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On the Hyers-Ulam Stability of Laguerre and Bessel Equations by Laplace Transform Method

E. Biçer [1*] and C. Tunç [2]

[1] *Department of Mathematics, Faculty of Arts and Sciences, Bingöl University, 12000, Bingöl, Turkey*
[2] *Department of Mathematics, Faculty of Sciences, Yuzuncu Yil University, 65080, Van, Turkey*

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Abstract: The purpose of this paper is to obtain new sufficient conditions guaranteeing the Hyers-Ulam stability of Laguerre differential equation

$$xy'' + (1 - x)y' + ny = 0$$

and Bessel differential equation of order zero

$$xy'' + y' + xy = 0.$$

Our findings make a contribution to the topic and complete those in the relevant literature.

Keywords: *Hyers-Ulam stability; Laguerre equation; Bessel equation; Laplace transform.*

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1 Introduction

Differential equations of second order can serve as excellent tools for description of mathematical modelling of systems and processes in the fields of engineering, physics, chemistry, economics, aerodynamics, and polymerrheology, etc. Therefore, the qualitative behaviors of solutions of differential equations of second order, stability, boundedness, oscillation, etc., play an important role in many real world phenomena related to the sciences and engineering technique fields. However, we would not like to give the details

* Corresponding author: mailto:cemtunc@yahoo.com

of the applications related to differential equations of second order here.This information indicates the importance of investigating the qualitative properties, Hyers-Ulam stability, Lyapunov stability, etc., of solutions of differential equations of second order.

The stability of functional equations was originally raised by Ulam in 1940 in a talk given at Wisconsin University (Ulam [17]). He discussed a number of unsolved important problems in that presentation. Later, Hyers [5] answered to the questions of Ulam [17]. Hence, the concepts related to the Hyers-Ulam stability arose in the literature. Later, the result of Hyers [5] has been generalized by Rassias [15]. In 1998, Alsina and Ger [3] studied the Hyers–Ulam stability of the fundamental linear differential equation. They proved that the linear differential equation has the Hyers–Ulam stability. After that, many researchers have studied the Hyers–Ulam stability of the various linear and partially differential equations. For more details on the Hyers–Ulam stability of various linear ordinary and partially differential equations, one can see Abdollahpour et al. [1], Alqifiary [2], Alsina and Ger [3], Biçer and Tunç [4], Hyers [5], Jung [6-11], Liu and Zhao [12], Lungu and Popa [13-14], Rassias [15], Tunç and Biçer [16], Ulam [17] and the references therein.

In these sources, the Hyers–Ulam stability of solutions to various linear ordinary, functional and partially differential equations was discussed by direct method, iteration method, fixed point method with a Lipschitz condition, integrating factor method, open mapping theorem, the Gronwall inequality, power series method, the Laplace transform method and etc.

The following works are notable. Jung [11] investigated general solution of the inhomogeneous Bessel differential equation of the form

$$x^2y''(x) + xy'(x) + (x^2 - \gamma^2)y(x) = \sum_{m=0}^{\infty} a_m x^m,$$

where the parameter γ is non-integral number.

Jung [10] solved the inhomogeneous differential equation of the form

$$xy'' + (1-x)y' + ny = \sum_{m=0}^{\infty} a_m x^m$$

by the power series method, where n is positive integer, and applied this result to obtain a partial solution to the Ulam stability of the differential equation

$$xy'' + (1-x)y' + ny = 0.$$

Abdollahpour at al. [1] discussed the Hyers-Ulam stability of the differential equation

$$xy'' + (1+v-x)y' + \lambda y = \sum_{m=0}^{\infty} a_m x^m$$

by means of the power series method. They studied the Hyers-Ulam stability of the associated homogeneous Laguerre differential equation in a subclass of analytic functions.

Alqifiary and Jung [2] investigated Hyers-Ulam stability of the differential equation

$$y^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k y^{(k)}(t) = f(t)$$

by applying the Laplace transform method, where α_k is a scalar.

