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Nonlinear Parabolic Equations with Singular Coefficient and Diffuse Data

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Abstract: In this paper we introduce a notion of renormalized solution for nonlinear parabolic problems whose model is $\frac{\partial b(u)}{\partial t} - \Delta A(u) - div\,(\Phi(x,t,u)Du) = \mu$ in Q, where b is a strictly increasing C^1-function defined on $\mathbb{R}$, and $A(z) = \int_0^z a(s)ds$. The function $a(s)$ is continuous on an interval $]-\infty, m[$ of $\mathbb{R}$ such that $a(u)$ blows up for a finite value m of the unknown u, Φ is a Carathéodory function and μ is a diffuse measure.

Keywords: *nonlinear parabolic equations; renormalized solutions; soft measure.*

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1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 1)$, T be a positive real number, and $Q = \Omega \times (0,T)$.

In this paper we deal with the existence of a renormalized solution for a class of nonlinear parabolic equations of the type

$$\frac{\partial b(u)}{\partial t} - \Delta A(u) - div\,(\Phi(x,t,u)Du) = \mu \quad \text{in } Q, \tag{1}$$

$$b(u(t=0)) = b(u_0) \quad \text{in } \Omega, \tag{2}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,T). \tag{3}$$

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In problem (1)-(3), the function b is assumed in $C^1(\mathbb{R})$, such that it is strictly increasing, and $A(z) = \int_0^z a(s)ds$, where the function $a \in C^0(]-\infty, m[, \mathbb{R}^+)$ (m is a positive real number) such that $\lim_{s \to m^-} a(s) = +\infty$. The function Φ is Carathéodory on $Q \times \mathbb{R}$ with values in $\mathbb{R}^+$ and $u_0 \in L^1(\Omega)$ such that $u_0 \leq m$ a.e. in Ω.

We study problem (1)-(3) in the presence of diffuse measure data μ. We call a finite measure μ diffuse if it does not charge sets of zero 2-capacity and $\mathcal{M}_0(Q)$ will denote the set of all diffuse measures in Q (see, [14]). In [9] the authors proved that for every $\mu \in \mathcal{M}_0(Q)$ there exist $f \in L^1(Q)$, $g \in L^2(0, T; H_0^1(\Omega))$ and $G \in L^2(0, T; H^{-1}(\Omega))$ such that $\mu = f + G + g_t$ in $\mathcal{D}'(Q)$. For $v = b(u) - g$, equation (1) is equivalent in $\mathcal{D}'(Q)$ to $\frac{\partial v}{\partial t} - div\Big(a(b^{-1}(v+g))D(b^{-1}(v+g))\Big) - div\Big(\Phi(x, t, b^{-1}(v+g))D(b^{-1}(v+g))\Big) = f + G$ with $f + G \in L^1(Q) + L^2(0, T; H^{-1}(\Omega))$. The first difficulty in solving this equation is defining the field $a(b^{-1}(v+g))D(b^{-1}(v+g))$ on the subset $\{(x, t);\ v+g = b(m)\}$ of Q, since on this set, $a(b^{-1}(v+g)) = +\infty$. In addition, the field $\Phi(x, t, b^{-1}(v+g))D(b^{-1}(v+g)) \notin \mathcal{D}'(Q)$ in general, since $g \notin L^\infty(Q)$ in general.

The second difficulty is represented here by the presence of the measure data μ and the nonlinear term $b(u)$. To overcome these difficulties, we use in this paper the framework of renormalized solutions. A large number of papers was then devoted to the study of renormalized (or entropy) solution of parabolic problems with rough data under various assumptions and in different contexts: in addition to the references already mentioned, see, e.g., [1, 3, 6–8, 10, 11].

The existence of a renormalized solution of (1)-(3) has been proved in [2] in the stationary case where $\Phi(x, t, u) = 0$ and $\mu \in L^2(\Omega)$.

The existence and uniqueness of renormalized solution of (1)-(3) have been proved in [9], in the case where $u_0 \in L^1(\Omega)$ and $\Delta A(u)$ is replaced by p-Laplacian operator $\Delta_p u$, $\Phi(x, t, u) = 0$ and for every measure $\mu \in \mathcal{M}_0(Q)$. In the case where $b(u) = u$, $\Delta A(u)$ is replaced by $-div(a(t, x, u, \nabla u))$, $\Phi(x, t, u) = \Phi(u)$ and $\mu = f + div\, g$ where $f \in L^1(Q)$ and $g \in (L^{p'}(Q))^N$, the existence of renormalized solution has been proved in [5].

When b is assumed to satisfy $0 < b_0 \leq b'(r) \leq b_1$, $\forall r \in \mathbb{R}$, and $\Delta A(u)$ is replaced by $div(a(x, t, \nabla u))$, $\Phi(x, t, u) = 0$ and $\mu \in \mathcal{M}_0(Q)$, the existence and uniqueness of renormalized solution have been established in [4].

In the stationary and evolution cases of $u_t - div(A(x, t, u)\nabla u) = f$ in Q, where the matrix $A(x, t, s)$ blows up (uniformly with respect to (x, t)) as $s \to m^-$ and where $f \in L^1(Q)$, the existence of renormalized solution has been proved in [3].

In the case of $u_t - div(d(u)Du) = \mu$, where the coefficients $d(s) = (d_i(s))$ are continuous on an interval $]-\infty, m[$ of $\mathbb{R}$ (with $m > 0$) with value in $\mathbb{R}^+$, $u_0 \in L^1(\Omega)$ and $\mu \in \mathcal{M}_0(Q)$, the existence of renormalized solution has been proved in [15]. Our goal is to extend the approach from [15].

