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CONTENTS

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CONTENTS

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Existence Result for Nonlinear Degenerated Parabolic Systems

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Abstract: An existence result of a solution for a class of nonlinear parabolic systems is established. The source term is less regular (bounded Radon measure) and no coercivity is made in the non-divergentiel lower order term $div(c(x,t)|u(x,t)|^{\gamma-2}u(x,t))$. The main contribution of our work is to prove the existence of a renormalized solution without the coercivity condition on the nonlinearities, so we used the Gagliardo-Nirenberg theorem to prove it.

Keywords: *Dirichlet problem; parabolic systems; Gagliardo-Nirenberg inequality; renormalized solutions.*

Mathematics Subject Classification (2010): Primary 35K41; Secondary 35K55, 35K65.

1 Introduction

Given a bounded-connected open set Ω of $\mathbb{R}^N$ $(N \geq 2)$, with Lipschitz boundary $\partial\Omega$, $Q_T = \Omega \times (0,T)$ is the generic cylinder of an arbitrary finite hight, $T < \infty$. We prove the existence of a renormalized solution for the nonlinear parabolic systems

$$\begin{cases} \frac{\partial b_i(x,u_i)}{\partial t} - \operatorname{div}(a(x,t,u_i,\nabla u_i) - \phi_i(x,t,u_i) - F_i) = f_i(x,u_1,u_2) \quad in \quad Q_T, \\ u_i(x,t) = 0 \quad on \quad \partial\Omega \times (0,T), \\ b_i(x,u_i(x,0)) = b_i(x,u_{0,i}(x)) \quad in \quad \Omega, \end{cases} \tag{1}$$

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where $i = 1, 2$. Here the vector field $a : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function such that $A(u_i) = -div\Big(a(x,t,u_i,\nabla u_i)\Big)$ is a Leray-Lions operator defined on $L^p(0,T;W_0^{1,p}(\nu))$, $\phi_i(x,t,u_i)$ is a Carathéodory function (see assumptions (13)–(14)), and $b_i : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that for every $x \in \Omega$, $b_i(x,.)$ is a strictly increasing C^1-function, the data $u_{0,i}$ is in $L^1(\Omega)$ such that $b_i(.,u_{0,i})$ in $L^1(\Omega)$. The data $f_i : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (see assumptions H4) and $F_i \in (L^{p'}(\nu))^N$. When problem (1) is investigated, there is a difficulty due to the fact that the data $b_1(x,u_0^1(x))$ and $b_2(x,u_0^2(x))$ only belong to L^1 and the functions $\Big(a(x,t,u_i,\nabla u_i)\Big)$, $\phi_i(x,t,u_i)$ and $f_i(x,u_1,u_2)$ do not belong to $(L^1_{loc}(Q_T))^N$ in general, so that proving existence of weak solution seems to be an arduous task, and we cannot use the Stocks formula in the a priori estimates of the nonlinearity $\phi_i(x,t,u_i)$. In order to overcome this difficulty, we work with the framework of renormalized solutions (see Definition 3.1). The notion of renormalized solutions was introduced by R.-J. DiPerna and P.-L. Lions [8] for the study of the Boltzmann equation. It was adapted to the study of some nonlinear elliptic or parabolic problems in fluid mechanics, see [6]. In the case where $b(x,u) = u$, the existence of renormalized solutions for (1) has been established by R.-Di Nardo [5]. In the case where $\phi(x;t;u) = 0$ and $f \in L^1(Q_T)$, the existence of renormalized solutions has been established by H. Redwane [13] in the classical Sobolev space, the existence results are already proved by the authors in the case where $f_i(x,u_1,u_2)$ is replaced by $f - div(g)$, where $f \in L^1(Q_T)$and $g \in (L^{p'}(Q_T))^N$. For the elliptic version of (1) we refer to [10].

One of the models of applications of these operators is the system of Boussinesq:

$$\frac{\partial u}{\partial t} + (u.\nabla))u - 2div(\mu(\theta)\varepsilon(u)) + \nabla p = F(\theta) \quad \text{in} \quad Q_T,$$

$$\frac{\partial b(\theta)}{\partial t} + u.\nabla b(\theta) - \triangle\theta = 2\mu(\theta)|\varepsilon(u)|^2 \quad \text{in} \quad Q_T,$$

$$u(t=0) = u_0,\ b(\theta)(t=0) = b(\theta_0) \quad \text{on} \quad \Omega,$$

$$u = 0 \quad \theta = 0 \quad \text{on} \quad \partial\Omega \times (0,T).$$

The first equation is the motion conservation equation, the unknowns are the fields of displacement $u : Q_T \to \mathbb{R}^N$ and temperature $\theta : Q_T \to \mathbb{R}$. The field $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^t)$ is the strain rate tensor.

It is our purpose, in this paper to generalize the result of [2, 5, 13] and we prove the existence of a renormalized solution of system (1).

The plan of the paper is as follows: In Section 2 we give basic assumptions. In Section 3 we give the definition of a renormalized solution of (1), and we establish (Theorem 3.1) the existence of such a solution.

2 Preliminaries and Auxiliary Results

We recall here some standard notations, properties and results which will be used throughout the paper.

