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Existence Results for Mild Solution for a Class of Impulsive Fractional Stochastic Problems with Nonlocal Conditions

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Abstract: This paper is concerned with the existence of mild solutions for nonlocal impulsive fractional order functional stochastic differential equations with delay. The existence and uniqueness results are shown by using the fixed point technique in a real Hilbert space. Finally, we illustrate the uniqueness result by an example involving partial derivatives.

Keywords: *fractional differential equation, existence and uniqueness, impulsive effects, stochastic differential equation.*

Mathematics Subject Classification (2010): 26A33, 34A12, 34A37, 34K50.

1 Introduction

The modeling with stochastic differential equations has attracted many authors due to its various applications in physics, biology, mathematical finance, etc (see [29, 31, 33] and references therein). The issues related to the existence and uniqueness for such model are widely studied by many authors and one can see the contribution in [5, 7, 18, 19, 34, 35, 37] and references therein. Recently, Das et al. [15] studied a fractional stochastic model with deviating argument and successfully applied the Faedo-Galerkin approximation method to prove the existence results. Benchaabane et al. [7] examined the Sobolev-type fractional stochastic model and established the existence and uniqueness of mild solutions via Picard's iteration technique.

Recently, the modeling with fractional differential equations has gained considerable importance due to its numerous applications in various fields of science and engineering,

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such as physics, chemistry, mechanics, system identification, etc (see the monographs [26, 30, 32]). A significant and systematic development on the existence and uniqueness of solutions for nonlinear type model of fractional differential equations can be seen in [3, 4, 10, 24, 25] and references therein. The model with impulsive nature is found in many real world problems which describe the phenomena of evolution of processes that are subject to sudden changes in their states, for details and update work, we cite the papers [12–14, 17, 20, 23, 28, 39, 40].

In some phenomena, the rate of change of the system and current status often depends not only on the current state but also on the history of the system. Such type of problem models are in the form of functional differential equations and arise in many important fields such as cell biology, electrodynamics, position control, etc. For more details, we refer the reader to the monographs [22, 27] and the papers [11, 35–37].

The nonlocal type initial condition, which is generalization of classical initial condition, was firstly initiated by Byszewski [8]. Further, Byszewski and Lakshmikantham in [9], remarked that the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. In [41] Jhou et al. considered more general nonlocal condition and established the existence and uniqueness of mild solutions by using Krasnoselskii's fixed point theorem and Banach contraction principle.

As far as solution technique is concerned, Feckan et al. [16] established a concept of solutions for the class of impulsive fractional differential equations which is claimed to be more suitable than the concept given by Agarwal et al. [2]. Recently, many authors followed this concept and improved the existing results (see [12, 14, 36]. In this work, we define the mild solution of the system (1)-(3) using the concept introduced in papers [16, 38]. The mild solution is associated with the solution operator reformed by Mittag–Leffler function on a Hilbert space.

Motivated by the above mentioned works as well as the papers [11, 16, 35, 36, 39, 41], we consider the following impulsive fractional functional stochastic differential equation with nonlocal condition:

$$ {}^{C}D_t^{\alpha}u(t) = Au(t) + t^n f(t, u_t) + t^n g(t, u_t)\frac{dw(t)}{dt}, t \in J, t \neq t_k, \tag{1} $$

$$ u(t) + (h(u_{t_1}, u_{t_2}, \dots, u_{t_p}))(t) = \phi(t), \quad t \in [-d, 0], \tag{2} $$

$$ \Delta u(t_k) = I_k(u(t_k^-)), \quad k = 1, 2, \dots, m, \tag{3} $$

where $J = [0, T], n \in Z^+$, and ${}^{C}D_t^{\alpha}$ denotes Caputo's fractional derivative of order $\alpha \in (0, 1)$. $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is a closed linear sectorial operator defined on a Hilbert space $(\mathbb{H}, \|\cdot\|)$ and $u(\cdot)$ takes the values in the real separable Hilbert space $\mathbb{H}$; $f : J \times PC_{\mathcal{L}}^0 \to \mathbb{H}$, $g : J \times PC_{\mathcal{L}}^0 \to \mathcal{L}(\mathbb{K}, \mathbb{H})$, $h : {PC_{\mathcal{L}}^0}^p \to \mathbb{H}$ and $I_k : \mathbb{H} \to \mathbb{H}$ are appropriate functions; $\phi(t)$ is $\mathbb{F}_0$- measurable $\mathbb{H}$-valued random variable independent of w. The functions u_θ are defined as $u_\theta(t) = u(\theta + t)$ for $\theta \in [-d, 0]$.

