Nonlinear Dynamics and Systems Theory, **17** (4) (2017) 369–375

Mathematical Models of Nonlinear Oscillations of Mechanical Systems with Several Degrees of Freedom

G.I. Melnikov [1], N.A. Dudarenko [1], K.S. Malykh [1], L.N. Ivanova [1] and V.G. Melnikov [1,2*]

[1] *ITMO University, St. Petersburg, Russia*

[2] *National Mineral Resources University, St. Petersburg, Russia*

Received: May 14, 2017; Revised: October 27, 2017

Abstract: A nonlinear dynamic system with several degrees of freedom, which is represented by a system of differential equations with polynomial structure, is considered. The system contains non-linear polynomials. It is assumed that the spectrum of the eigenvalues of the linear part matrix starts with a pair of complex conjugate eigenvalues having negative real parts with minimum modulus. A polynomial transformation of the equations is performed in order to simplify the mathematical model by reducing the number of non-linear terms in the differential equations. Nonlinear oscillations of an object with constant parameters are investigated. Estimations of motion are obtained by the method of differential inequalities for positive definite Lyapunov function at different ratios between the constant parameters of the system. An example is presented .

Keywords: *autonomous dynamical system; degrees of freedom; phase state variables; nonlinear oscillations; polynomial transformation of variables; Lyapunov function; differential inequality.*

Mathematics Subject Classification (2010): 74H45, 70K75, 70K05, 34C10, 34C15, 45G10, 41A10, 37B25, 34K13.

1 Introduction

The paper deals with nonlinear analysis in classical and modern mechanics [1–5].

We use a Poincare-Dulac approach [6–9] and consider a nonoscillatory nonlinear stationary mechanical system with one degree of freedom. The system has autonomous nonlinear polynomial characteristics associated with its phase variables. This fact leads to the linear form, alternative to the extended model method shown in [10].

* Corresponding author: mailto:v.g.melnikov@yandex.ru

2 Transformation of Polynomial Equations

We consider a nonlinear autonomous system of equations of perturbed motion in the case when all roots of the characteristic equation of the corresponding linear system are different [11–13]. Let us transform it into a canonical form

$$\dot{y}_s = \lambda_s y_s + \sum_{k=2}^{m} \sum_{\nu_1+\ldots+\nu_n=k} p_s^{(\nu_1,\ldots,\nu_n)} y_1^{\nu_1} \ldots y_n^{\nu_n} \quad \left(s = \overline{1,n}\right), \tag{1}$$

where y_s are real and complex variables; λ_s are roots of the characteristic equation of the linear part of the system; $p_s^{(\nu_1,\ldots,\nu_n)}$ are small coefficients; m are odd numbers.

Suppose λ_1 and λ_2 are complex-conjugate pure imaginary roots or roots with real parts much less than those of the other roots and the imaginary parts of these roots

$$\lambda_{1,2} = \alpha \pm \beta i, \qquad \text{where} \quad \alpha \gtrless 0, \;\; |\alpha| < \beta.$$

The real parts of the other roots are essentially negative $|Re\lambda_s| < 0, \quad s = \overline{3,n}$. It should be noted that in such case the variables y_1, y_2 will be complex-conjugated. Such systems are often used to describe nonlinear oscillations in engineering and physics. Suppose the roots $\lambda_1 \ldots \lambda_n$ are such that within the limits of some number of digit order numbers $k = \overline{3,m}$ they do not vanish at any values of indices $\nu_1, \ldots, \nu_n$ complying with the condition (2), except for the values (3) where $Re\,\lambda_1$, $Re\,\lambda_2$ are small.