The organization of the paper is the following. In Section 2, we give some preliminaries on the concept of p-capacity and set out the main notation we will use throughout the paper. Section 3 will be devoted to the exposition of our main assumptions and to the definition of renormalized solution of (1)-(3). In Section 4 (Theorem 4.1) we establish the existence of such a solution. In Section 5 (Appendix), we provide the proof of Theorem 2.2. Section 6 is devoted to an example which illustrates our abstract result.

2 Preliminaries on Parabolic Capacity and Measures

For every open subset $U \subset Q$ the 2-parabolic capacity of U is given by (for further details see, [9, 14]): $cap_2(U) = inf\Big\{\|u\|_W \; : \; u \in W, \; u \geq \chi_U \; a.e. \; in \; Q\Big\}$, where $W = \Big\{u \in L^2(0,T;H_0^1(\Omega)), \; u_t \in L^2(0,T;H^{-1}(\Omega))\Big\}$, endowed with the norm $\|u\|_W = \|u\|_{L^2(0,T;H_0^1(\Omega))} + \|u_t\|_{L^2(0,T;H^{-1}(\Omega))}$. The 2-parabolic capacity is then extended to arbitrary Borel set $B \subseteq Q$ as $cap_2(B) = inf\Big\{cap_2(U) : \; U \; open \; set \; of \; Q, \; B \subseteq U\Big\}$. We will denote by $\mathcal{M}(Q)$ the set of all Radon measures with bounded variation on Q, while, as we have already mentioned, $\mathcal{M}_0(Q)$ will denote the set of all measures with bounded variation over Q that do not charge the sets of zero 2−capacity, that is: if $\mu \in \mathcal{M}_0(Q)$ then $\mu(E) = 0$ for all $E \subseteq Q$ such that $cap_2(E) = 0$.

In [9] the authors proved the following decomposition theorem:

Theorem 2.1 *Let μ be a bounded measure on Q. If $\mu \in \mathcal{M}_0(Q)$, then there exists (f, G, g) such that $f \in L^1(Q)$, $G \in L^2(0,T;H^{-1}(\Omega))$, $g \in L^2(0,T;H_0^1(\Omega))$ and*

$$\int_Q \phi \, d\mu = \int_Q f\phi \, dx \, dt + \int_0^T \langle G, \phi\rangle \, dt - \int_0^T \langle \phi_t, g\rangle \, dt \quad \phi \in C_c^\infty(\Omega \times [0,T]).$$

Such a triplet (f, G, g) will be called a decomposition of μ.

Note that the decomposition of μ is not uniquely determined.

The following theorem will be a key point in the existence result given in the next section. The proof follows the arguments in Theorem 1.2 in [13].

Theorem 2.2 *Let $a \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $b \in C^1(\mathbb{R})$ with $0 < \beta \leq b' \leq \gamma$, Φ be a Carathéodory function such that $\Phi \in L^\infty(Q \times \mathbb{R})$, $\mu \in \mathcal{M}_0(Q) \cap L^2(0,T;H^{-1}(\Omega))$ and $u_0 \in L^2(\Omega)$, let $u \in W$ be the (unique) weak solution of*

$$\begin{cases} \dfrac{\partial b(u)}{\partial t} - \Delta A(u) - div\,(\Phi(x,t,u)Du) = \mu & in \; Q, \\ b(u\,(t=0)) = b(u_0) & in \; \Omega, \\ u = 0 & on \; \partial\Omega \times (0,T). \end{cases} \tag{4}$$

Then, $cap_2\{|u| > K\} \leq \dfrac{C}{\sqrt{K}} \quad \forall \; K \geq 1$, where $C > 0$ is a constant depending on $\|\mu\|_{\mathcal{M}(Q)}$ and $\|u_0\|_{L^2(\Omega)}$.

Proof. The proof of Theorem 2.2 is postponed to the Appendix in Section 5. □

Definition 2.1 A sequence of measures (μ_n) in Q is equidiffuse if for every $\eta > 0$ there exists $\delta > 0$ such that $cap_2(E) < \delta \Longrightarrow |\mu_n|(E) < \eta \quad \forall \; n \geq 1$.

The following result is proved in [13]:

Lemma 2.1 *Let ρ_n be a sequence of mollifiers on Q. If $\mu \in \mathcal{M}_0(Q)$, then the sequence $(\rho_n * \mu_n)$ is equidiffuse.*

Here are some notations we will use throughout the paper. For any nonnegative real number K we denote by $T_K(r) = \min(K, \max(r, -K))$ the truncation function at level K for every $r \in \mathbb{R}$. We consider the following smooth approximation of $T_K(s)$: for $m > 0$, $\eta \in]0,1[$ and $\sigma \in]0,1[$, we define $S^m_{K,\sigma}$, T^m_K : $\mathbb{R} \longrightarrow \mathbb{R}$ by

$$S^{m,\eta}_{K,\sigma}(s) = \begin{cases} 1 & \text{if } -K \le s \le m-\sigma, \\ 0 & \text{if } s \le -K-\eta \text{ or } s \ge m, \\ \text{affine} & \text{otherwise,} \end{cases} \text{ and } T^m_K(s) = \begin{cases} s & \text{if } -K \le s \le m, \\ -K & \text{if } s \le -K, \\ m & \text{if } s \ge m, \end{cases} \tag{5}$$

and let us denote $T^{m,\eta}_{K,\sigma}(z) = \displaystyle\int_0^z S^{m,\eta}_{K,\sigma}(s)\,ds$.