In the problem under consideration, the equation (1) is very important due to its appearance in the mathematical modeling of viscoelasticity. This fact prompts us to study the existence and uniqueness of solutions of system (1)-(3). To the best of our knowledge, the study of sufficient conditions for the existence of the problem (1)-(3) in Hilbert space is an untreated topic yet.

This work has been divided in four sections, the second section provides some basic definitions and preliminary results. The third section is equipped with main results for the problem (1)-(3) and in the last section an example is presented to verify the established results.

2 Preliminaries

Let $\mathbb{H}, \mathbb{K}$ be two real separable Hilbert spaces and $\mathcal{L}(\mathbb{K}, \mathbb{H})$ be the space of bounded linear operators from $\mathbb{K}$ into $\mathbb{H}$. For convenience, we will use the same notation $\|\cdot\|$ to denote the norms in $\mathbb{H}, \mathbb{K}$ and $\mathcal{L}(\mathbb{K}, \mathbb{H})$, and use $(\cdot,\cdot)$ to denote the inner product of $\mathbb{H}$ and $\mathbb{K}$ without any confusion. Let $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space satisfying the condition that $\mathbb{F}_0$ contains all $\mathbb{P}$-null sets of $\mathbb{F}$. An $\mathbb{H}-$valued random variable is an $\mathbb{F}$- measurable function $u(t) : \Omega \to \mathbb{H}$ and a collection of random variables $S = \{u(t, \omega) : \Omega \to \mathbb{H} \setminus t \in J\}$ is called the stochastic process. Usually, we write $u(t)$ instead of $u(t, \omega)$ and $u(t) : J \to \mathbb{H}$ in the space of S. Let $\mathbb{W} = (\mathbb{W}_t)_{t\geq 0}$ be a $\mathbb{Q}$-Wiener process defined on $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathbb{P})$ with the covariance operator $\mathbb{Q}$ such that $Tr\mathbb{Q} < \infty$. We assume that there exist a complete orthonormal system $\{e_k\}_{k\geq 1}$ in $\mathbb{K}$, a bounded sequence of nonnegative real numbers λ_k such that $\mathbb{Q}e_k = \lambda_k e_k,\ k = 1, 2, \ldots,$ and a sequence of independent Brownian motions $\{\beta_k\}_{k\geq 1}$ such that

$$(w(t), e)_{\mathbb{K}} = \sum_{k=1}^{\infty} \sqrt{\lambda_k}(e_k, e)_{\mathbb{K}}\beta_k(t), e \in \mathbb{K}, t \geq 0.$$

Let $\mathcal{L}_0^2 = \mathcal{L}^2(\mathbb{Q}^{\frac{1}{2}}\mathbb{K}, \mathbb{H})$ be the space of all Hilbert-Schmidt operators from $\mathbb{Q}^{\frac{1}{2}}\mathbb{K}$ to $\mathbb{H}$ with the inner product $< \varphi, \psi >_{\mathcal{L}_0^2}= Tr[\varphi\mathbb{Q}\psi*]$.

The collection of all strongly measurable, square integrable, $\mathbb{H}$-valued random variables, denoted by $\mathcal{L}^2(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathbb{P}; \mathbb{H}) = \mathcal{L}^2(\Omega; \mathbb{H})$, is a Banach space equipped with the norm $\|u(\cdot)\|^2_{\mathcal{L}^2} = E\|u(\cdot, w)\|^2_{\mathbb{H}}$, where E denotes expectation defined by $E(h) = \int_\Omega h(w)d\mathbb{P}$. An important subspace is given by $\mathcal{L}_0^2(\Omega; \mathbb{H}) = \{f \in \mathcal{L}^2(\Omega, \mathbb{H}) : f$ is $\mathbb{F}_0$- is measurable$\}$.

We consider the space

$$PC^0_{\mathcal{L}} = PC([-d, 0], \mathcal{L}^2(\Omega; \mathbb{H}))$$

as a Banach space of all continuous functions $u : [-d, 0] \to \mathcal{L}^2(\Omega; \mathbb{H})$, endowed with the norm

$$\|u\|^2_{PC^0_{\mathcal{L}}} = \sup_{t\in J} \left\{E\|u(t)\|^2_{\mathbb{H}}, u \in PC^0_{\mathcal{L}}\right\}.$$

To study the impulsive conditions, we consider

$$PC_{\mathcal{L}} = PC([-d, T], \mathcal{L}^2(\Omega; \mathbb{H}))$$

as a Banach space of all such continuous functions $u : [-d, T] \to \mathcal{L}^2(\Omega; \mathbb{H})$, which are continuous on $[0, T]$ except for a finite number of points $t_i \in (0, T),\ i = 1, 2, \ldots, m,$ at which $u(t_i^+)$ and $u(t_i^-) = u(t_i)$ exist, endowed with the norm

$$\|u\|^2_{PC_{\mathcal{L}}} = \sup_{t\in J} \left\{E\|u(t)\|^2_{\mathbb{H}}, u \in PC_{\mathcal{L}}\right\}.$$