$$\begin{aligned} &\lambda_s^{(\nu_1,\ldots,\nu_n)} = \nu_1\lambda_1 + \nu_2\lambda_2 + \ldots + (\nu_s - 1)\,\lambda_s + \nu_{s+1}\lambda_{s+1} + \ldots + \nu_n\lambda_n, \\ &\left(s = \overline{1,n}; \quad \nu_1 + \ldots + \nu_n = k, \quad \nu_i \geq 0\right). \end{aligned} \tag{2}$$

$$\begin{aligned} &s = 1, \;\; \nu_1 = (k+1)/2, \;\; \nu_2 = (k-1)/2, \quad k - \text{odd}, \\ &s = 2, \;\; \nu_1 = (k-1)/2, \;\; \nu_2 = (k+1)/2, \;\; \nu_3 = \ldots = \nu_n = 0 \;\;, \;\; k = \overline{3,m}, \\ &s = \overline{3,n}; \;\; \nu_1 = \nu_2 = (k-1)/2, \;\; \nu_s = 1, \\ &\nu_3 = \ldots = \nu_{s-1} = \nu_{s+1} = \nu_{s+2} = \ldots = \nu_n = 0. \end{aligned} \tag{3}$$

With these hypotheses, we approximately integrate system (1) in the neighborhood $\sum_{s=1}^{n} |y_s|^2 \leq \varepsilon^2$. Let us make polynomial transformation of variables.

$$z_s = y_s + \sum_{k=3}^{m} \sum_{\nu_1+\ldots+\nu_n=k} A_s^{(\nu_1,\ldots,\nu_n)} y_1^{\nu_1} \ldots y_n^{\nu_n} \qquad \left(s = \overline{1,n}\right). \tag{4}$$

The transformation coefficients are constant and are defined from the condition that the system (1) in new variables has the following form

$$\dot{z}_s = \left(\lambda_s + \sum_{k=3}^{m} a_s^{(k)} r^{k-1}\right) z_s + Z_s^{(m+1)}, \;\; r = |z_1| = \sqrt{z_1 z_2}. \tag{5}$$

The right-hand parts of this system contain linear as well as nonlinear terms corresponding to special values of indices with undefined coefficients and remainder terms of $(m+1)$ smallness order. We make coefficients $A_s^{(\nu_1,\ldots,\nu_n)}$ corresponding to special values (3) equal to zero (instead of them the coefficients $a_s^{(k)}$ are introduced). In order to calculate all undefined coefficients, we apply

$$r^2 = \sum B^{(\nu_1,\ldots,\nu_n)} y_1^{\nu_1} \ldots y_n^{\nu_n}, \; B^{(\nu_1,\ldots,\nu_n)} = \sum_{\nu_r'} A_1^{\left(\nu_1',\ldots,\nu_n'\right)} A_2^{\left(\nu_1-\nu_1',\ldots,\nu_n-\nu_n'\right)}. \tag{6}$$

Having put every sum in the form $\sum Q^{(\nu_1,\ldots,\nu_n)}\, y_1^{\nu_1}\ldots y_n^{\nu_n}$ and equating the coefficients of similar powers $C_1^{\nu_1}\ldots\, C_n^{\nu_n}$, we obtain the following equations:

$$\lambda_s^{(\nu_1,\ldots,\nu_n)} A_s^{(\nu_1,\ldots,\nu_n)} + p_s^{(\nu_1,\ldots,\nu_n)} = \sum_{k'=\nu}^{k} a_s^{(k')} \sum_{\left(\nu_r^{(i)},\nu_r''\right)} \prod_{i=1}^{\frac{k'-1}{2}} B^{\left(\nu_1^{(i)},\ldots,\nu_n^{(i)}\right)} A_s^{\left(\nu_1'',\ldots,\nu_n''\right)} -$$
$$- \sum_{i=1}^{n} \sum_{\nu_j'} \nu_i' p_i^{\left(\nu_1-\nu_1',\ldots,\nu_i-\nu_i'+1,\nu_{i+1}-\nu_{i+1}',\ldots,\nu_n-\nu_n'\right)} A_s^{\left(\nu_1',\ldots,\nu_n'\right)}, \quad s=\overline{1,n}, \quad k' \, is\ odd. \tag{7}$$

We designate the sum of upper indices in undefined coefficients $A_s^{(\nu_1,\ldots,\nu_n)}$ as a coefficient decade. In the right-hand part of the equations the sums depend on the coefficients with decade smaller than k, as every factor "takes" its decade from the total "stock" of k. The high-order digit $\{k'\}$ of the coefficients $a_s^{(k')}$ is reached when the number of factors under the product sign is the largest, which is possible if every factor has the lowest order. By adding correlations we define that $k' = k$, and by analysing every correlation we can make sure that indexes ν have special values (3). So the high-order digit of the coefficients equals k; besides, it may be obtained only with special values of indexes. It is obvious that the coefficient $a_s^{(k)}$ equals one.