Let Ω be a bounded open set of $\mathbb{R}^N$ and $Q_T = \Omega \times (0,T)$, T is a positive real number. Let $\nu(x)$ be a nonnegative function on Ω such that $\nu(x) \in L^r(\Omega)$, $r \geq 1$, $\nu(x)^{-1} \in L^t(\Omega)$,

$p \geq 1 + 1/t$. We denote by $L^p(\Omega, \nu)$, or simply $L^p(\nu)$ if there is no confusion, $p \geq 1$, the space of measurable functions u on Ω such that

$$\|u\|_{L^p(\nu)} = \Big(\int_\Omega |u|^p \nu(x) dx \Big)^{\frac{1}{p}} < +\infty, \tag{2}$$

and by $W^{1,p}(\nu)$ the completion of the space $C^1(\overline{\Omega})$ with respect to the norm

$$\|u\|_{W^{1,p}(\nu)} = \|u\|_{L^p(\nu)} + \|\nabla u\|_{L^p(\nu)}. \tag{3}$$

Moreover, we denote by $W_0^{1,p}(\nu)$ the closure of $C_0^1(\overline{\Omega})$ in $W^{1,p}(\nu)$, provided with the induced topology defined by the induced norm, and by $W^{-1,p'}(\nu^{1-p'})$, $p' = \frac{p}{p-1}$, its dual space. $W^{1,p}(\nu)$ and $W_0^{1,p}(\nu)$ are reflexive Banach spaces if $1 < p < \infty$, (see [11]).

Denote $V = W_0^{1,p}(\nu)$, $H = L^2(\nu)$ and $V^* = W_0^{-1,p'}(\nu^{1-p'})$, with $p \geq 2$. The dual space of $X := L^p(0,T; W_0^{1,p}(\nu))$ denoted X^* is identified with $L^{p'}(0,T;V^*)$. Define $W_p^1(0,T,V,H) = \{v \in X : v' \in X^*\}$. Endowed with the norm

$$\|u\|_{W_p^1} = \|u\|_X + \|u'\|_{X^*},$$

$W_p^1(0,T,V,H)$ is a Banach space. Here u' stands for the generalized time derivative of u, that is,

$$\int_0^T u'(t)\varphi(t)dt = -\int_0^T u(t)\varphi'(t)dt \text{ for all } \varphi \in C_0^\infty(0,T).$$

Lemma 2.1 *[14]*

1. *The evolution triple $V \hookrightarrow H \hookrightarrow V^*$ is verified.*
2. *The imbedding $W_p^1(0,T,V,H) \hookrightarrow C(0,T,H)$ is continuous.*
3. *The imbedding $W_p^1(0,T,V,H) \hookrightarrow L^p(Q_T,\nu)$ is compact.*

Lemma 2.2 *[1] Let $\{v_n\}$ be a bounded sequence in $L^p(0,T;V)$ such that*

$$\frac{\partial v_n}{\partial t} = \alpha_n + \beta_n \text{ in } \mathcal{D}'(Q_T)$$

with $\{\alpha_n\}$ and $\{\beta_n\}$ being two bounded sequences respectively in X^* and in $L^1(Q_T)$. Then $v_n \to v$ in $L^p_{loc}(Q_T,\nu)$. Furthermore, $v_n \to v$ strongly in $L^1(Q_T)$.

From now on, we assume that the following assumptions hold true

$$\nu(x)^{-1} \in L^t(\Omega),\ t \geq \frac{N}{p},\ 1 + \frac{1}{t} < p < N(1 + \frac{1}{t}), \tag{4}$$

$$\nu(x) \in L^r(\Omega),\ r > \frac{Nt}{pt - N}. \tag{5}$$

An important tool that we will use here, is the following weighted version of the Sobolev inequality (see Theorem 3.1 and Corollary 3.5 in [11]).

Proposition 2.1 *[11] Assume that* (4) *and* (5) *hold true. Let $\tilde{p}$ denote the number associated with p defined by*

$$\frac{1}{\tilde{p}} = r'\Big(\frac{1}{p}(1+\frac{1}{t}) - \frac{1}{N}\Big).$$

Then the imbedding of $W_0^{1,p}(\nu)$ into $L^{\tilde{p}}(\nu)$ is continuous. moreover, there exists a constant $C_0 > 0$ depending on N, p, ν, t, such that

$$\|u\|_{L^{\tilde{p}}(\nu)} \leq C_0 \||\nabla u|\|_{L^p(\nu)}, \forall u \in W_0^{1,p}(\nu). \tag{6}$$

Using this proposition, we can prove the following interpolation result.

Proposition 2.2 *Assume that* (4) *and* (5) *hold true. Let v be a function in $W_0^{1,p}(\nu) \cap L^s(\Omega)$ with $2 \leq p < N$ and $s > r'$. Then there exists a positive constant C, depending on N, p, ν, t and q, such that*

$$\|v\|_{L^\sigma(\nu)} \leq C \|\nabla v\|^{1-\theta}_{L^p(\nu)} \|v\|^\theta_{L^s(\Omega)}$$

for every θ and σ satisfying

$$0 \leq \theta \leq 1,\ 1 \leq \sigma \leq +\infty,\ \frac{1}{\sigma} = \theta + r'(1-\theta)\Big((1+\frac{1}{t})\frac{1}{p} - \frac{1}{N}\Big),\ r > \frac{Nt}{pt-N}.$$

Proof. For every $1 \leq \sigma \leq \tilde{p}$, we can write $\dfrac{1}{\sigma} = \theta + \dfrac{1-\theta}{\tilde{p}}$ for some $0 \leq \theta \leq 1$. then by the Hölder inequality and (6), one has

$$\|v\|_{L^\sigma(\nu)} \quad \leq C_0 \||\nabla v|\|^{1-\theta}_{L^p(\nu)} \|v\|^\theta_{L^1(\nu)} \leq C_0 \||\nabla v|\|^{1-\theta}_{L^p(\nu)} \|\nu\|^\theta_{L^{s'}(\Omega)} \|v\|^\theta_{L^s(\Omega)},$$

which gives the desired result.