Remark 2.1 ([21]) If $\alpha \in (0, 1)$ and $A \in \mathbb{A}^\alpha(\theta_0, \omega_0)$, then for any $u \in \mathbb{H}$ and $t > 0$ we have $\|T_\alpha(t)\| \leq Me^{\omega t}$ and $\|S_\alpha(t)\| \leq Ce^{\omega t}(1 + t^{\alpha-1}), \omega > \omega_0$. Thus we have

$$\|T_\alpha(t)\| \leq \widetilde{M}_T \text{ and } \|S_\alpha(t)\| \leq t^{\alpha-1}\widetilde{M}_S,$$

where $\widetilde{M}_T = \sup_{0\leq t\leq T} \|T_\alpha(t)\|$ and $\widetilde{M}_S = \sup_{0\leq t\leq T} Ce^{\omega t}(1 + t^{1-\alpha})$.

Now, we state the definition of mild solution of the system (1)-(3) based on the concept introduced in [38].

Definition 2.1 A measurable $\mathbb{F}_t-$ adapted stochastic process $u : [-d, T] \to \mathbb{H}$ such that $u \in PC_{\mathcal{L}}$ is called a mild solution of the system (1)-(3) if $u(0) = \phi(0) - (h(u_{t_1}, u_{t_2}, \ldots, u_{t_p}))(0)$ on $[-d, 0], \Delta u|_{t=t_k} = I_k(u(t_k^-))$, $k = 1, 2, \ldots, m$, the restriction of $u(\cdot)$ to the interval $[0, T) \backslash t_1, \ldots, t_m$, is continuous and $u(t)$ satisfies the following integral equation

$$u(t) = \begin{cases} S_\alpha(t)[\phi(0) - (h(u_{t_1}, u_{t_2}, \ldots, u_{t_p}))(0)] & \\ + \int_0^t T_\alpha(t-s)s^n f(s, u_s)ds & \\ + \int_0^t T_\alpha(t-s)s^n g(s, u_s)dw(s), & t \in (0, t_1], \\ S_\alpha(t)[\phi(0) - (h(u_{t_1}, u_{t_2}, \ldots, u_{t_p}))(0)] & \\ + S_\alpha(t - t_1)I_1(u(t_1^-)) + \int_0^t T_\alpha(t-s)s^n f(s, u_s)ds & \\ + \int_0^t T_\alpha(t-s)s^n g(s, u_s)dw(s), & t \in (t_1, t_2], \\ \vdots & \\ S_\alpha(t)[\phi(0) - (h(u_{t_1}, u_{t_2}, \ldots, u_{t_p}))(0)] & \\ + \sum_{i=1}^m S_\alpha(t - t_i)I_i(u(t_i^-)) + \int_0^t T_\alpha(t-s)s^n f(s, u_s)ds & \\ + \int_0^t T_\alpha(t-s)s^n g(s, u_s)dw(s), & t \in (t_m, T]. \end{cases}$$

To avoid the repetitions of the basic definitions, we cite them from appropriate papers and books: for Reimann-Liouville integral operator, Mittag–Lefller function and Caputo's derivative see [32], for α-resolvent family see [4], for sectorial operator see [21] and for solution operator see [1].

3 Existence and Uniqueness of Solutions

For the forthcoming analysis, we introduce the following assumption.

(H1) Functions $f; g; h$ and I_k are continuous and there exist positive constants $L_f; L_g; L_h$ and L_I such that

$$\begin{aligned} E\|f(t, \phi) - f(t, \varphi)\|^2_{\mathbb{H}} &\leq L_f \|\phi - \varphi\|^2_{PC^0_{\mathcal{L}}}, \\ E\|g(t, \phi) - g(t, \varphi)\|^2_{\mathbb{H}} &\leq L_g \|\phi - \varphi\|^2_{PC^0_{\mathcal{L}}}, \\ E\|(h(u_{t_1}, u_{t_2}, \ldots, u_{t_p}))(t) - (h(v_{t_1}, v_{t_2}, \ldots, v_{t_p}))(t)\|^2_{\mathbb{H}} &\leq L_h E\|u - v\|^2_{\mathbb{H}}, \\ E\|I_k(u) - I_k(v)\|^2_{\mathbb{H}} &\leq L_I E\|u - v\|^2_{\mathbb{H}}, \end{aligned}$$

for all $u, v \in \mathbb{H}$ and $\phi, \varphi \in PC^0_{\mathcal{L}}$.