3 Main Assumptions and Definition of Renormalized Solution

Throughout the paper, we assume that the following assumptions hold true: Ω is a bounded open set on $\mathbb{R}^N$ $(N \ge 2)$, $T > 0$ is given and we set $Q = \Omega \times (0,T)$.

$$b : \mathbb{R} \to \mathbb{R} \text{ is a strictly increasing } \mathcal{C}^1 - \text{function such that } 0 < \beta \le b' \text{ and } b(0) = 0, \tag{6}$$

$$a \in C^0(]-\infty, m[\,, \mathbb{R}^+) \text{ with } a(s) < +\infty \quad \forall s < m, \tag{7}$$

$$\exists\ \alpha > 0 \text{ such that} : a(s) \ge \alpha\ ,\ \forall s \in]-\infty, m[, \tag{8}$$

$$\lim_{s \to m^-} a(s) = +\infty \text{ and } \int_0^m a(s)\,ds < +\infty, \tag{9}$$

$$\Phi : Q \times \mathbb{R} \to \mathbb{R}^+ \text{ is a Carathéodory function such that } \Phi(x,t,0) = 0, \tag{10}$$

$$\max_{\{|r|<K\}} |\Phi(x,t,r)| \in L^\infty(Q) \quad \text{for all } K > 0, \tag{11}$$

$$\mu \in \mathcal{M}_0(Q), \tag{12}$$

$$u_0 \in L^1(\Omega) \text{ such that } u_0 \le m \text{ a.e. in } \Omega. \tag{13}$$

We now give the definition of a renormalized solution of problem (1)-(3).

Definition 3.1 A function $u \in L^1(Q)$ is a renormalized solution of problem (1)-(3) if

$$u \le m \text{ a.e. in } Q \text{ and } T_K(u) \in L^2(0,T; H^1_0(\Omega)) \ \ \forall K > 0, \tag{14}$$

$$a(u) DT^m_K(u) \chi_{\{u<m\}} \in (L^2(Q))^N \quad \forall K > 0, \tag{15}$$

if there exist a sequence of nonnegative measures $\Lambda_K \in \mathcal{M}(Q)$ and a nonnegative measure $\Gamma_m \in \mathcal{M}(Q)$ such that

$$\lim_{K \to +\infty} \|\Lambda_K\|_{\mathcal{M}(Q)} = 0, \tag{16}$$

$$\int_Q \varphi\, d\Gamma_m = 0 \quad \forall \varphi \in \mathcal{C}^1_0([0,T[), \tag{17}$$

and if, for every $K > 0$

$$\frac{\partial B_K^m(u)}{\partial t} - div\Big(a(u)DT_K^m(u)\chi_{\{u<m\}}\Big) - div\Big(\Phi(x,t,T_K^m(u))DT_K^m(u)\Big) \tag{18}$$
$$= \mu + \Lambda_K + \Gamma_m \quad \text{in } \mathcal{D}'(Q),$$

where $B_K^m(z) = \displaystyle\int_0^z b'(s)(T_K^m)'(s)\,ds$.

Remark 3.1 1/ Note that, in view of (14), (15) and (16), all terms in (18) are well defined. 2/ Let us point out that, in (17), the function $\varphi \in \mathcal{C}_0^1([0,T[)$ does not depend on the variable x, we are not able to prove (17) with any function $\varphi \in L^2(0,T;H^1(\Omega)) \cap L^\infty(Q)$ such that $D\varphi = 0$ a.e. in $\{(x,t)\,; u(x,t) = m\}$ because of a lack of regularity on u with respect to t in the parabolic case.

4 Existence of a Renormalized Solution

This section is devoted to establishing the following existence theorem.

Theorem 4.1 *Under assumptions (6)-(13) there exists at least one renormalized solution of problem (1)-(3) in the sense of Definition 3.1.*

Proof. The proof is divided into 4 steps. At Step 1, we introduce an approximate problem. Step 2 is devoted to establishing a few *a priori* estimates and we prove that u satisfies (14) and (15) of Definition 3.1. At last, Step 3 and Step 4 are aimed to prove that u satisfies (16), (17) and (18) of Definition 3.1. □

⋆ **Step 1**. A regularized problem.

Let us introduce the following regularization of the data: for $n \geq 1$ fixed

$$b_n(s) = b\left(T_n(s)\right) + \frac{1}{n}s \text{ and } a^n(s) = a\left(T_{\frac{1}{n}}^{m-\frac{1}{n}}(s)\right) \quad \forall\, s \in \mathbb{R}, \tag{19}$$

$$u_0^n \in C_c^\infty(\Omega): \; b_n(u_0^n) \to b(u_0) \text{ strongly in } L^1(\Omega) \text{ as } n \text{ tends to } +\infty, \tag{20}$$

$$\Phi_n(x,t,s) = \Phi\left(x,t,T_n(s)\right) \quad \forall s \in \mathbb{R}. \tag{21}$$

We consider a sequence of mollifiers (ρ_n), and we define the convolution $\rho_n * \mu$ for every $(x,t) \in Q$ by $\mu^n(x,t) = \rho_n * \mu(x,t) = \displaystyle\int_Q \rho_n(x-y,t-s)d\mu(y,s)$. Let us now consider the following regularized problem

$$\frac{\partial b_n(u^n)}{\partial t} - \Delta A^n\left(u^n\right) - div\left(\Phi_n(x,t,u^n)Du^n\right) = \mu^n \quad \text{in } Q, \tag{22}$$

$$b_n(u^n(t=0)) = b_n(u_0^n) \quad \text{in } \Omega, \tag{23}$$

$$u^n = 0 \quad \text{on } \partial\Omega \times (0,T). \tag{24}$$

As a consequence, proving existence of a weak solution $u^n \in L^2(0,T;H_0^1(\Omega))$ of (22)-(24) is an easy task (see e.g. [12]).