As an immediate consequence of the previous result, we get

Corollary 2.1 *Let $v \in L^p((0,T), W_0^{1,p}(\nu)) \cap L^\infty((0,T), L^s(\Omega))$, with $2 \leq p < N$ and $s > r'$. Then $v \in L^\sigma(\nu)$ with $\sigma = \frac{p\tilde{p}+\tilde{p}-p}{\tilde{p}}$. Moreover,*

$$\int_{Q_T} \nu(x)|v|^\sigma dxdt \leq C \parallel v \parallel^{\frac{\tilde{p}-p}{\tilde{p}}}_{L^\infty(0,T,L^s(\Omega))} \int_{Q_T} \nu(x)|\nabla v|^p dxdt.$$

Proof. By virtue of Proposition 2.2, we can write

$$\int_\Omega \nu(x)|v|^\sigma dx \leq C \||\nabla v|\|^{(1-\theta)\sigma}_{L^p(\nu)} \parallel v \parallel^{\theta\sigma}_{L^s(\Omega)}.$$

Integrating between 0 and T, we get

$$\int_0^T \int_\Omega \nu(x)|v|^\sigma dxdt \leq C \int_0^T \||\nabla v|\|^{(1-\theta)\sigma}_{L^p(\nu)} \|v\|^{\theta\sigma}_{L^s(\Omega)} dt. \tag{7}$$

Since $v \in L^p((0,T), W_0^{1,p}(\nu)) \cap L^\infty((0,T), L^s(\Omega))$, we have

$$\int_0^T \int_\Omega \nu(x)|v|^\sigma dxdt \leq C \|v\|^{\theta\sigma}_{L^\infty(0,T,L^s(\Omega))} \int_0^T \||\nabla v(t)|\|^{(1-\theta)\sigma}_{L^p(\nu)} dt.$$

Now we choose θ such that $(1-\theta)\sigma = p$ and $\theta\sigma = \frac{\tilde{p}-p}{\tilde{p}}$. This choice yields

$$\theta = \frac{\tilde{p}-p}{p\tilde{p}+\tilde{p}-p}, \quad \sigma = \frac{p\tilde{p}+\tilde{p}-p}{\tilde{p}}.$$

Then, (7) becomes

$$\int_0^T \int_\Omega \nu(x)|v|^\sigma dxdt \leq C \parallel v \parallel^{\frac{\tilde{p}-p}{\tilde{p}}}_{L^\infty(0,T,L^s(\Omega))} \int_0^T \||\nabla v(t)|\|^p_{L^p(\nu)} dt.$$

3 Assumptions on Data

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$, T be a positive real number, and $Q_T = \Omega \times (0,T)$.

3.1 Assumptions

Throughout this paper, we assume that the following assumptions hold true:

Assumptions (H1)

$$b_i : \Omega \times \mathbb{R} \to \mathbb{R} \quad \text{is a Carathéodory function such that for every } x \in \Omega, \tag{8}$$

$b_i(x,.)$ is a strictly increasing $\mathcal{C}^1(\mathbb{R})$-function with $b_i(x,0) = 0$, for any $k > 0$, there exist a constant $\lambda_i > 0$ and functions $A^i_k \in L^\infty(\Omega)$ and $B^i_k \in L^p(\Omega)$ such that: for almost every x in Ω

$$\lambda_i \leq \frac{\partial b_i(x,s)}{\partial s} \leq A^i_k(x) \quad \text{and} \quad \Big|\nabla_x\Big(\frac{\partial b_i(x,s)}{\partial s}\Big)\Big| \leq B^i_k(x) \quad \forall\ |s| \leq k. \tag{9}$$

Assumptions (H2) Let $a: \ Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function such that, for any $k > 0$, there exist ν_k and a function $h_k \in L^{p'}(\nu)$ with

$$|a(x,t,s,\xi)| \leq \nu(x)\Big(h_k(x,t) + |\xi|^{p-1}\Big) \quad \forall\ |s| \leq k, \tag{10}$$

$$a(x,t,s,\xi)\xi \geq \alpha\nu(x)|\xi|^p \ \text{ with } \alpha > 0, \tag{11}$$

$$(a(x,t,s,\xi) - a(x,t,s,\eta))(\xi - \eta) > 0 \ \text{ with } \xi \neq \eta. \tag{12}$$

Assumptions (H3) Let $\phi_i: \ Q_T \times \mathbb{R} \to \mathbb{R}^N$ be a Carathéodory function such that

$$|\phi_i(x,t,s)| \leq c_i(x,t)|s|^\gamma \nu(x), \tag{13}$$

$$c_i(x,t) \in L^\tau(\nu) \quad \text{with} \quad \tau = \frac{p(3\tilde{p}-p)}{(p-1)(\tilde{p}-p)}, \ \gamma = \frac{2(p-1)(p\tilde{p}+\tilde{p}-p)}{p(3\tilde{p}-p)} \tag{14}$$

for almost every $(x,t) \in Q_T$, for every $s \in \mathbb{R}$ and every $\xi, \ \eta \in \mathbb{R}^N$.

Assumptions (H4) We suppose for that for i=1,2 $\quad f_i : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $f_1(x,0,s) = f_2(x,s,0) = 0$ a.e $x \in \Omega$,
$\forall s \in \mathbb{R}$. And for almost every $x \in \Omega$, for every $s_1, s_2 \in \mathbb{R}$

$$signe(s_i)f_i(x,s_1,s_2) \geq 0. \tag{15}$$

The growth assumptions on f_i are as follows: for each $k > 0$ there exist $\sigma_k > 0$ and a function F_k in $L^1(\Omega)$ such that

$$|f_1(x, s_1, s_2)| \leq F_k + \sigma_k |b_2(x, s_2)|. \quad \text{a.e} \quad x \in \Omega, \forall |s_1| \leq k, \quad \forall s_2 \in \mathbb{R}. \tag{16}$$

for each $k > 0$ there exist $\mu_k > 0$ and a function G_k in $L^1(\Omega)$ such that

$$|f_2(x, s_1, s_2)| \leq G_k(x) + \mu_k |b_1(x, s_1)|. \quad \text{a.e} \quad x \in \Omega, \forall |s_2| \leq k, \quad \forall s_1 \in \mathbb{R}. \tag{17}$$

$u_{0,i}$ is a measurable function such that $b_i(x, u_{0,i}) \in L^1(\Omega)$ for $i = 1, 2$.