System (7) represents a chain of linear algebraic equations which is solved starting from the lower order $k = 2$ and from the lower number $s = 1$ to further ones. Indeed, all equations corresponding to non-special values of indices are satisfied when choosing undefined coefficients from the first term of the formula (7), and all "special" equations where the factor $A_s^{(\nu_1,\ldots,\nu_n)}$ equals zero or is very small are satisfied when choosing $a_s^{(k)}$.

The remainder functions $Z_s^{(m+1)}$ should be equated to nonlinear terms of not lower than $(m+1)$ order that are contained in the equations obtained by means of formulas (4) and (1) in (5). These functions may be transformed to z_s variables by correlations (4) previously solved with respect to y_s.

3 Transformed System Analysis

Suppose that by means of (4), (7) the system (1) is transformed to (5). The latter system may be integrated if the remainder terms of $(m+1)$ order are ignored. From the first two equations we obtain the equation for variables module:

$$\dot{r} = \alpha r + \sum_{k=3}^{m} \alpha^{(k)} r^k \quad \text{where} \quad \alpha = Re\lambda_1 < 0, \qquad \alpha^{(k)} = Re\; a_1^{(k)}. \tag{8}$$

The special points of the equation (8) are defined in $[t_0, t]$ by equating the right-hand part to zero, and general solution is defined by means of variables separation.

$$\int_{r_0}^{r} \left(\alpha r + \sum \alpha^{(k)} r^k\right)^{-1} dr \; = t - t_0 \;, \qquad (k \, \text{are odd numbers}). \tag{9}$$

The second way of equality (8) integration is given in [14]. Suppose that using one of the methods, we found the solution $r = r(t, r_0, t_0)$. Then the solutions of the first and second equations of system (5) are as follows:

$$z_{1,2} = re^{\pm i\theta} \qquad \text{at} \qquad \theta = \beta(t-t_0) + \sum_{k=3}^{m} \beta^{(k)} \int_{t_0}^{t} r^{k-1} dt + \theta_0, \;\; \beta^{(k)} = \operatorname{Im} a_1^{(k)}. \tag{10}$$

The solution of the other equations is obtained according to the following formulas:

$$z_s = z_{s0} \exp \left(\lambda_s (t - t_0) + \sum_k a_s^{(k)} \int_{t_0}^{t} r^{k-1} dt \right), \quad s = \overline{3, n}. \tag{11}$$

In order to find an approximate solution in variables y_s we must solve the transformation (4) with respect to y_s:

$$y_s \approx z_s + \sum B_s^{(\nu_1, \ldots, \nu_n)} z_1^{\nu_1} \ldots z_n^{\nu_n}, \quad s = \overline{1, n}. \tag{12}$$

The coefficients $B_s^{(\nu_1, \ldots, \nu_n)}$ may be expressed in terms of $A_s^{(\nu_1, \ldots, \nu_n)}$ by using (12) in (4) and equating the coefficients with similar terms. Namely, the coefficients of lower orders in (12) differ from the coefficients in (4) only by sign.

With regard to the obtained approximated solution, there is an idea that: if it is allowed to ignore or add the terms of $(m+1)$ order and more in equations (1), then it is always possible to select terms such that the obtained system will be integrated quite accurately. These terms are selected from the condition of $Z_s^{(m+1)}$ being equal to zero.