⋆ **Step 2**. A priori estimates. Taking $T_K(u^n)$ as a test function in (22) gives

$$\int_\Omega B_K^n(u^n)(T)\,dx + \int_Q DA^n\left(u^n\right)DT_K(u^n)\,dx\,dt \tag{25}$$

$$+\int_Q \Phi_n\,(x,t,u^n)\,Du^n DT_K(u^n)\,dx\,dt = \int_Q \mu^n T_K(u^n)\,dx\,dt + \int_\Omega B_K^n(u_0^n)\,dx,$$

where $B_K^n(z) = \displaystyle\int_0^z b_n'(s)T_K(s)\,ds$. We deduce

$$\int_\Omega B_K^n(u^n)(T)\,dx + \int_Q \left(a^n(u^n) + \Phi_n(x,t,u^n)\right)\left|DT_K(u^n)\right|^2\,dx\,dt \le CK \tag{26}$$

since $\|\mu^n\|_{L^1(Q)}$ and $\|b_n(u_0^n)\|_{L^1(\Omega)}$ are bounded. We deduce for any $K \ge 0$

$$T_K(u^n) \text{ is bounded in } L^2(0,T;H_0^1(\Omega)), \tag{27}$$

and

$$a^n(u^n)^{\frac{1}{2}}DT_K(u^n) \text{ is bounded in } (L^2(Q))^N. \tag{28}$$

Now, using $\frac{1}{r}T_r(u^n)\chi_{(0,t)}$ as a test function in (22) we obtain

$$\int_\Omega \frac{1}{r}B_r^n(u^n)\,dx + \frac{1}{r}\int_0^t\int_\Omega \left(a^n(u^n) + \Phi_n(x,t,u^n)\right)\left|DT_r(u^n)\right|^2 dx\,dt \le C, \tag{29}$$

where $B_r^n(z) = \displaystyle\int_0^z b_n'(s)T_r(s)\,ds$. The second term in the left-hand side of the above inequality is nonnegative. Taking the limit in (29) as r tends to 0 we obtain $b_n(u^n)$ is bounded in $L^\infty(0,T;L^1(\Omega))$. According to (7)-(9), we have for any $K \ge 0$, $\left|\displaystyle\int_0^{u^n} a^n(s)\chi_{\{-K\le s\le m\}}\,dx\right| \le \displaystyle\int_{-K}^m a(s)\,ds \equiv C_K < +\infty$, then we can use $\displaystyle\int_0^{u^n} a^n(s)\chi_{\{-K\le s\le m\}}\,ds$ in $L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$ as a test function in (22), we have

$$\beta\int_\Omega\int_0^{u^n}\int_0^z a^n(s)\chi_{\{-K\le s\le m\}}\,ds\,dz\,dx \tag{30}$$

$$+\int_Q \Big((a^n(u^n))^2 + \Phi_n(x,t,u^n)a^n(u^n)\Big)\left|DT_K^m(u^n)\right|^2 \le \left(\|\mu^n\|_{L^1} + \|b_n(u_0^n)\|_{L^1}\right)\int_{-K}^m a(s)\,ds.$$

Since $\displaystyle\int_\Omega\int_0^{u^n}\int_0^z a^n(s)ds\,dz\,dx$ and $\displaystyle\int_Q \Phi_n(x,t,u^n)a^n(u^n)\left|DT_K^m(u^n)\right|^2 dx\,dt$ are positives, $\|\mu^n\|_{L^1(Q}$ and $\|b_n(u_0^n)\|_{L^1(\Omega)}$ are bounded, we deduce from (30) that

$$a^n(u^n)DT_K^m(u^n) \text{ is bounded in } (L^2(Q))^N. \tag{31}$$

For any integer $M \ge 1$, let S_M be an increasing function of $C^\infty(\mathbb{R})$ and such $S_M(r) = r$ for $|r| \le \frac{M}{2}$ and $S_M(r) = Msg(r)$ for $|r| \ge M$. Note that for any M, $supp\, S_M' \subset [-M,M]$. We will show that for any fixed integer M the sequence $S_M(b_n(u^n))$ satisfies

$$S_M(b_n(u^n)) \text{ is bounded in } L^2(0,T;H_0^1(\Omega)), \tag{32}$$

and

$$\frac{\partial S_M(b_n(u^n))}{\partial t} \text{ is bounded in } L^1(Q) + L^2(0,T;H^{-1}(\Omega)) \tag{33}$$

independently of n. Due to the definition of b_n, it is clear that for $|b_n(u^n)| \leq M$ we have $|b(T_n(u^n)| \leq M$ and $|u^n| < K_M$ as soon as $n > K_M$ and where $K_M = max\left\{b^{-1}(M), \left|b^{-1}(-M)\right|\right\}$. As a first consequence we obtain $DS_M(b_n(u^n)) = S'_M(b_n(u^n))b'_n(u^n)DT_{K_M}(u^n)$ as soon as $n > K_M$, since $S'_M(b_n(u^n)) = 0$ on the set $\{|b_n(u^n)| > M\}$, and $K_M = max\{-b^{-1}(M), |b^{-1}(-M)|\}$. Secondly, the following estimate holds true $\|S'_M(b_n(u^n))b'_n(u^n)\|_{L^\infty(Q)} \leq \|S'_M\|_{L^\infty(\mathbb{R})} \left(\max_{|r|\leq K_M} |b'(r)| + 1\right)$ as soon as $n > K_M$. Since b' is continuous on $\mathbb{R}$, it follows that for any integer M, $S'_M(b_n(u^n))b'_n(u^n)$ is bounded in $L^\infty(Q)$ independently of n as soon as $n > K_M$. As a consequence of (27) we then obtain (32).