In this paper, we investigate the Hyers-Ulam stability of Laguerre differential equation of the form

$$xy'' + (1-x)y' + ny = 0, \tag{1}$$

where n is positive integer, and Bessel differential equation of order zero

$$xy'' + y' + xy = 0. \tag{2}$$

Motivated by the mentioned sources, the aim of this paper is to prove the Hyers-Ulam stability of Laguerre and Bessel equations given by (1) and (2) by the Laplace transform method. It is worth mentioning that, to the best of our knowledge, the Laplace transform method is a very effective method to discuss the Hyers-Ulam stability of these equations, equation (1) and equation (2). In addition, to the best of our information till now, the Hyers-Ulam stability of equation (1) and equation (2) was not discussed in the literature by the Laplace transform method. This paper is the first attempt in the literature on the topic for the mentioned equations. Our results will also be differ from those obtained in the literature (see, [1-19] and the references therein). By this way, we mean that this paper has made a contribution to the subject in the literature, and the paper may be useful for researchers working on the qualitative behaviors of solutions like the Hyers-Ulam stability to various differential and partially differential equations. In view of all the mentioned information, the novelty and originality of the current paper can be checked.

2 Hyers-Ulam Stability of Laguerre Equation

Let $I = (0, \infty)$. Our first main result is the following theorem.

Theorem 1. If the function y satisfies the differential inequality

$$\mid xy'' + (1-x)y' + ny \mid \leq \varepsilon \tag{3}$$

for all $x \in I$ and for some $\varepsilon > 0$, then there exists a solution $y_0 : I \to \Re$ of equation (1) such that

$$\mid y(x) - y_0(x) \mid \leq \frac{1}{n}\varepsilon.$$

Proof. It is clear from (3) that

$$-\varepsilon \leq xy'' + (1-x)y' + ny \leq \varepsilon.$$

If we apply the Laplace transform to the last inequality, then we have

$$L(-\varepsilon) \leq L[xy'' + (1-x)y' + ny] \leq L(\varepsilon).$$

Hence, since a Laplace transform is linear, it is clear that

$$L(-\varepsilon) \leq L(xy'') + L((1-x)y') + L(ny) \leq L(\varepsilon).$$

In view of the basic information related to the properties of a Laplace transform, it can be written that

$$-\frac{\varepsilon}{s} \leq -\frac{d}{ds}[s^2Y(s) - sY(0) - Y'(0)] + sY(s) - Y(0) + \frac{d}{ds}[sY(s) - Y(0)] + nY(s) \leq \frac{\varepsilon}{s}$$

and
$$-\frac{\varepsilon}{s} \le -2sY(s) - s^2\frac{dY}{ds} + sY(s) + Y(s) + s\frac{dY}{ds} + nY(s) \le \frac{\varepsilon}{s}$$
so that
$$-\frac{\varepsilon}{s} \le (s-s^2)\frac{dY}{ds} + (n+1-s)Y(s) \le \frac{\varepsilon}{s}.$$