Our first result is based on the Banach contraction principle.

Theorem 3.1 *Let the assumption (H1) hold with the positive constant*

$$\Theta = \begin{cases} [4\widetilde{M}_S^2 L_h + 4m\widetilde{M}_S^2 L_I + 4\widetilde{M}_T^2 \frac{T^{2\alpha+n}}{\alpha} \frac{\Gamma(\alpha)\Gamma(n+1)}{\Gamma(\alpha+n+1)} L_f \\ +4\widetilde{M}_T^2 T^{2\alpha-1+n} \frac{\Gamma(2\alpha-1)\Gamma(n+1)}{\Gamma(2\alpha+n)} L_g] \end{cases} < 1,$$

then the system (1)-(3) has a unique mild solution.

Proof. Define the operator $P : PC_{\mathcal{L}} \to PC_{\mathcal{L}}$ so that

$$(Pu)(t) = \begin{cases} S_\alpha(t)[\phi(0) - (h(u_{t_1}, u_{t_2}, \ldots, u_{t_p}))(0)] \\ + \int_0^t T_\alpha(t-s)s^n f(s, u_s)ds \\ + \int_0^t T_\alpha(t-s)s^n g(s, u_s)dw(s), & t \in (0, t_1], \\ S_\alpha(t)[\phi(0) - (h(u_{t_1}, u_{t_2}, \ldots, u_{t_p}))(0)] \\ + S_\alpha(t-t_1)I_1(u(t_1^-)) + \int_0^t T_\alpha(t-s)s^n f(s, u_s)ds \\ + \int_0^t T_\alpha(t-s)s^n g(s, u_s)dw(s), & t \in (t_1, t_2], \\ \vdots \\ S_\alpha(t)[\phi(0) - (h(u_{t_1}, u_{t_2}, \ldots, u_{t_p}))(0)] \\ + \sum_{i=1}^m S_\alpha(t-t_i)I_i(u(t_i^-)) + \int_0^t T_\alpha(t-s)s^n f(s, u_s)ds \\ + \int_0^t T_\alpha(t-s)s^n g(s, u_s)dw(s), & t \in (t_m, T]. \end{cases}$$

Now, we show that P is a contraction map.To this end we take two points $u, u^* \in PC_{\mathcal{L}}$, then for all $t \in (0, t_1]$, we have

$$\begin{aligned} E\|(Pu)(t) - (Pu^*)(t)\|_{\mathbb{H}}^2 \leq\ & 3E\|S_\alpha(t)[(h(u_{t_1}, u_{t_2}, \ldots, u_{t_p}))(0) \\ & -(h(u_{t_1}^*, u_{t_2}^*, \ldots, u_{t_p}^*))(0)]\|_{\mathbb{H}}^2 \\ & +3E\|\int_0^t T_\alpha(t-s)s^n[f(s, u_s) - f(s, u_s^*)]ds\|_{\mathbb{H}}^2 \\ & +3E\|\int_0^t T_\alpha(t-s)s^n[g(s, u_s) - g(s, u_s^*)]dw(s)\|_{\mathbb{H}}^2, \\ \leq\ & [3\widetilde{M}_S^2 L_h + 3\widetilde{M}_T^2 \frac{T^{2\alpha+n}}{\alpha} \frac{\Gamma(\alpha)\Gamma(n+1)}{\Gamma(\alpha+n+1)} L_f \\ & +3\widetilde{M}_T^2 T^{2\alpha-1+n} \frac{\Gamma(2\alpha-1)\Gamma(n+1)}{\Gamma(2\alpha+n)} L_g]\|u - u^*\|_{PC_{\mathcal{L}}}. \end{aligned}$$

For $t \in (t_1, t_2]$, we get the estimate

$$\begin{aligned} E\|(Pu)(t) - (Pu^*)(t)\|_{\mathbb{H}}^2 \leq\ & 4E\|S_\alpha(t)[(h(u_{t_1}, u_{t_2}, \ldots, u_{t_p}))(0) \\ & -(h(u_{t_1}^*, u_{t_2}^*, \ldots, u_{t_p}^*))(0)]\|_{\mathbb{H}}^2 \\ & +4E\|S_\alpha(t-t_1)[I_1(u(t_1^-)) - I_1(u^*(t_1^-))\|_{\mathbb{H}}^2 \\ & +4E\|\int_0^t T_\alpha(t-s)s^n[f(s, u_s) - f(s, u_s^*)]ds\|_{\mathbb{H}}^2 \\ & +4E\|\int_0^t T_\alpha(t-s)s^n[g(s, u_s) - g(s, u_s^*)]dw(s)\|_{\mathbb{H}}^2, \\ \leq\ & [4\widetilde{M}_S^2 L_h + 4\widetilde{M}_S^2 L_I + 4\widetilde{M}_T^2 \frac{T^{2\alpha+n}}{\alpha} \frac{\Gamma(\alpha)\Gamma(n+1)}{\Gamma(\alpha+n+1)} L_f \\ & +4\widetilde{M}_T^2 T^{2\alpha-1+n} \frac{\Gamma(2\alpha-1)\Gamma(n+1)}{\Gamma(2\alpha+n)} L_g]\|u - u^*\|_{PC_{\mathcal{L}}}^2. \end{aligned}$$