To show that (33) holds true, we multiply the equation (22) by $S'_M(b_n(u^n))$ to obtain

$$\frac{\partial S_M(b_n(u^n))}{\partial t} = div\Big(S'_M(b_n(u^n))a^n(u^n)Du^n\Big) - S''_M(b_n(u^n))b'_n(u^n)a^n(u^n)\,|Du^n|^2 \quad (34)$$

$$+div\Big(S'_M(b_n(u^n))\Phi_n(x,t,u^n)Du^n\Big) - S''_M(b_n(u^n))b'_n(u^n)\Phi_n(x,t,u^n)\,|Du^n|^2 + \mu^n S'_M(b_n(u^n))$$

in $\mathcal{D}'(Q)$. Each term in the right-hand side of (34) is bounded either in $L^2(0,T;H^{-1}(\Omega))$ or in $L^1(Q)$. Indeed, since $suppS'_M$ and $suppS''_M$ are both included in $[-M,M]$, u^n may be replaced by $T_{K_M}(u^n)$ in each of these terms.

Proceeding as in [5] we see that estimates (32) and (33) imply that, for a subsequence still indexed by n, $b_n(u^n) \to \chi$ almost everywhere in Q. Since b^{-1} is continuous on $\mathbb{R}$, b_n^{-1} converges everywhere to b^{-1} when n goes to ∞, so that $u^n \to u = b^{-1}(\chi)$ a.e. in Q and using (27), (28) and (31), we obtain

$$b_n(u^n) \longrightarrow b(u) \text{ almost everywhere in } Q, \quad (35)$$

$$T_K(u^n) \rightharpoonup T_K(u) \text{ weakly in } L^2(0,T;H_0^1(\Omega)), \quad (36)$$

$$(a^n(u^n))^{\frac{1}{2}} DT_K(u^n) \rightharpoonup X_K \text{ weakly } in\ (L^2(Q))^N, \quad (37)$$

$$a^n(u^n)DT_K^m(u^n) \rightharpoonup Y_K \text{ weakly } in\ (L^2(Q))^N. \quad (38)$$

By using the admissible test function $T_{2m}^{n+}(u^n) - T_m^{n+}(u^n)$ in (22) we have

$$\int_Q (a^n(u^n) + \Phi_n(x,t,u^n)) \left|D\left(T_{2m}^{n+}(u^n) - T_m^{n+}(u^n)\right)\right|^2 dx\,dt \leq Cm. \quad (39)$$

Now, since $\Phi_n(x,t,u^n) \geq 0$, and in view of (19) and the Poincaré inequality we deduce

$$a(m - \frac{1}{n}) \int_Q \left|T_{2m}^{n+}(u^n) - T_m^{n+}(u^n)\right|^2 dx\,dt \leq Cm. \quad (40)$$

According to (9) and (20) (since $d_p(m - \frac{1}{n}) \to +\infty$ as n tends to $+\infty$) passing to the limit in (40) as n tends to $+\infty$, we deduce that $T_{2m}^+(u) - T_m^+(u) = 0$ a.e. in Q, hence

$$u \leq m \quad \text{a.e. in } Q. \quad (41)$$

In view of (37), (38) and (41) we deduce for any $K \geq 0$

$$X_K = (a(u))^{\frac{1}{2}}\, DT_K(u) \text{ and } Y_K = a(u)DT_K^m(u) \text{ a.e. in } \{(x,t) \in Q \ /\ u(x,t) < m\}. \quad (42)$$

We define, for any fixed $K \geq 1$, $0 < \eta < 1$ and $0 < \sigma < 1$, the functions $H_{K,\eta}$ and $Z_{m,\sigma}$ by

$$H_{K,\eta}(s) = \begin{cases} -1, & \text{if } s \leq -K-\eta, \\ 0, & \text{if } s \geq -K, \\ \text{affine}, & \text{otherwise}, \end{cases} \quad \text{and} \quad Z_{m,\sigma}(s) = \begin{cases} 0, & \text{if } s \leq m-\sigma, \\ 1, & \text{if } s \geq m, \\ \text{affine}, & \text{otherwise}. \end{cases} \tag{43}$$

We use the admissible test functions $H_{K,\eta}(u^n)$ and $Z_{m,\sigma}(u^n)$ in (22) to get

$$\int_\Omega \overline{H}_{K,\eta}(u^n)(T)\,dx + \int_Q DA^n(u^n)DH_{K,\eta}(u^n)\,dx\,dt \tag{44}$$

$$+\int_Q \Phi_n(x,t,u^n)\,Du^n DH_{K,\eta}(u^n)\,dx\,dt = \int_Q H_{K,\eta}(u^n)\mu^n\,dx\,dt + \int_\Omega \overline{H}_{K,\eta}(u_0^n)\,dx,$$

and

$$\int_\Omega \overline{Z}_{m,\sigma}(u^n)(T)\,dx + \int_Q DA^n(u^n)DZ_{m,\sigma}(u^n)\,dx\,dt \tag{45}$$

$$+\int_Q \Phi_n(x,t,u^n)\,Du^n DZ_{m,\sigma}(u^n)\,dx\,dt = \int_Q Z_{m,\sigma}(u^n)\mu^n\,dx\,dt + \int_\Omega \overline{Z}_{m,\sigma}(u_0^n)\,dx,$$

where $\overline{H}_{K,\eta}(r) = \int_0^r b_n'(s)H_{K,\eta}(s)ds \geq 0$ for $r \leq 0$ and $\overline{Z}_{m,\sigma}(r) = \int_0^r b_n'(s)Z_{m,\sigma}(s)\,ds \geq 0$ for $r \geq 0$. Hence, using (43) and dropping a nonnegative term, we obtain