Remark 3.1 An additional condition for the characteristic coefficients (2) may be annulled, if we introduce additional terms corresponding to the special values of indices into the transformed system (5), as Dulak did in a non-special case. In this case, instead of system (5), we have:

$$\dot{z}_s = \left(\lambda_s + \sum_{k=1}^{m} a_s^{(k)} r^{k-1} \right) z_s + \sum_{\tilde{\nu}_j}^{m} a_s^{(\tilde{\nu}_1, \ldots, \tilde{\nu}_{s-1})} z_1^{\tilde{\nu}_1} \ldots z_{s-1}^{\tilde{\nu}_{s-1}} + Z_s^{(m+1)}. \tag{13}$$

The equations that must be in accord with the undefined coefficients are calculated from formulas (4) and (1) in (13) and by equating the coefficients of the corresponding terms; they differ from equations (7) by the additional terms. The system (13) also represents a chain of consequent approximately integrated equations.

We note that I.G. Malkin [15] analyzed the transformation of two equations system to the form similar to the first two equations of system (5) by means of substitution reverse to substitution (4).

From the first two equations we can obtain an equation similar to (8) but with an additional term of $(m+1)$ order

$$\dot{r} = r f\left(r^2\right) + R^{(m+1)}, \;\; f\left(r^2\right) = \alpha + \sum_{k=3}^{m} \alpha^{(k)} r^{k-1}, \tag{14}$$

where k, m are odd. Let us take Lyapunov's function and its derivative

$$V = r^2 + \sum_{s=3}^{n} z_s \bar{z}_s, \quad \dot{V} = 2r\dot{r} + \sum_{s=3}^{m} \left(\dot{z}_s \bar{z}_s + \dot{\bar{z}}_s z_s \right).$$

Taking into consideration the equations (14) and (5), we obtain the inequality:

$$\dot{V} < 2 \left[f\left(V\right) + K V^{\frac{m}{2}} \right] V, \qquad 0 < V \le \varepsilon^2. \tag{15}$$

Function V decreases in this ring in accordance with the law

$$\int_{V_0}^{V} \frac{dV}{\left[f\left(V + K V^{\frac{m}{2}}\right) \right] V} > 2(t - t_0), \qquad 0 < V_0 < \varepsilon^2. \tag{16}$$

Example 3.1 Let us integrate approximately the Van-der-Pol equation

$$\dot{x} = \varepsilon x - y - \varepsilon x y^2, \qquad \dot{y} = x. \tag{17}$$

Here, $\lambda_{1,2} \approx \alpha \pm i$, where $\alpha = \varepsilon/2 > 0$ are the complex-conjugated roots of the characteristic equation of the corresponding linear system with small real part. Let us put system (17) into the canonical form:

$$\dot{y}_1 = \lambda_1 y_1 + p_1^{(3,0)} y_1^3 + p_1^{(2,1)} y_1^2 y_2 + p_1^{(1,2)} y_1 y_2^2 + p_1^{(0,3)} y_2^3, \;\; y_2 = \overline{y_1}. \tag{18}$$

Accurate to the ε^2 order, we have:

$$p_1^{(3,0)} = p_1^{(0,3)} = -p_1^{(2,1)} = -p_1^{(1,2)} = \frac{\varepsilon}{2}; \;\; y_1 = x + \left(-\frac{\varepsilon}{2} + i\right) y \,.$$

Let us make transformation of variables

$$z_1 = y_1 + \; A_1^{(3,0)} y_1^3 + A_1^{(1,2)} y_1 y_2^2 + A_1^{(0,3)}, y_2^3 \tag{19}$$

where the coefficients are defined from formulas (7), (2)

$$(\nu_1 \lambda_1 + \nu_2 \lambda_2 - \lambda_1)\, A_1^{(\nu_1,\nu_2)} + p_1^{(\nu_1,\nu_2)} = 0, \;\; p_1^{(2,1)} = a_1^{(3)} \quad (\nu_1 = 3, 1, 0; \;\; \nu_2 = 3 - \nu_1)\,.$$