Similarly, for general $t \in (t_i, t_{i+1}], i = 2, \ldots, m$, we obtain

$$\begin{aligned} E\|(Pu)(t) - (Pu^*)(t)\|^2_{\mathbb{H}} &\leq [4\widetilde{M}_S^2 L_h + 4k\widetilde{M}_S^2 L_I + 4\widetilde{M}_T^2 \frac{T^{2\alpha+n}}{\alpha} \frac{\Gamma(\alpha)\Gamma(n+1)}{\Gamma(\alpha+n+1)} L_f \\ &\quad + 4\widetilde{M}_T^2 T^{2\alpha-1+n} \frac{\Gamma(2\alpha-1)\Gamma(n+1)}{\Gamma(2\alpha+n)} L_g]\|u - u^*\|^2_{PC_{\mathcal{L}}}. \end{aligned}$$

Thus for all $t \in [0, T]$, we have

$$\begin{aligned} E\|(Pu)(t) - (Pu^*)(t)\|^2_{PC_{\mathcal{L}}} &\leq [4\widetilde{M}_S^2 L_h + 4m\widetilde{M}_S^2 L_I + 4\widetilde{M}_T^2 \frac{T^{2\alpha+n}}{\alpha} \frac{\Gamma(\alpha)\Gamma(n+1)}{\Gamma(\alpha+n+1)} L_f \\ &\quad + 4\widetilde{M}_T^2 T^{2\alpha-1+n} \frac{\Gamma(2\alpha-1)\Gamma(n+1)}{\Gamma(2\alpha+n)} L_g]\|u - u^*\|^2_{PC_{\mathcal{L}}}, \\ &\leq \Theta\|u - u^*\|^2_{PC_{\mathcal{L}}}. \end{aligned}$$

Since $\Theta < 1$ implies that the map P is a contraction map, it has a unique fixed point $u \in PC_{\mathcal{L}}$ which is the unique mild solution of the problem (1)-(3) on J. This completes the proof of the theorem.

Now, to prove the next result, we use Schaefer's fixed point theorem [35] and assume the following conditions:

(H2) Functions f and g are continuous and there exist continuous functions $\widetilde{L}_f, \widetilde{L}_g : J \to (0, \infty)$ such that

$$\begin{aligned} E\|f(t, u_t)\|^2_{\mathbb{H}} &\leq \widetilde{L}_f(t)\psi(E\|u\|^2_{\mathbb{H}}), \\ E\|g(t, u_t)\|^2_{\mathcal{L}_2^0} &\leq \widetilde{L}_g(t)\varphi(E\|u\|^2_{\mathbb{H}}), \end{aligned}$$

for all $\phi, \varphi \in PC^0_{\mathcal{L}}$.

(H3) Functions h and I_k are continuous and there exist positive constant M_1 and Δ such that

$$E\|(h(u_{t_1}, u_{t_2}, \ldots, u_{t_p}))(t)\|^2_{\mathbb{H}} \leq M_1; \max_{1 \leq k \leq m}\{E\|I_k(u)\|^2_{\mathbb{H}}\} = \Delta,$$

for all $u, v \in \mathbb{H}$.

Theorem 3.2 *Let the assumptions (H2) and (H3) hold with*

$$\int_0^T \eta(s)ds \leq \int_c^\infty \frac{ds}{\psi(s) + \varphi(s)}, \tag{4}$$

where

$$\begin{aligned} \eta(t) &= \max\{5\widetilde{M}_T^2 \frac{T^\alpha}{\alpha}(t)^{\alpha-1} t^n \widetilde{L}_f(t), 5\widetilde{M}_T^2 (t)^{2(\alpha-1)} t^n \widetilde{L}_g(t)\}, \\ c &= 5\widetilde{M}_S^2[E\|\phi(0)\|^2_{\mathbb{H}} + M_1], \end{aligned}$$

then the equation (1)-(3) has at least one mild solution on J.