$$\frac{1}{\eta}\int_{\{-K-\eta\leq u^n\leq -K\}} (a^n(u^n)+\Phi_n(x,t,u^n))\,|Du^n|^2\,dx\,dt \tag{46}$$

$$\leq \int_{\{u^n\leq -K\}} |\mu^n|\,dx\,dt + \int_{\left\{u_0^n\leq -K\right\}} |b_n(u_0^n)|\,dx \leq C_1,$$

and

$$\frac{1}{\sigma}\int_{\{m-\sigma\leq u^n\leq m\}} (a^n(u^n)+\Phi_n(x,t,u^n))\,|Du^n|^2\,dx\,dt \leq \|\mu^n\|_{L^1(Q)} + \|b_n(u_0^n)\|_{L^1(\Omega)} \leq C_2. \tag{47}$$

Thus, there exists a bounded Radon measure Ψ_K^n, as η tends to zero

$$\Psi_{K,\eta}^n \equiv \frac{1}{\eta}\,(a^n(u^n)+\Phi_n(x,t,u^n))\,|Du^n|^2\,\chi_{\{-K-\eta\leq u^n\leq -K\}} \rightharpoonup \Psi_K^n \ast -\text{ weakly in } \mathcal{M}(Q). \tag{48}$$

$\star$ **Step 3**. At this step we prove that u satisfies (18). Let $S_{K,\sigma}^{m,\eta}$ be the function defined by (5) for all real numbers $\sigma > 0$, $\eta > 0$ and $K > 0$. Since $supp\,(S_{K,\sigma}^{m,\eta})' \subset [-K-\eta,-K] \cup [m-\sigma,m]$, we multiply the equation (22) by $S_{K,\sigma}^{m,\eta}(u^n)$ to get

$$\frac{\partial B_{K,\sigma}^{n,m,\eta}(u^n)}{\partial t} - div\left(DA^n(u^n)\,S_{K,\sigma}^{m,\eta}(u^n)\right) + DA^n(u^n)DS_{K,\sigma}^{m,\eta}(u^n) \tag{49}$$

$$-div\left(\Phi_n(x,t,u^n)Du^n S_{K,\sigma}^{m,\eta}(u^n)\right) + \Phi_n(x,t,u^n)\,Du^n DS_{K,\sigma}^{m,\eta}(u^n) = \mu^n S_{K,\sigma}^{m,\eta}(u^n) \text{ in } \mathcal{D}'(Q),$$

where $B^{n,m,\eta}_{K,\sigma}(z) = \displaystyle\int_0^z b'_n(s) S^{m,\eta}_{K,\sigma}(s)ds$. Let

$$\lambda^n_{m,\sigma} \equiv \frac{1}{\eta}\left(a^n(u^n) + \Phi_n(x,t,u^n)\right)|Du^n|^2 \chi_{\{m-\sigma\le u^n\le m\}}. \tag{50}$$

From (48), (50) and (49), we deduce that

$$\frac{\partial B^{n,m,\eta}_{K,\sigma}(u^n)}{\partial t} - div\Big(DA^n(u^n)\, S^{m,\eta}_{K,\sigma}(u^n)\Big) - div\Big(\Phi_n(x,t,u^n) DT^{m,\eta}_{K,\sigma}(u^n)\Big) \tag{51}$$
$$= \mu^n + \Big(S^{m,\eta}_{K,\sigma}(u^n) - 1\Big)\mu^n - \Psi^n_{K,\eta} + \lambda^n_{m,\sigma} \quad \text{in } \mathcal{D}'(Q).$$

Passing to the limit in (51) as η tends to zero, we deduce

$$\frac{\partial B^{n,m}_{K,\sigma}(u^n)}{\partial t} - div\left(DA^n(u^n) S^m_{K,\sigma}(u^n)\right) - div\left(\Phi_n(x,t,u^n) DT^m_{K,\sigma}(u^n)\right) \tag{52}$$
$$= \mu^n - \mu^n\chi_{\{u^n<-K\}} - Z_{m,\sigma}(u^n)\mu^n - \Psi^n_K + \lambda^n_{m,\sigma} \text{ in } \mathcal{D}'(Q).$$

We define the measures $\Lambda^n_K = -\mu^n\chi_{\{u^n<-K\}} - \Psi^n_K$ and $\Gamma^n_{m,\sigma} = -Z_{m,\sigma}(u^n)\mu^n + \lambda^n_{m,\sigma}$. Now, using the properties of convolution $\mu_n = \rho_n * \mu$ and in view of (46), (47), (48) and (50), we deduce that Λ^n_K and $\Gamma^n_{m,\sigma}$ are bounded in $L^1(Q)$ independently of n, so that there exist bounded measures Λ_K and $\Gamma_{m,\sigma}$ such that $\Lambda^n_K \rightharpoonup \Lambda_K$ $*-$weakly in $\mathcal{M}(Q)$ and $\Gamma^n_{m,\sigma} \rightharpoonup \Gamma_{m,\sigma}$ $*-$weakly in $\mathcal{M}(Q)$. We deduce from (35), (36), (38), (41) (42) and (52) that u satisfies

$$B^m_{K,\sigma}(u)_t - div\Big(a(u) DT^m_K(u) S^m_{K,\sigma}(u)\chi_{\{u<m\}}\Big) \tag{53}$$
$$-div\Big(\Phi(x,t,T^m_K(u)) DT^m_{K,\sigma}(u)\Big) = \mu + \Lambda_K + \Gamma_{m,\sigma} \quad \text{in } \mathcal{D}'(Q).$$