Assume that $(s^2-s)>0$. Dividing the above inequality by (s^2-s) and then multiplying the last inequality by the term $\frac{s^{n+1}}{(s-1)^n}$, we obtain
$$-\frac{\varepsilon s^{n-1}}{(s-1)^{n+1}} \le \frac{dY}{ds}\frac{s^{n+1}}{(s-1)^n} + \frac{(s-n-1)}{(s^2-s)}\frac{s^{n+1}}{(s-1)^n}Y(s) \le \frac{\varepsilon s^{n-1}}{(s-1)^{n+1}}.$$
From this, we have
$$-\frac{\varepsilon s^{n-1}}{(s-1)^{n+1}} \le \frac{d}{ds}[\frac{s^{n+1}}{(s-1)^n}Y(s)] \le \frac{\varepsilon s^{n-1}}{(s-1)^{n+1}}.$$
For any $s_1 > s$, integrating the above inequality from s to s_1, we get
$$-\frac{\varepsilon}{n}[(\frac{s}{s-1})^n - (\frac{s_1}{s_1-1})^n] \le \frac{s_1^{n+1}}{(s_1-1)^n}Y(s_1) - \frac{s^{n+1}}{(s-1)^n}Y(s) \le \frac{\varepsilon}{n}[(\frac{s}{s-1})^n - (\frac{s_1}{s_1-1})^n]$$
so that
$$\begin{aligned}-\frac{\varepsilon}{n}[(\frac{s}{s-1})^n - 2(\frac{s_1}{s_1-1})^n] &\le \frac{s_1^{n+1}}{(s_1-1)^n}Y(s_1) - \frac{s^{n+1}}{(s-1)^n}Y(s) + \frac{\varepsilon}{n}(\frac{s_1}{s_1-1})^n \\ &\le \frac{\varepsilon}{n}(\frac{s}{s-1})^n,\end{aligned}$$
$$-\frac{\varepsilon}{n}(\frac{s}{s-1})^n \le \frac{s_1^{n+1}}{(s_1-1)^n}Y(s_1) - \frac{s^{n+1}}{(s-1)^n}Y(s) + \frac{\varepsilon}{n}(\frac{s_1}{s_1-1})^n \le \frac{\varepsilon}{n}(\frac{s}{s-1})^n.$$
Multiplying the last inequality by the term $\frac{(s-1)^n}{s^{n+1}}$, we obtain
$$-\frac{\varepsilon}{ns} \le \frac{s_1^{n+1}}{(s_1-1)^n}Y(s_1)\frac{(s-1)^n}{s^{n+1}} + \frac{\varepsilon}{n}(\frac{s_1}{s_1-1})^n\frac{(s-1)^n}{s^{n+1}} - Y(s) \le \frac{\varepsilon}{ns}.$$
Appling the inverse Laplace transform, we have
$$\begin{aligned}L^{-1}(-\frac{\varepsilon}{ns}) &\le L^{-1}[\frac{s_1^{n+1}}{(s_1-1)^n}Y(s_1)\frac{(s-1)^n}{s^{n+1}}] + L^{-1}[\frac{\varepsilon}{n}(\frac{s_1}{s_1-1})^n\frac{(s-1)^n}{s^{n+1}}] - L^{-1}[Y(s)] \\ &\le L^{-1}(\frac{\varepsilon}{ns})\end{aligned}$$
and
$$-\frac{\varepsilon}{n} \le [\frac{\varepsilon}{n} + s_1Y(s_1)](\frac{s_1}{s_1-1})^n L^{-1}[\frac{(s-1)^n}{s^{n+1}}] - y(x) \le \frac{\varepsilon}{n}.$$
Since
$$L^{-1}[\frac{(s-1)^n}{s^{n+1}}] = 1 - nx + \binom{n}{2}\frac{x^2}{2!} - \binom{n}{3}\frac{x^3}{3!} + ... + (-1)^{n+1}\frac{x^n}{n!},$$

it follows that

$$-\frac{\varepsilon}{n} \leq [\frac{\varepsilon}{n} + s_1 Y(s_1)](\frac{s_1}{s_1 - 1})^n [1 - nx + \binom{n}{2}\frac{x^2}{2!} - \binom{n}{3}\frac{x^3}{3!} + ... + (-1)^{n+1}\frac{x^n}{n!}] - y(x) \leq \frac{\varepsilon}{n}.$$

Then, we can write

$$| y(x) - y_0(x) | \leq \frac{\varepsilon}{n},$$

where

$$y_0(x) = (s_1 Y(s_1) - \frac{\varepsilon}{n})(\frac{s_1}{s_1 - 1})^n (1 - nx + \binom{n}{2}\frac{x^2}{2!} - \binom{n}{3}\frac{x^3}{3!} + ... + (-1)^{n+1}\frac{x^n}{n!}).$$

This completes the proof of Hyers-Ulam stability of solutions of equation (1).

Our second and last main result is the following theorem.