Proof. Consider the closed subspace $H_2 = \{u : u \in PC_{\mathcal{L}}\}$ of all continuous processes u, which are F_t-adapted measurable processes such that the F_0-adapted processes $u(0)$ are endowed with a norm defined by

$$\|u\|_{H_2} = (\sup_{t\in J} \|u(t)\|^2_{\mathcal{L}^2})^{\frac{1}{2}}.$$

Now, we define the operator $N : H_2 \to H_2$ in the same way as in Theorem 3.1. Now, we have to prove that the operator N has at least one fixed point for general interval $t \in (t_k, t_{k+1}], k = 0, 1, \ldots, m$.

With this in mind, consider a sequence $\{u^n\}_{n=0}^{\infty}$ such that $u^n \to u$ in H_2. Then for $t \in (t_k, t_{k+1}], k = 0, 1, \ldots, m$, we have

$$\begin{aligned} E\|(Nu^n)(t) - (Nu)(t)\|^2_{\mathbb{H}} \quad \le \quad & 4E\|S_\alpha(t)[(h(u^n_{t_1}, u^n_{t_2}, \ldots, u^n_{t_p}))(0) \\ & -(h(u_{t_1}, u_{t_2}, \ldots, u_{t_p}))(0)]\|^2_{\mathbb{H}} \\ & +4kE\|S_\alpha(t-t_k)[I_k(u^n(t_k^-)) - I_k(u(t_k^-))\|^2_{\mathbb{H}} \\ & +4E\|\int_0^t T_\alpha(t-s)s^n[f(s,u^n_s) - f(s,u_s)]ds\|^2_{\mathbb{H}} \\ & +4E\|\int_0^t T_\alpha(t-s)s^n[g(s,u^n_s) - g(s,u_s)]dw(s)\|^2_{\mathbb{H}}, \end{aligned}$$

since the functions f, g, h and $I_k, k = 1, 2, \ldots, m$, are continuous, we get

$$\lim_{n\to\infty} E\|Nu^n - Nu\|^2_{\mathbb{H}} = 0,$$

which implies that the operator N is continuous on H_2.

Now, we show that N maps bounded sets into bounded sets in H_2. Consider

$$B_r = \{u \in H_2 : E\|u\|^2_{\mathbb{H}} \le r\} \text{ for } r > 0, \exists\, \xi > 0, \text{such that } E\|(Nu)(t)\|^2_{\mathbb{H}} \le \xi.$$

It is clear that B_r is a closed bounded convex subset of H_2. Let $u \in B_r$. Then, we have

$$\begin{aligned} E\|(Nu)(t)\|^2_{\mathbb{H}} \quad \le \quad & 5E\|S_\alpha(t)\|^2[\|\phi(0)\|^2_{\mathbb{H}} + \|(h(u_{t_1}, u_{t_2}, \ldots, u_{t_p}))(0)\|]^2_{\mathbb{H}} \\ & +5E\|\sum_{i=1}^m S_\alpha(t-t_i)I_i(u(t_i^-))\|^2_{\mathbb{H}} \\ & +5E\|\int_0^t T_\alpha(t-s)s^n f(s,u_s)ds\|^2_{\mathbb{H}} \\ & +5E\|\int_0^t T_\alpha(t-s)s^n g(s,u_s)dw(s)\|^2_{\mathbb{H}}, \\ \le \quad & 5\widetilde{M}_S^2[\|\phi(0)\|^2_{\mathbb{H}} + M_1] + 5m\widetilde{M}_S^2\Delta \\ & +5\widetilde{M}_T^2\frac{T^\alpha}{\alpha}\psi(r)\int_0^t (t-s)^{\alpha-1}s^n\widetilde{L}_f(s)ds \\ & +5\widetilde{M}_T^2\varphi(r)\int_0^t (t-s)^{2(\alpha-1)}s^n\widetilde{L}_g(s)ds, \\ = \quad & \xi. \end{aligned}$$

Next, we prove that N maps bounded sets into equicontinuous sets of B_r. Let $t_k < x < y \leq t_{k+1}$, for each $u \in B_r$, we have

$$\begin{aligned}
E\|(Nu)(x) - (Nu)(y)\|^2_{\mathbb{H}} &\leq 5\|S_\alpha(x) - S_\alpha(y)\|^2[E\|\phi(0)\|^2_{\mathbb{H}} \\
&\quad + E\|(h(u_{t_1}, u_{t_2}, \ldots, u_{t_p}))(0)\|]^2_{\mathbb{H}} \\
&\quad + 5\sum_{i=1}^{m} \|S_\alpha(x - t_i) - S_\alpha(y - t_i)\|^2 E\|I_i(u(t_i^-))\|^2_{\mathbb{H}} \\
&\quad + 5E\|\int_0^t [T_\alpha(x - s) - T_\alpha(y - s)] \times s^n f(s, u_s)ds\|^2_{\mathbb{H}} \\
&\quad + 5E\|\int_0^t [T_\alpha(x - s) - T_\alpha(y - s)]s^n g(s, u_s)dw(s)\|^2_{\mathbb{H}}.
\end{aligned}$$