4 Main Results

In this section, we study the existence of renormalized solutions to systems (1).

Definition 4.1 A couple of measurable functions (u_1, u_2) defined on Q_T is called a *renormalized* solution of (1) if for i=1,2. the function u_i satisfies

$$b_i(x, u_i) \in L^\infty(0, T; L^1(\Omega)), \tag{18}$$

$$T_k(u_i) \in L^p(0, T; W_0^{1,p}(\nu)) \text{ for any } k > 0, \tag{19}$$

$$\lim_{m \to +\infty} \frac{1}{m} \int_{\{(x,t) \in Q_T:\ |u_i(x,t)| \leq m\}} a(x, t, u_i, \nabla u_i) \nabla u_i \, dx \, dt = 0, \tag{20}$$

and if for every function S in $W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support

$$\begin{aligned} \frac{\partial B_{i,S}(x, u_i)}{\partial t} &- div\Big(a(x, t, u_i, \nabla u_i) S'(u_i)\Big) + S^{''}(u_i) a(x, t, u_i, \nabla u_i) \nabla u_i \\ &+ div\Big(\phi_i(x, t, u_i) S'(u_i)\Big) - S''(u_i) \phi_i(x, t, u_i) \nabla u_i \\ &= f_i(x, u_1, u_2) S'(u_i) - div(S'(u_i) F_i) + S''(u_i) F_i \nabla u_i \ in \ D^{'}(Q_T), \end{aligned} \tag{21}$$

and

$$B_{i,S}(x, u_i)(t = 0) = B_{i,S}(x, u_{i,0}) \quad in \quad \Omega, \tag{22}$$

where $B_{i,S}(x, z) = \displaystyle\int_0^z \frac{\partial b_i(x, s)}{\partial s} S^{'}(s) ds.$

Equation (21) is formally obtained through pointwise multiplication of equation (1) by $S'(u)$. However meanwhile $a(x, t, u_i, \nabla u_i)$ and $\phi_i(x, t, u_i)$ do not in general make sense in (1). Recall that for a renormalized solution, due to (19), each term in (21) has a meaning in $L^1(Q_T) + L^{p'}(0, T; W^{-1,p'}(\nu^{1-p'}))$ (see e.g. [6]).We have

$$\frac{\partial B_{i,S}(x, u_i)}{\partial t} \text{ belongs to } L^{p'}(0, T; W^{-1,p'}(\nu^{1-p'})) + L^1(Q_T). \tag{23}$$

$$B_{i,S}(x, u_i) \text{ belongs to } L^p(0, T; W_0^{1,p}(\nu)). \tag{24}$$

Then (23) and (24) imply that $B_{i,S}(x, u_i)$ belongs to $C^0([0, T]; L^1(\Omega))$ (for the proof of this trace result see [12],) so that the initial condition (22) makes sense.

Theorem 4.1 *Let $b(x,u_0) \in L^1(\Omega)$, assume that (H1)-(H4) hold true, then there exists at least a renormalized solution (u_1,u_2) of problem* (1) *in the sense of Definition* (4.1).

Proof. **Step 1.** Let us introduce the following regularization of the data: for i=1,2. For each $n > 0$

$$b_{i,n}(x,r) = b(x,T_n(r)) + \frac{r}{n} \quad \forall\ r \in \mathbb{R}, \tag{25}$$

$$a_n(x,t,s,\xi) = a(x,t,T_n(s),\xi) \quad \text{a.e. } (x,t) \in Q_T,\ \forall\ s \in \mathbb{R},\ \forall\ \xi \in \mathbb{R}^N, \tag{26}$$

$$\phi_{i,n}(x,t,r) = \phi_i(x,t,T_n(r)) \quad \text{a.e. } (x,t) \in Q_T,\ \forall\ r \in \mathbb{R}. \tag{27}$$

Let $f_{1,n}(x,s_1,s_2) = f_1(x,T_n(s_1),T_n(s_2))$ a.e $x \in \Omega$, $\forall s_1,s_2 \in \mathbb{R}$.

$$\text{and} \quad f_{2,n}(x,s_1,s_2) = f_2(x,T_n(s_1),T_n(s_2)) \quad \text{a.e } x \in \Omega, \forall s_1,s_2 \in \mathbb{R}. \tag{28}$$

Let $u_{i,0n} \in \mathcal{C}_0^\infty(\Omega)$ such that

$$b_{i,n}(x,u_{i,0n}) \to b_i(x,u_{i,0}) \text{ strongly in } L^1(\Omega). \tag{29}$$

In view of (25), for i=1,2 $b_{i,n}$ is a Carathéodory function and satisfies (9), there exists $\lambda_i > 0$ such that:

$$\lambda_i + \frac{1}{n} \leq \frac{\partial b_{i,n}(x,s)}{\partial s} \text{and} \quad |b_{i,n}(x,s)| \leq max_{|s|\leq n}|b_i(x,s)| \quad a.e.\ x \in \Omega,\ \forall s \in \mathbb{R}.$$

Let us now consider the regularized problem

$$\begin{cases} \frac{\partial b_{i,n}(x,u_{i,n})}{\partial t} - div(a_n(x,t,u_{i,n},\nabla u_{i,n}) - \phi_{i,n}(x,t,u_{i,n}) - F_i) = f_{i,n}(x,u_1,u_2) \ \ in \ \ Q_T, \\ u_{i,n}(x,t) = 0 \quad on \quad \partial\Omega \times (0,T), \\ b_{i,n}(x,u_{i,n})(t=0) = b_{i,n}(x,u_{i,0n}) \quad in \quad \Omega. \end{cases} \tag{30}$$

In view of (16)-(17), there exist $F_{1,n} \in L^1(\Omega)$ and $F_{2,n} \in L^1(\Omega)$ and $\sigma_n > 0$,$\mu_n > 0$ such that :

$$|f_{1,n}(x,s_1,s_2)| \leq F_{1,n}(x) + \sigma_n \max_{|s|\leq n} |b_i(x,s)|. \quad \text{a.e} \quad x \in \Omega, \quad \forall s_1,s_2 \in \mathbb{R}.$$

$$|f_{2,n}(x,s_1,s_2)| \leq F_{2,n}(x) + \mu_n \max_{|s|\leq n} |b_i(x,s)|. \quad \text{a.e} \quad x \in \Omega, \quad \forall s_1,s_2 \in \mathbb{R}.$$

As a consequence, proving the existence of weak solution $u_{i,n} \in L^p(0,T;W_0^{1,p}(\nu))$ of (30) is an easy task (see e.g. [**?**, 9]).