Theorem 2. Let $\varepsilon \in \Re, \varepsilon > 0$. If the function y satisfies the differential inequality

$$| xy'' + y' + xy | \leq \varepsilon \tag{4}$$

for all $x \in I$, then there exists a solution $y_0 : I \to \Re$ of equation (2) such that

$$| y(x) - y_0(x) | \leq 2\varepsilon.$$

Proof. It is clear from (4) that

$$-\varepsilon \leq xy'' + y' + xy \leq \varepsilon.$$

When we apply the Laplace transform to the last inequality, we get

$$L(-\varepsilon) \leq L(xy'') + L(y') + L(xy) \leq L(\varepsilon).$$

Then, it follows that

$$-\frac{\varepsilon}{s} \leq -\frac{d}{ds}[s^2 Y(s) - sY(0) - Y'(0)] + sY(s) - Y(0) - \frac{d}{ds}Y(s) \leq \frac{\varepsilon}{s}.$$

Hence

$$-\frac{\varepsilon}{s} \leq -s^2 Y'(s) - 2sY(s) + Y(0) + sY(s) - Y(0) - Y'(s) \leq \frac{\varepsilon}{s}$$

so that

$$-\frac{\varepsilon}{s} \leq -(s^2 + 1)Y'(s) - sY(s) \leq \frac{\varepsilon}{s}.$$

Multiplying the last inequality with the term $-\frac{1}{\sqrt{s^2+1}}$, we arrive at

$$-\frac{\varepsilon}{s\sqrt{s^2 + 1}} \leq \sqrt{s^2 + 1}Y'(s) + \frac{s}{\sqrt{s^2 + 1}}Y(s) \leq \frac{\varepsilon}{s\sqrt{s^2 + 1}}$$

so that

$$-\frac{\varepsilon}{s\sqrt{s^2 + 1}} \leq \frac{d}{ds}(\sqrt{s^2 + 1}Y(s)) \leq \frac{\varepsilon}{s\sqrt{s^2 + 1}}.$$

For any $s_1 > s$, integrating the above inequality from s to s_1, we get

$$\begin{aligned} -\varepsilon[\ln(\frac{\sqrt{s^2+1}+1}{s}) - \ln(\frac{\sqrt{s_1^2+1}+1}{s_1})] &\leq \sqrt{s_1^2+1}Y(s_1) - \sqrt{s^2+1}Y(s) \\ &\leq \varepsilon[\ln(\frac{\sqrt{s^2+1}+1}{s}) - \ln(\frac{\sqrt{s_1^2+1}+1}{s_1})]. \end{aligned}$$

In view of the last inequality, we can write

$$-\varepsilon\ln(\frac{\sqrt{s^2+1}+1}{s}) \leq \sqrt{s_1^2+1}Y(s_1) - \sqrt{s^2+1}Y(s) \leq \varepsilon\ln(\frac{\sqrt{s^2+1}+1}{s}).$$

Multiplying the last inequality with term $\frac{1}{\sqrt{s^2+1}}$, we obtain

$$-\frac{\varepsilon}{\sqrt{s^2+1}}\ln(\frac{\sqrt{s^2+1}+1}{s}) \leq \frac{\sqrt{s_1^2+1}}{\sqrt{s^2+1}}Y(s_1) - Y(s) \leq \frac{\varepsilon}{\sqrt{s^2+1}}\ln(\frac{\sqrt{s^2+1}+1}{s}).$$

Since $s > 0$, we can write

$$-\frac{\varepsilon}{\sqrt{s^2+1}}\frac{\sqrt{s^2+1}+1}{s} \leq \frac{\sqrt{s_1^2+1}}{\sqrt{s^2+1}}Y(s_1) - Y(s) \leq \frac{\varepsilon}{\sqrt{s^2+1}}\frac{\sqrt{s^2+1}+1}{s}$$

so that

$$-\frac{2\varepsilon}{s} \leq \frac{\sqrt{s_1^2+1}}{\sqrt{s^2+1}}Y(s_1) - Y(s) \leq \frac{2\varepsilon}{s}.$$

If we apply the inverse Laplace transform, then we obtain

$$L^{-1}(-\frac{2\varepsilon}{s}) \leq L^{-1}(\frac{\sqrt{s_1^2+1}}{\sqrt{s^2+1}}Y(s_1)) - L^{-1}(Y(s)) \leq L^{-1}(\frac{2\varepsilon}{s})$$

so that

$$-2\varepsilon \leq \sqrt{s_1^2+1}Y(s_1)J_0(x) - y(x) \leq 2\varepsilon,$$

where

$$J_0(x) = 1 - \frac{1}{1!}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \dots.$$

From this,we can obtain

$$\mid y(x) - y_0(x) \mid \leq 2\varepsilon,$$

where

$$y_0(x) = -\sqrt{s_1^2+1}Y(s_1)J_0(x).$$

This completes the proof of Hyers-Ulam stability of solutions of equation (2).