Since $T_\alpha(t)$ and $S_\alpha(t)$ are strongly continuous,$\|S_\alpha(x) - S_\alpha(y)\| \to 0; |S_\alpha(x - t_i) - S_\alpha(y - t_i)\| \to 0$ and $\|T_\alpha(x - s) - T_\alpha(y - s)\| \to 0$ as $x \to y$. Therefore, from the above inequality, we get $\lim_{x \to y} E\|(Nu)(x) - (Nu)(y)\|^2_{\mathbb{H}} = 0$. Hence, the set $\{Nu,\ u \in B_r\}$ is equicontinuous. Now by Arzela-Ascoli's theorem, we conclude that the operator N is compact.

Finally, we will prove that the set

$$R = \{u \in H_2 \text{ such that } u = qNu(t) \text{ for some } 0 < q < 1\}$$

is bounded. Let $u \in R$, then $u(t) = qNu(t)$ for some $0 < q < 1$. Therefore for each $t \in J$, we have

$$\begin{aligned}
u(t) &= q(S_\alpha(t)[\phi(0) - (h(u_{t_1}, u_{t_2}, \ldots, u_{t_p}))(0)] + \sum_{i=1}^{m} S_\alpha(t - t_i)I_i(u(t_i^-)) \\
&\quad + \int_0^t T_\alpha(t - s)s^n f(s, u_s)ds + \int_0^t T_\alpha(t - s)s^n g(s, u_s)dw(s)),
\end{aligned}$$

which shows that

$$\begin{aligned}
E\|u(t)\|^2_{\mathbb{H}} &\leq 5E\|S_\alpha(t)[\phi(0) + (h(u_{t_1}, u_{t_2}, \ldots, u_{t_p}))(0)]\|^2_{\mathbb{H}} \\
&\quad + 5E\|\sum_{i=1}^{m} S_\alpha(t - t_i)I_i(u(t_i^-))\|^2_{\mathbb{H}} \\
&\quad + 5E\|\int_0^t T_\alpha(t - s)s^n f(s, u_s)ds\|^2_{\mathbb{H}} \\
&\quad + 5E\|\int_0^t T_\alpha(t - s)s^n g(s, u_s)dw(s)\|^2_{\mathbb{H}}, \\
&\leq 5\widetilde{M}_S^2[E\|\phi(0)\|^2_{\mathbb{H}} + M_1] + 5m\widetilde{M}_S^2\Delta \\
&\quad + 5\widetilde{M}_T^2\frac{T^\alpha}{\alpha}\int_0^t (t - s)^{\alpha - 1}s^n\widetilde{L}_f(s)\psi(E\|u(s)\|^2_{\mathbb{H}})ds \\
&\quad + 5\widetilde{M}_T^2\int_0^t (t - s)^{2(\alpha - 1)}s^n\widetilde{L}_g(s)\varphi(E\|u(s)\|^2_{\mathbb{H}})ds.
\end{aligned}$$

Let the function $\lambda(t)$ be defined as

$$\begin{aligned}
\lambda(t) &= \sup\{E\|u(s)\|_{\mathbb{H}}^2, 0 \leq s \leq t\},\ 0 \leq t \leq T,\\
\lambda(t) &\leq 5\widetilde{M}_S^2[E\|\phi(0)\|_{\mathbb{H}}^2 + M_1] + 5m\widetilde{M}_S^2\Delta\\
&\quad +5\widetilde{M}_T^2\frac{T^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}s^n\widetilde{L}_f(s)\psi(\lambda(s))ds\\
&\quad +5\widetilde{M}_T^2\int_0^t (t-s)^{2(\alpha-1)}s^n\widetilde{L}_g(s)\varphi(\lambda(s))ds.
\end{aligned}$$

The last inequality in the right-hand side is denoted by $\mu(t)$, then we have

$$\mu(0) = c = 5\widetilde{M}_S^2[E\|\phi(0)\|_{\mathbb{H}}^2 + M_1], \lambda(t) \leq \mu(t).$$