To end the proof of (18), we use

$$\int_Q |\Gamma_{m,\sigma}|\, dx\, dt \le \liminf_{n\to+\infty}\int_Q \left|\Gamma^n_{m,\sigma}\right|\, dx\, dt \le 2\,\|\mu\|_{\mathcal{M}(Q)} + \|b(u_0)\|_{L^1(\Omega)}$$

so that there exists a bounded measure Γ_m such that $\Gamma_{m,\sigma}$ converges to Γ_m $*-$weakly in $\mathcal{M}(Q)$. Therefore, as σ tends to zero in (53), we obtain in $\mathcal{D}'(Q)$

$$\frac{\partial B^m_K(u)}{\partial t} - div\left(a(u) DT^m_K(u)\,\chi_{\{u<m\}}\right) - div\left(\Phi(x,t,T^m_K(u) DT^m_K(u)\right) = \mu + \Lambda_K + \Gamma_m, \tag{54}$$

where $B^m_K(z) = \displaystyle\int_0^z b'(s)(T^m_K)'(s)ds$, and (18) is then established.

⋆ **Step 4**. At this step we prove that Λ_K and Γ_m satisfy (16) and (17). From (46) and (48), it follows that

$$\|\Lambda^n_K\|_{L^1(Q)} = \|-\mu^n\chi_{\{u^n<-K\}} + \Psi^n_K\|_{L^1(Q)} \le 2\int_{\{u^n<-K\}} |\mu^n|\, dx\, dt + \int_{\{u^n_0<-K\}} |b_n(u^n_0)|\, dx. \tag{55}$$

Since $\|\Lambda_K\|_{\mathcal{M}(Q)} \le \liminf_{n\to+\infty}\|\mu^n\chi_{\{u^n<-K\}} + \Psi^n_K\|_{\mathcal{M}(Q)}$, the sequence (μ^n) is equidiffuse, and the function $b_n(u^n_0)$ converges to $b(u_0)$ strongly in $L^1(\Omega)$, we deduce from theorem

2.2 and (55) that $\|\Lambda_K\|_{\mathcal{M}(Q)}$ tends to zero as K tends to infinity, then we obtain (16). To prove (17), we can write for all $\varphi \in C_0^1([0,T[)$

$$\int_Q \varphi \, d\Gamma_m = \lim_{\sigma\to 0}\int_Q \varphi \, d\Gamma_{m\sigma} = \lim_{\sigma\to 0}\lim_{n\to+\infty}\int_Q \varphi \Gamma_\sigma^n \, dx\, dt, \tag{56}$$

where $\Gamma_{m,\sigma}^n = \lambda_{m,\sigma}^n - Z_{m,\sigma}(u^n)\mu^n$. Taking the admissible test function $Z_{m,\sigma}(u^n)\varphi$ in (22), we have

$$-\int_Q \overline{Z}_{m,\sigma}(u^n)\varphi_t \, dx\, dt - \int_\Omega \overline{Z}_{m,\sigma}(u_0^n)\varphi(0)\, dx + \int_Q DA^n(u^n)D(Z_{m,\sigma}(u^n)\varphi)\, dx\, dt \tag{57}$$

$$+\int_Q \Phi(x,t,u^n)D(Z_{m,\sigma}(u^n)\varphi)\, dx\, dt = \int_Q Z_{m,\sigma}(u^n)\mu^n\varphi \, dx\, dt,$$

where $\overline{Z}_{m,\sigma}(r) = \displaystyle\int_0^r b_n'(s)Z_{m,\sigma}(s)ds$. We deduce from (57) that

$$-\int_Q \overline{Z}_{m,\sigma}(u^n)\varphi_t \, dx\, dt - \int_\Omega \overline{Z}_{m,\sigma}(u_0^n)\varphi(0)\, dx \tag{58}$$

$$= \int_{\{m-\sigma\le u^n\le m\}} \frac{1}{\sigma}\left(a^n(u^n) + \Phi_n(x,t,u^n)\right)|Du^n|^2 \varphi \, dx\, dt - \int_Q Z_{m,\sigma}(u^n)\mu^n\varphi\, dx\, dt.$$

In the sequel we pass to the limit in (58) when n tends to infinity and then σ tends to zero. Note that $\overline{Z}_{m,\sigma}(u^n)$ converges to $\overline{Z}_{m,\sigma}(u)$ strongly in $L^1(Q)$ and $\overline{Z}_{m,\sigma}(u_0^n)$ converges to $\overline{Z}_{m,\sigma}(u_0)$ strongly in $L^1(\Omega)$ as n tends to infinity. Moreover, since $\overline{Z}_{m,\sigma}(u)$ converges to $(b(u)-b(m))^+$ as σ tends to zero, $u \le m$ and $u_0 \le m$ almost everywhere, then it is easy to see that

$$\lim_{\sigma\to 0}\lim_{n\to+\infty}\int_Q \overline{Z}_{m,\sigma}(u^n)\varphi_t\, dx\, dt = 0 \text{ and } \lim_{\sigma\to 0}\lim_{n\to+\infty}\int_\Omega \overline{Z}_{m,\sigma}(u_0^n)\varphi(0)\, dx = 0. \tag{59}$$

Then, from (56), (58) and (59) we deduce (17).

As a conclusion of step 1 to step 4, the proof of Theorem 4.1 is complete.

5 Appendix

Here we prove Theorem 2.2.