Step 2: The estimates derived in this step rely on standard techniques for problems of type (30). So we just sketch the proof of them (the reader is referred to [4]) for the elliptic version. Let $\tau_1 \in (0,T)$ and t be fixed in $(0,\tau_1)$. For i=1,2, using $T_k(u_{i,n})\chi_{(0,t)}$ as a test function in (30), we integrate between $(0,\tau_1)$, and by the condition (13) we have

$$\int_\Omega B_{i,k}^n(x,u_{i,n}(t))dx + \int_{Q_t} a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla T_k(u_{i,n})\,dx\,ds \tag{31}$$

$$\leq \int_{Q_t} c(x,t)|u_{i,n}|^\gamma \nu(x)|\nabla T_k(u_{i,n})|\,dx\,ds + \int_{Q_t} f_{i,n}(x,u_1^n,u_2^n)T_k(u_{i,n})\,dx\,ds$$

$$+\int_\Omega B_k^{i,n}(x,u_{i,0}^n)dx+\int_{Q_t}F_i\nabla T_k(u_i^n)dxds,$$

where $B_{i,k}^n(x,r)=\int_0^r T_k(s)\frac{\partial b_{i,n}(x,s)}{\partial s}ds$. Due to definition of $B_{i,k}^n$ we have:

$$0\le\int_\Omega B_{i,k}^n(x,u_{i,0n})dx\le k\int_\Omega|b_{i,n}(x,u_{i,0n})|dx=k||b_i(x,u_{i,0n})||_{L^1(\Omega)}\quad\forall k>0.\tag{32}$$

Using (31), (11) and (28) we obtain:

$$\int_\Omega B_{i,k}^n(x,u_{i,n}(t))dx+\alpha\int_{Q_t}\nu(x)|\nabla T_k(u_{i,n})|^p\,dx\,ds\le\int_{Q_t}c(x,t)|u_{i,n}|^\gamma\nu(x)|\nabla T_k(u_{i,\epsilon})|\,ds\,dx$$

$$+k(\|b_i(x,u_{i,0n})\|_{L^1(\Omega)}+\|f_{i,n}\|_{L^1(Q_T)})+\int_{Q_t}F_i\nabla T_k(u_{i,n})dxds.\tag{33}$$

Let $M_i=\Big(sup_n||f_{i,n}||_{L^1(Q_T)}+||b_i(x,u_{i,0n})||_{L^1(\Omega)}\Big)$. Noting that

$$B_{i,k}^n(x,s)=\int_0^s T_k(\sigma)\frac{\partial b_{i,n}(x,\sigma)}{\partial\sigma}d\sigma\ge\frac{\lambda_i+\frac{1}{n}}{2}|T_k(s)|^2>\frac{\lambda_i}{2}|T_k(s)|^2$$

we deduce from (31) and (32) that

$$\frac{\lambda_i}{2}\int_\Omega|T_k(u_{i,n})|^2\,dx+\alpha\int_{Q_t}\nu(x)|\nabla T_k(u_{i,n})|^p\,dx\,ds\tag{34}$$

$$\le M_ik+\int_{Q_t}c_i(x,t)|u_{i,n}|^\gamma\nu(x)|\nabla T_k(u_{i,n})|\,dx\,ds+\int_{Q_t}F_i\nabla T_k(u_{i,n})\,dx\,ds.$$

By Gagliardo-Nirenberg and Young inequalities we have:

$$\int_{Q_t}c_i(x,t)|u_{i,n}|^\gamma\nu(x)|\nabla T_k(u_{i,n})|\,dx\,ds$$

$$\le C_i\frac{\gamma(\tilde p-p)}{2(p\tilde p+\tilde p-p)}||c_i(x,t)||_{L^\tau(Q_{\tau_1},\nu)}\sup_{t\in(0,\tau_1)}\int_\Omega|T_k(u_{i,n})|^2\,dx$$

$$+C_i\frac{2p\tilde p+(2-\gamma)(\tilde p-p)}{2(p\tilde p+\tilde p-p)}||c_i(x,t)||_{L^\tau(Q_{\tau_1},\nu)}$$

$$\Big(\int_{Q_{\tau_1}}\nu(x)|\nabla T_k(u_{i,n})|^p\,dx\,ds\Big)^{(\frac{1}{p}+\frac{\gamma\tilde p}{p\tilde p+\tilde p-p})\frac{2(p\tilde p+\tilde p-p)}{2p\tilde p+(2-\gamma)(\tilde p-p)}}.\tag{35}$$

