinements of many existing results. These inequalities help us in the study of some classes of integral and integro-differential equations. They can be used in the stability analysis of some classes of dynamical systems on time scales.

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