Therefrom, accurate to the 2nd order, we find:

$$A_1^{(3,0)} = A_1^{(1,2)} = -A_1^{(0,3)} = -\frac{\varepsilon i}{32}, \;\; a_1^{(3)} = -\frac{\varepsilon}{8}, \tag{20}$$

z_1 being a variable module, according to (8), satisfies the equation:

$$\frac{dr}{dt} \approx \frac{\varepsilon}{2} r - \frac{\varepsilon}{8} r^3,$$

which coincides with the equation for amplitude obtained by the method of Krylov and Bogolyubov [16]. General solution is as follows:

$$r = 2\left(1 + c e^{-\varepsilon t}\right)^{-\frac{1}{2}}, \qquad \text{where} \qquad c = \frac{4}{r_0^2} - 1. \tag{21}$$

From formula (10), we obtain:

$$z_1 = r e^{i\theta}, \;\; z_2 = \overline{z_1} = r e^{-i\theta}, \qquad \text{where} \qquad \theta = t + \theta_0, \tag{22}$$

where the initial value θ_0 can be defined on the basis of (18), (20), (21).

Formulas (21), (22) show that in complex plane $z_1 = \xi + i\eta$ the paths of representation point and spiral coil from the inside and outside on the circumference $r = 2$. As this takes place, the angular speed of vector radius r is $\theta = 1$. Based on (19) and (21) we have:

$$y_1 \approx z_1 - A_1^{(3,0)} z_1^3 - A_1^{(1,2)} z_1 z_2^2 - A_1^{(0,3)} z_2^3 = r\, e^{i\theta} - \frac{\varepsilon i}{32} r^3 \left(2e^{3i\theta} + 2e^{-i\theta} - e^{-3i\theta}\right).$$

The original variable is determined as:

$$y = \frac{y_1 - y_2}{2i} = Im\,(y_1) = r \sin\theta - \frac{\varepsilon}{32} r^3 \left(cos 2\theta + 2 cos\theta\right), \;\; \theta = t + \theta_0.$$

The results are shown in Figure 1.

As $t \to \infty$, all solutions, except for zero, asymptotically tend to the periodic one

$$y = 2\, \sin\theta - \frac{\varepsilon}{2} \cos\theta - \frac{\varepsilon}{4} \cos 3\theta, \quad \theta = t + \theta_0.$$

This solution accurate to ε^2 terms coincides with the solution defined by the method of Krylov and Bogolyubov [6, 14, 17].

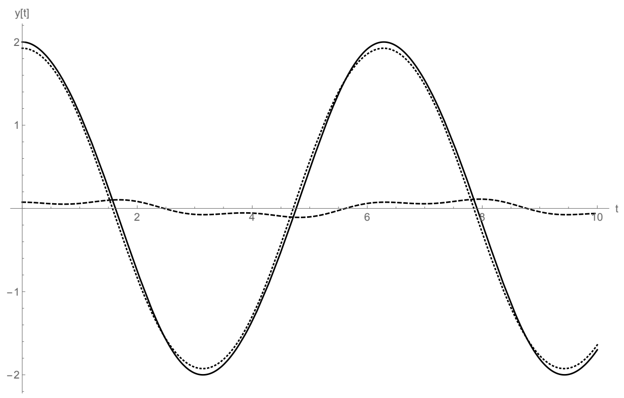

Figure 1: (—) denotes an exact solution $y(t)$ of (3.1), $(\cdots\cdots)$ stands for an approximate solution,$(- - -)$ means an approximation error.

4 Conclusion

Using the non-linear transformation of the polynomial model with adopted precision we investigate a nonlinear vibrational autonomous system with finite degrees of freedom at different ratios between the constants.

This transformation simplifies the form of differential equations, ultimately reduces the number of non-linear terms in the model and forms a small number of high-quality constant coefficients of monomials. The method is modified in order to exclude small divisors. Nonlinear oscillations are investigated by means of analytical integration of the transformed recurrence equations, as well as by integrating the differential inequalities for the Lyapunov function. This method can be applied to a wide range of problems.

Acknowledgment

The authors gratefully acknowledge the reviewers' comments and the contribution of the RFFI research projects no. 17-01-00672 and 16-08-00997, the state financial support of the leading universities, project no. 074-U01, the Ministry of Education and Science, projects no. 14.Z50.31.0031, 14.578.21.0203.