Proof. Let $b(u) = v$, then equation (4) is equivalent to

$$\begin{cases} v_t - div\,(G(x,t,v)Dv) = \mu & \text{in } Q,\\ v(x,0) = b(u(x,0)) & \text{in } \Omega,\\ v = 0 & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{60}$$

where $G(x,t,v) = \dfrac{a\left(b^{-1}(v)\right) + \Phi\left(x,t,b^{-1}(v)\right)}{b'\left(b^{-1}(v)\right)}$. For simplicity we assume that $\mu \ge 0$ and $u_0 \ge 0$. We use the admissible test function $T_K(u)$ in (60) to get

$$\int_\Omega \overline{T}_K(v)\, dx + \int_Q \left|(G(x,t,v)^{\frac{1}{2}} DT_K(v)\right|^2 dx\, dt \le K\left(\|\mu\|_{\mathcal{M}(Q)} + \|b(u_0)\|_{L^1(\Omega)}\right) \equiv KM, \tag{61}$$

where $\overline{T}_K(r) = \int_0^r T_K(s)ds$. Since $\frac{1}{2}T_K^2(r) \leq \overline{T}_K(r) \leq Kr$, $\beta \leq b' \leq \gamma$ and $G(x,t,v) \geq \frac{\alpha}{\gamma}$, we deduce that $\max\left\{\|T_K(v)\|^2_{L^\infty(0,T;L^2(\Omega))}; \|G(x,t,v)^{\frac{1}{2}}DT_K(v)\|^2_{L^2(Q)}\right\} \leq KM$ and $\|T_K(v)\|^2_{L^2(0,T;H^1_0(\Omega))} \leq \gamma\frac{KM}{\alpha}$. Let $z \in W$ be the solution of

$$\begin{cases} -z_t - div\,(G(x,t,v)Dz) = -2\; div\,(G(x,t,v)DT_K(v)) & in\; Q, \\ z = 0 & on\; (0,T)\times\partial\Omega, \\ z(t=T) = T_K(v(t=T)) & in\; \Omega. \end{cases} \tag{62}$$

Taking the admissible test function z in (62) and integrating between τ and T, we have by Young's inequality that $\max\left\{ \|z\|^2_{L^\infty(0,T;L^2(\Omega))};\ \|Dz\|^2_{L^2(Q)} \right\} \leq CKM$. Moreover, the equation (62) implies that $\|z_t\|_{L^2(0,T;H^{-1}(\Omega))} \leq C\Big(\|z\|_{L^2(0,T;H^1_0(\Omega))} + \|T_K(v)\|_{L^2(0,T;H^1_0(\Omega))}\Big)$. Hence we deduce that $\|z\|_W \leq C\sqrt{K}$. Since $\mu \geq 0$, $b(u_0) \geq 0$ and $G(x,t,v) \geq 0$, we have $v_t - div\,(G(x,t,v)Dv) \geq 0$ and $v \geq 0$ in Q, and by a non-linear version of Kato's inequality for parabolic equations (see [13]), we deduce that $T_K(v)_t - div\,(G(x,t,v)DT_K(v)) \geq 0$. Then we conclude that $-z_t - div(G(x,t,v)Dz) \geq -T_K(v)_t - div(G(x,t,v)DT_K(v))$ in $\mathcal{D}'(Q)$. Now, using the standard comparison argument, we easily see that $z \geq T_K(v)$ a.e. in Q, hence $z \geq K$ a.e. on $\{v > K\}$, and we conclude that $cap_2\{v > K\} \leq \left\|\frac{z}{K}\right\|_W \leq \frac{C}{\sqrt{K}}$, the proof of Theorem 2.2 is complete. □

6 Example

Let us consider the following special case: $b(s) = s(e^s+1)$, $a(s) = \frac{1}{(m-s)^{\frac{1}{3}}}$ for $s < m$ and $\Phi(x,t,s) = L(x,t)e^{s^2}$, where $L(x,t) \in L^\infty(Q)$. Note that $A(s) = \int_0^s a(r)\,dr = \frac{3}{2}(m^{\frac{2}{3}} - (m-s)^{\frac{2}{3}})$ and $A(m) = \frac{3}{2}m^{\frac{2}{3}} < +\infty$. Finally, it is easy to show that the hypotheses of Theorem 4.1 are satisfied. Therefore, for all $\mu \in \mathcal{M}_0(Q)$ and $u_0 \in L^1(\Omega)$ with $u_0 \leq m$, there exists at least one renormalized solution of problem (1)-(3), and then u satisfies

$$u \in L^1(Q),\ u \leq m \text{ a.e. in } Q \text{ and } T_K(u) \in L^2(0,T;H^1_0(\Omega))\ \ \forall K > 0, \tag{63}$$

$$\frac{1}{(m-u)^{\frac{1}{3}}}DT^m_K(u)\chi_{\{u<m\}} \in (L^2(Q))^N \quad \forall K > 0. \tag{64}$$

There exist a sequence of nonnegative measures $\Lambda_K \in \mathcal{M}(Q)$ and a nonnegative measure $\Gamma_m \in \mathcal{M}(Q)$ such that

$$\lim_{K\to+\infty} \|\Lambda_K\|_{\mathcal{M}(Q)} = 0 \text{ and } \int_Q \varphi\, d\Gamma_m = 0 \quad \forall \varphi \in \mathcal{C}^1_0([0,T[), \tag{65}$$

and for every $K > 0$

$$\frac{\partial B^m_K(u)}{\partial t} - div\Big(\frac{1}{(m-u)^{\frac{1}{3}}}DT^m_K(u)\chi_{\{u<m\}}\Big) - div\Big(L(x,t)e^{(T^m_K(u))^2}DT^m_K(u)\Big) \tag{66}$$

$$= \mu + \Lambda_K + \Gamma_m \quad \text{in } \mathcal{D}'(Q),$$

where $B_K^m(z) = \displaystyle\int_0^z (1 + e^s + se^s)(T_K^m)'(s)\, ds.$

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