Since $\gamma=\frac{2(p-1)(p\tilde p+\tilde p-p)}{p(3\tilde p-p)}$ and by using (34) and (35), we obtain

$$\frac{\lambda_i}{2}\int_\Omega|T_k(u_{i,n})|^2\,dx+\alpha\int_{Q_t}\nu(x)|\nabla T_k(u_{i,n})|^p\,dx\,ds\le M_ik+$$

$$C_i\frac{\gamma(\tilde p-p)}{2(p\tilde p+\tilde p-p)}||c_i(x,t)||_{L^\tau(Q_{\tau_1},\nu)}\sup_{t\in(0,\tau_1)}\int_\Omega|T_k(u_{i,n})|^2\,dx+\frac{\alpha^{\frac{-p'}{p}}}{p'}\|F_i\|_{(L^{p'}(\nu))^N}$$

$$+C_i \frac{2p\tilde{p}+(2-\gamma)(\tilde{p}-p)}{2(p\tilde{p}+\tilde{p}-p)} ||c_i(x,t)||_{L^\tau(Q_{\tau_1},\nu)} \int_{Q_{\tau_1}} \nu(x)|\nabla T_k(u_{i,n})|^p \,dx\,ds$$

$$+\frac{\alpha}{p}\int_{Q_t} \nu(x)|\nabla T_k(u_{i,n})|^p \,dx\,ds$$

which is equivalent to

$$\Big(\frac{\lambda_i}{2} - C_i \frac{\gamma(\tilde{p}-p)}{2(p\tilde{p}+\tilde{p}-p)} ||c_i(x,t)||_{L^\tau(Q_{\tau_1},\nu)}\Big) \sup_{t\in(0,\tau_1)} \int_\Omega |T_k(u_{i,n})|^2\,dx$$

$$+\alpha \int_{Q_{\tau_1}} \nu(x)|\nabla T_k(u_{i,n})|^p \,dx\,ds$$

$$-\Big(C_i \frac{2p\tilde{p}+(2-\gamma)(\tilde{p}-p)}{2(p\tilde{p}+\tilde{p}-p)} ||c_i(x,t)||_{L^\tau(Q_{\tau_1},\nu)} + \frac{\alpha}{p}\Big) \int_{Q_{\tau_1}} \nu(x)|\nabla T_k(u_{i,n})|^p \,dx\,ds \le M_i k.$$

If we choose τ_1 such that

$$\Big(\frac{\lambda_i}{2} - C_i \frac{\gamma(\tilde{p}-p)}{2(p\tilde{p}+\tilde{p}-p)} ||c_i(x,t)||_{L^\tau(Q_{\tau_1},\nu)}\Big) > 0, \tag{36}$$

$$\Big(\frac{\alpha}{p'} - C_i \frac{2p\tilde{p}+(2-\gamma)(\tilde{p}-p)}{2(p\tilde{p}+\tilde{p}-p)} ||c_i(x,t)||_{L^\tau(Q_{\tau_1},\nu)}\Big) > 0, \tag{37}$$

and then denote by C_i the minimum between the constants $\Big(\frac{\lambda_i(p\tilde{p}+\tilde{p}-p)}{\gamma(\tilde{p}-p)||c_i(x,t)||_{L^\tau(Q_{\tau_1})}}\Big)$ and $\Big(\frac{2\alpha(p\tilde{p}+\tilde{p}-p)}{p'[2p\tilde{p}+(2-\gamma)(\tilde{p}-p)]||c_i(x,t)||_{L^\tau(Q_{\tau_1})}}\Big)$, we obtain

$$\sup_{t\in(0,\tau_1)} \int_\Omega |T_k(u_{i,n})|^2\,dx + \int_{Q_{\tau_1}} \nu(x)|\nabla T_k(u_{i,n})|^p \,dx\,dt \le C_i M_i k. \tag{38}$$

Then, by (38) and Lemma 3.1([**?**, 2]), we conclude that $T_k(u_{i,n})$ is bounded in $L^p(0,T,W_0^{1,p}(\nu))$ independently of n and for any $k \ge 0$, so there exists a subsequence still denoted by $u_{i,n}$ such that

$$T_k(u_{i,n}) \rightharpoonup H_{i,k} \quad \text{weakly in} \quad L^p(0,T,W_0^{1,p}(\nu)). \tag{39}$$

Lemma 4.1 *(see [2])*

$$u_{i,n} \to u_i \ \ a.e.\ Q_T, \ \ b_i(x,u_i) \in L^\infty(0,T;L^1(\Omega)), \tag{40}$$

where u_i is a measurable function defined on Q_T for i=1,2.

$$\lim_{m\to+\infty} \limsup_{n\to+\infty} \frac{1}{m} \int_{\{|u_{i,n}|\le m\}} a(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\,dx\,dt = 0. \tag{41}$$

Step 4: In this step we prove that the weak limit $X_{i,k}$ of $a(x,t,T_k(u_{i,n})\nabla T_k(u_{i,n}))$ can be identified with $a(x,t,T_k(u_i),\nabla T_k(u_i))$, for i=1,2. In order to prove this result we recall the following lemma.

Lemma 4.2 *For i=1,2, the subsequence of $u_{i,n}$ satisfies for any $k \geq 0$:*

$$\limsup_{n\to+\infty} \int_{Q_T} \int_0^t a(x,s,u_{i,n},\nabla T_k(u_{i,n}))\nabla T_k(u_{i,n})\,ds\,dx\,dt \leq \int_{Q_T}\int_0^t X_{i,k}\nabla T_k(u_i)\,dx\,ds\,dt, \tag{42}$$

$$\lim_{n\to+\infty} \int_{Q_T}\int_0^t \Big(a(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n})) - a(x,t,T_k(u_{i,n}),\nabla T_k(u_i))\Big) \Big(\nabla T_k(u_{i,n}) - \nabla T_k(u_i)\Big) = 0, \tag{43}$$

$$X_{i,k} = a(x,t,T_k(u_i),\nabla T_k(u_i)) \quad a.e.\ in\ Q_T, \tag{44}$$

and as n tends to $+\infty$

$$a(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n}))\nabla T_k(u_{i,n}) \rightharpoonup a(x,t,T_k(u_i),\nabla T_k(u_i))\nabla T_k(u_i) \tag{45}$$

weakly in $L^1(Q_T)$.