On the other hand

$$\begin{aligned}
\mu'(t) &= 5\widetilde{M}_T^2\frac{T^\alpha}{\alpha}(t)^{\alpha-1}t^n\widetilde{L}_f(t)\psi(\lambda(t)) + 5\widetilde{M}_T^2(t)^{2(\alpha-1)}t^n\widetilde{L}_g(t)\varphi(\lambda(t)).\\
&\leq 5\widetilde{M}_T^2\frac{T^\alpha}{\alpha}(t)^{\alpha-1}t^n\widetilde{L}_f(t)\psi(\mu(t)) + 5\widetilde{M}_T^2(t)^{2(\alpha-1)}t^n\widetilde{L}_g(t)\varphi(\mu(t)),
\end{aligned}$$

or by equation (4) we have

$$\int_{\mu(0)}^{\mu(t)}\frac{ds}{\psi(s)+\varphi(s)} \leq \int_0^T \eta(s)ds < \int_c^\infty \frac{ds}{\psi(s)+\varphi(s)}.$$

This inequality shows that there is a constant $\mathbb{C}$ such that $\mu(t) \leq \mathbb{C}, t \in J$, and hence, $\lambda(t) \leq \mathbb{C}$, for every $t \in J$. Further, we get $\|u(t)\| \leq \lambda(t) \leq \mu(t) \leq \mathbb{C},\ t \in J$. As the consequence of Schaefer's fixed point theorem, we deduce that N has a fixed point on J which is a solution to (1)-(3). This completes the proof of the theorem.

4 Application

Consider the following nonlocal impulsive fractional partial differential equation of the form

$$\frac{\partial^\alpha}{\partial t^\alpha}u(t,x) = \frac{\partial^2}{\partial y^2}u(t,x) + t\frac{\|u(s-d,x)\|}{36+\|u(s-d,x)\|} + t\frac{\|u(s-d,x)\|}{49+\|u(s-d,x)\|}\frac{dw(t)}{dt},$$

$$u(t,0) = u(t,\pi) = 0,\quad t \geq 0, \tag{5}$$

$$u(t,x) + \sum_{i=0}^n \int_0^\pi k(x,y)u_{t_i}(t,y)dy = (\phi(t))(x), t \in [-d,0], x \in [0,\pi], \tag{6}$$

$$\Delta u(t_i)(x) = \int_{-\infty}^{t_i} q_i(t_i - s)u(s,x)ds, x \in [0,\pi], \tag{7}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is Caputo's fractional derivative of order $\alpha \in (0,1),\ 0 < t_1 < 1$ are prefixed numbers and $\phi \in PC_{\mathcal{L}^2}$. Let $\mathbb{H} = L^2[0,\pi]$ and define the operator $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ by $A\omega = \omega''$ with the domain $D(A) := \{\omega \in X : \omega, \omega'\text{are absolutely continuous}, \omega'' \in \mathbb{H}, \omega(0) = 0 = \omega(\pi)\}$. Then

$A\omega = \sum_{n=1}^{\infty} n^2(\omega, \omega_n)\omega_n, \omega \in D(A)$, where $\omega_n(x) = \sqrt{\frac{2}{\pi}} sin(nx), n \in N$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ in $\mathbb{H}$ and is given by

$$T(t)\omega = \sum_{n=1}^{\infty} e^{-n^2 t}(\omega, \omega_n)\omega_n, \text{ for all } \omega \in \mathbb{H}, \text{ and every } t > 0.$$

The subordination principle of solution operator (Theorem 3.1 in [6]) implies that A is the infinitesimal generator of a solution operator $\{S_\alpha(t)\}_{t\geq 0}$. Since $S_\alpha(t)$ is strongly continuous on $[0, \infty)$, by uniformly bounded theorem, there exists a constant $M > 0$, such that $\|S_\alpha(t)\|_{L(\mathbb{H})} \leq M$ for $t \in [0, 1]$.

Furthermore, we can see

$$E\|f(t, x_t) - f(t, y_t)\|_{\mathbb{H}}^2 \leq \frac{1}{36} E\|x - y\|_{\mathbb{H}}^2.$$

Hence the function f satisfies $(H1)$. Similarly, we can show that the functions g, I_k, h satisfy $(H1)$. Furthermore, we have

$$L_f = \frac{1}{36}, L_g = \frac{1}{49}, L_h = L_I = \frac{1}{25}, \widetilde{M_S} = \widetilde{M_T} = 1, \alpha = \frac{3}{4}, n = 1.$$

It can be calculated that $\Theta = .37 < 1$. Hence the condition of Theorem 3.1 is fulfilled, so we deduce that the system (5)-(7) has a unique mild solution on $[0, 1]$.

5 Conclusion

Fractional order stochastic differential equation is an equation in which randomness is included. In this paper, we established the sufficient conditions for the existence results for a class of impulsive fractional functional stochastic differential equations with nonlocal initial condition. To prove the stated theorems we utilized the well known fixed point theorems with suitable setting of abstract spaces. In our subsequent study, we will try to addressed the existence and uniqueness issue for the class of stochastic fractional neutral integro-differential equation with non-instantaneous impulsive conditions.

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