References

[1] Kovalev, A.M., Martynyuk, A.A., Boichuk, O.A., Mazko, A.G., Petryshyn, R.I., Slyusarchuk, V.Y., Zuyev, A.L. and Slyn'ko, V.I. Novel qualitative methods of nonlin-

ear mechanics and their application to the analysis of multifrequency oscillations, stability, and control problems. *Nonlinear Dynamics and Systems Theory* **9** (2) (2009) 117–145.

[2] Dshalalow, J.H., Izobov, N.A. and Vassilyev, S.N. Academician A.A. Martynyuk to the 70th birthday anniversary. *Nonlinear Dynamics and Systems Theory* **11** (1) (2011) 1–5.

[3] Martynyuk, A.A., Dshalalow, J.H. and Zhukovskii, V.I. Professor V.M. Starzhinskii. *Nonlinear Dynamics and Systems Theory* **8** (1) (2008) 1–6.

[4] Vassilyev, S.N., Martynyuk, A.A. and Siljak, D.D. Academician V.M. Matrosov in memoriam. *Nonlinear Dynamics and Systems Theory* **13** (4) (2013) 325–331.

[5] Martynyuk, A.A., Mishchenko, E.F., Samoilenko, A.M. and Sukhanov, A.D. Personage in science:academician N.N. Bogoliubov (to the 100th birthday anniversary). *Nonlinear Dynamics and Systems Theory* **9** (2) (2009) 109–115.

[6] Kyzio, J. and Okniski, A. The Duffing-van der Pol equation: Metamorphoses of resonance curves. *Nonlinear Dynamics and Systems Theory* **15** (1) (2015) 25–31.

[7] Leonov, G. Strange attractors and classical stability theory. *Nonlinear Dynamics and Systems Theory* **8** (1) (2008) 49–96.

[8] Melnikov, V. Chebyshev economization in Poincare-Dulac transformations of nonlinear systems. *Nonlinear Analysis, Theory, Methods and Applications* **63** (5-7) (2005) e1351–e1355.

[9] Melnikov, G., Ivanov, S. and Melnikov, V. The modified Poincare-Dulac method in analysis of autooscillations of nonlinear mechanical systems. *J. Phys. Conf. Ser.* **570** (2014) 022002.

[10] Melnikov, V. Chebyshev economization in transformations of nonlinear systems with polynomial structure. *Int. Conf. on Systems – Proc.* **1** (2010) 301–303.

[11] Aleksandrov, A.Y., Martynyuk, A.A. and Zhabko, A.P. The problem of stability by nonlinear approximation. *Nonlinear Dynamics and Systems Theory* **15** (3) (2015) 221–230.

[12] Melnikov, V.G., Melnikov, G.I., Malykh, K.S. and Dudarenko, N.A. Poincare-Dulac method with Chebyshev economization in autonomous mechanical systems simulation problem. *2015 International Conference on Mechanics – Seventh Polyakhov's Reading* (2015) 7106757.

[13] Ivanov, S. Mathematical modeling of nonlinear dynamic system of the truck crane. *Contemporary Engineering Sciences* **9** (10) (2016) 487–495.

[14] Mitropol'skii, Y.A. and Martynyuk, A.A. Several results and real problems in the theory of periodic motions and nonlinear mechanics (review). *Soviet Applied Mechanics* **24** (3) (1988) 205–216.

[15] Mitropol'skii, Y.A. and Martynyuk, A.A. Certain directions of investigations into stability of periodic motions and the theory of nonlinear vibrations. *Soviet Applied Mechanics* **14** (3) (1978) 223–232.

[16] Martynyuk, A.A. and Nikitina, N.V. Complex oscillations revisited. *International Applied Mechanics* **41** (2) (2005) 179–186.

[17] Martynyuk, A.A. On an averaging theorem of the first N. N. Bogolyubov type. *Polish Academy of Sciences, Inst. of Fundamental Technical Research, Nonlinear Vibration Probl.* (21) (1983) 7–10.