3 Conclusion

A kind of linear differential equations of second order, namely Laguerre and Bessel equations, is considered. Sufficient conditions are established guaranteeing the Hyers -Ulam stability of solutions of these equations. To prove the main results here, we benefit from the Laplace transform method. The results obtained essentially complement the results in the literature.

References

[1] Abdollahpour, M. R.and Aghayari, R. and Rassias, M. Th. Hyers-Ulam stability of associated Laguerre differential equations in a subclass of analytic functions. *J. Math. Anal. Appl.* **1** (2016) 605–612.

[2] Alqifiary, Q.H. and Jung, S.M. Laplace transform and generalized Hyers-Ulam stability of linear differential equations. *Electron. J. Differential Equations* **80** (2014) 11 pp.

[3] Alsina, C. and Ger, R. On some inequalities and stability results related to the exponential function. *J. Inequal. Appl.* **2** (1998) 373–380.

[4] Bicer, E. and Tunç, C. On the Hyers-Ulam stability of certain partial differential equations of second order. *Nonlinear Dyn. Syst. Theory* **17** (2) (2017) 150–157.

[5] Hyers, D.H. On the Stability of the Linear Functional Equation. *Proc. Nat. Acad. Sci.* **27** (1941) 222–224.

[6] Jung, S.M. Hyers Ulam stability of linear differential equations of first order (III). *J. Math. Anal. Appl.* **311** (2005) 139–146.

[7] Jung, S.M. Hyers Ulam stability of linear differential equations of first order (II). *Appl. Math. Lett.* **19** (2006) 854–858.

[8] Jung, S.M. Hyers Ulam stability of first order linear partial differential equations with constant coefficients. *Math. Inequal. Appl.* **10** (2007) 261–266.

[9] Jung, S.M. Hyers Ulam stability of linear partial differential equations of first order. *Appl. Math. Lett.* **22** (2009) 70–74.

[10] Jung, S.M. Approximation of analytic functions by Legendre functions. *Nonlinear Anal.* **71** (2009) 103–108.

[11] Jung, S.M. Approximation of analytic functions by Laguerre functions. *Appl. Math. Comput.* **218** (2011) 832–835.

[12] Liu, H. and Zhao, X. Hyers-Ulam-Rassias stability of second order partial differential equations. *Ann. Differential Equations* **29** (4) (2013) 430–437.

[13] Lungu, N. and Popa, D. Hyers-Ulam stability of a first order partial differential equation. *J. Math. Anal. Appl.* **385** (2012) 86–91.

[14] Lungu, N. and Popa, D. Hyers-Ulam stability of some partial differential equations. Carpathian *J. Math.* **30** (3) (2014) 327–334.

[15] Rassias, T.M. On the Stability of the Linear Mapping in Banach Spaces. Proc. Amer. *Math. Soc.* **72** (2) (1978) 297–300.

[16] Tunc, C. and Bicer, E. Hyers-Ulam-Rassias stability for a first order functional differential equation. *J. Math. Fundam. Sci.* **47** (2) (2015) 143–153.

[17] Ulam, S.M. *Problems in Modern Mathematics*. Science Editions John Wiley& Sons, Inc. New York. 1964.

[18] Vlasov, V. Asymptotic behavior and stability of the solutions of functional differential equations in Hilbert space. *Nonlinear Dyn. Syst. Theory* **2** (2) (2002) 215–232.

[19] Vasundhara D.J. Stability in terms of two measures for matrix differential equations and graph differential equations. *Nonlinear Dyn. Syst. Theory* **16** (2) (2016) 179–191.