For i=1,2. We introduce a time regularization of the $T_k(u_i)$ for $k > 0$ in order to perform the monotonicity method.

Lemma 4.3 *(see H. Redwane [13]) Let $k \geq 0$ be fixed. Let S be an increasing $C^\infty(\mathbb{R})$−function such that $S(r) = r$ for $|r| \leq k$, and suppS' is compact. Then*

$$\liminf_{\mu\to+\infty} \lim_{n\to+\infty} \int_0^T \int_0^t < \frac{\partial b_{i,n}(x,u_{i,n})}{\partial t}, S'(u_{i,n})(T_k(u_{i,n}) - (T_k(u_i))_\mu) > \geq 0,$$

where $< .,. >$ denotes the duality pairing between $L^1(\Omega) + W^{-1,p'}(\nu^{1-p'})$ and $L^\infty(\Omega) \cap W_0^{1,p}(\nu)$.

Let S_m be a sequence of increasing C^∞-function such that:

$$S_m(r) = r \text{ for } |r| \leq m,\ supp(S'_m) \subset [-2m, 2m] \text{ and } \|S''_m\|_{L^\infty(\mathbb{R})} \leq \frac{3}{m} \text{ for any } m \geq 1.$$

For i=1,2. We use the sequence $(T_k(u_i))_\mu$ of approximation of $T_k(u_i)$, and plug the test function $S'_m(u_{i,n})(T_k(u_{i,n}) - (T_k(u_i))_\mu)$ for $m > 0$ and $\mu > 0$. For fixed $k \geq 0$, let $W^n_\mu = T_k(u_{i,n}) - (T_k(u_i))_\mu$. We obtain upon integration over $(0,t)$ and then over $(0,T)$:

$$\begin{aligned}
&\int_0^T\int_0^t < \frac{\partial b_{i,n}(x,u_{i,n})}{\partial t}, S'_m(u_{i,n})W^n_\mu > \,ds\,dt \\
&+ \int_{Q_T}\int_0^t a_n(x,s,u_{i,n},\nabla u_{i,n})S'_m(u_{i,n})\nabla W^n_\mu\,ds\,dt\,dx \\
&+ \int_{Q_T}\int_0^t a_n(x,s,u_{i,n},\nabla u_{i,n})S''_m(u_{i,n})\nabla u_{i,n}\nabla W^n_\mu\,ds\,dt\,dx \\
&- \int_{Q_T}\int_0^t \phi_{i,n}(x,s,u_{i,n})S'_m(u_{i,n})\nabla W^n_\mu\,ds\,dt\,dx \\
&- \int_{Q_T}\int_0^t S''_m(u_{i,n})\phi_{i,n}(x,s,u_{i,n})\nabla u_{i,n}\nabla W^n_\mu\,ds\,dt\,dx = \int_{Q_T}\int_0^t f_{i,n}S'_m(u_{i,n})W^n_\mu\,dx\,ds\,dt
\end{aligned} \tag{46}$$

$$+\int_{Q_T}\int_0^t F_i S'_m(u_{i,n})\nabla W_\mu^n \,ds\,dt\,dx + \int_{Q_T}\int_0^t F_i S''_m(u_{i,n})\nabla u_{i,n}\nabla W_\mu^n \,ds\,dt\,dx.$$

We pass to the limit in (46) as $n\to+\infty$, $\mu\to+\infty$ and then $m\to+\infty$ for k being a fixed real number. We use Lemma (4.3) and proceed as in ([4,13]), then it possible to conclude that

$$\liminf_{\mu\to+\infty}\lim_{n\to+\infty}\int_0^T\int_0^t < \frac{\partial b_{i,n}(x,u_{i,n})}{\partial t}, W_\mu^n > ds\,dt \geq 0 \qquad \text{for any } m\geq k, \tag{47}$$

$$\lim_{m\to+\infty}\limsup_{\mu\to+\infty}\limsup_{n\to+\infty}\int_{Q_T}\int_0^t a_n(x,t,u_{i,n},\nabla u_{i,n})S''_m(u_{i,n})\nabla u_{i,n}\nabla W_\mu^n\,ds\,dt\,dx = 0, \tag{48}$$

$$\lim_{\mu\to+\infty}\lim_{n\to+\infty}\int_{Q_T}\int_0^t f_{i,n}S'_m(u_{i,n})W_\mu^n \quad ds\,dt\,dx = 0, \tag{49}$$

$$\lim_{\mu\to+\infty}\int_{Q_T}\int_0^t F_iS'_m(u_{i,n})\nabla W_\mu^n \quad ds\,dt\,dx = 0, \tag{50}$$

$$\lim_{\mu\to+\infty}\int_{Q_T}\int_0^t F_iS''_m(u_{i,n})\nabla u_{i,n}W_\mu^n \quad ds\,dt\,dx = 0. \tag{51}$$

$$\lim_{\mu\to+\infty}\lim_{n\to+\infty}\int_{Q_T}\int_0^t \phi_{i,n}(x,t,u_{i,n})S'_m(u_{i,n})\nabla W_\mu^n\,ds\,dt\,dx = 0, \tag{52}$$

$$\lim_{\mu\to+\infty}\lim_{n\to+\infty}\int_{Q_T}\int_0^t S''_m(u_n)\phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n}\nabla W_\mu^n\,ds\,dt\,dx = 0. \tag{53}$$

For the proof of(52) and (53) the reader is referred to ([2]),(44) and (45) hold true. Note that, taking the limit as n tends to $+\infty$ in (41) and using (45) show that u satisfies (20). Now we want to prove that u satisfies the equation (21).

Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that $suppS'\subset[-k,k]$ where k is a real positive number. Pointwise multiplication of the approximate equation (30) by $S'(u_n)$ leads to

$$\frac{\partial B_{i,S}^n(x,u_{i,n})}{\partial t} - div\Big(a_n(x,t,u_{i,n},\nabla u_{i,n})S'(u_{i,n})\Big) + S''(u_{i,n})a(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n} \tag{54}$$

$$+\,div\Big(\phi_{i,n}(x,t,u_{i,n})S'(u_{i,n})\Big) - S''(u_{i,n})\phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n} = f_{i,n}S'(u_{i,n})$$

$$-div(F_iS'(u_{i,n})) + S''(u_{i,n})F_i\nabla u_{i,n} \quad \text{in } D'(Q_T),$$

where $B_{i,S}^n(x,r) = \displaystyle\int_0^r \frac{\partial b_{i,n}(x,s)}{\partial s}S'(s)\,ds$. In what follows we pass to the limit as n tends to $+\infty$ in each term of (54). Since the fact that $u_{i,n}$ converges to u_i a.e. in Q_T implies that $B_{i,S}^n(x,u_{i,n})$ converges to $B_{i,S}(x,u_i)$ a.e. in Q_T and $L^\infty(Q_T)$ is weak-$*$, we have that $\dfrac{\partial B_{i,S}^n(x,u_{i,n})}{\partial t}$ converges to $\dfrac{\partial B_{i,S}(x,u_i)}{\partial t}$ in $D'(Q_T)$. We observe that the term $a_n(x,t,u_{i,n},\nabla u_{i,n})S'(u_{i,n})$ can be identified with $a(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n}))S'(u_{i,n})$ for $n\geq k$, so using the pointwise convergence of $u_{i,n}$ to u_i in Q_T and the weak convergence of $T_k(u_{i,n})$to $T_k(u_i)$in $L^p(0,T;W_0^{1,p}(\nu))$, we get
$a_n(x,t,u_{i,n},\nabla u_{i,n})S'(u_{i,n}) \rightharpoonup a(x,t,T_k(u_{i,n}),\nabla T_k(u_i))S'(u_i)$ in $L^{p'}(\nu^{1-p'})$, and $S''(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n} \rightharpoonup S''(u_i)a(x,t,T_k(u_{i,n}),\nabla T_k(u_i))\nabla T_k(u_i)$

in $L^1(Q_T)$. Furthermore, $\phi_{i,n}(x,t,u_{i,n})S'(u_{i,n}) = \phi_{i,n}(x,t,T_k(u_{i,n}))S'(u_{i,n})$ a.e. in Q_T. By (27) we obtain $|\phi_{i,n}(x,t,T_k(u_{i,n}))S'(u_{i,n})| \leq \nu(x)|c_i(x,t)|k^\gamma$, it follows that $\phi_{i,n}(x,t,T_k(u_{i,n}))S'(u_{i,n}) \to \phi_{i,n}(x,t,T_k(u_i))S'(u_i)$ strongly in $L^{p'}(\nu^{1-p'})$.

In a similar way

$$S''(u_{i,n})\phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n} = S''(T_k(u_{i,n}))\phi_{i,n}(x,t,T_k(u_{i,n}))\nabla T_k(u_{i,n}) \quad \text{a.e. in } Q_T.$$

Using the weak convergence of $T_k(u_{i,n})$ in $L^p(0,T;W_0^{1,p}(\nu))$ it is possible to prove that $S''(u_{i,n})\phi_n(x,t,u_{i,n})\nabla u_{i,n} \to S''(u_i)\phi_i(x,t,u_i)\nabla u_i$ in $L^1(Q_T)$, and $S''(u_{i,n})F_i\nabla u_{i,n}$ converges to $S''(u_i)F_i\nabla u_i$ in $L^1(Q_T)$. Since $|S'(u_{i,n})| \leq C$, it follows that $F_iS''(u_{i,n})$ converges to $F_iS''(u_i)$ strongly in $L^{p'}(\nu)$. Finally by (28) we deduce that $f_nS'(u_{i,n})$ converges to $f_iS'(u_i)$ in $L^1(Q_T)$. It remains to prove that $B_{i,S}(x,u_i)$ satisfies the initial condition $B_{i,S}(x,u_i)(t=0) = B_{i,S}(x,u_{i,0})$ in Ω. To this end, firstly note that $B^n_{i,S}(x,u_{i,n})$ is bounded in $L^p(0,T;W_0^{1,p}(\nu))$. Secondly, the above considerations of the behavior of the terms of this equation show that $\frac{\partial B^n_{i,S}(x,u_{i,n})}{\partial t}$ is bounded in $L^1(Q_T) + L^{p'}(0,T;W^{-1,p'}(\nu^{1-p'}))$. As a consequence, $B^n_{i,S}(u_{i,n})(t=0) = B^n_{i,S}(x,u_{i,0n})$ converges to $B_{i,S}(x,u_i)(t=0)$ strongly in $L^1(\Omega)$ (for the proof of this trace result see [12]). On the other hand, the smoothness of S implies that $B_{i,S}(x,u_i)(t=0) = B_{i,S}(x,u_{i,0})$ in Ω. The proof of Theorem 3.1 is complete.

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Existence of Positive Periodic Solutions for a Second-Order Nonlinear Neutral Differential Equation by the Krasnoselskii's Fixed Point Theorem

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Abstract: This work is devoted to the study of the existence of positive periodic solutions of the second order nonlinear neutral differential equation

$$\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)x(t)=\frac{d^2}{dt^2}Q(t,x(t-\tau(t)))+f(t,h\left(x(t)\right),g\left(x(t-\tau(t))\right)).$$

The method used here is one of the most efficient techniques for studying this type of equations since it combines some useful properties of Green's function together with Krasnoselskii's fixed point theorem.

Keywords: *positive periodic solutions; nonlinear neutral differential equations; fixed point theorem.*

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1 Introduction

In this work we are essentially interested in the study of the existence of positive periodic solutions for certain classes of second order nonlinear neutral differential equations which are ubiquitous in different scientific disciplines and arise specially in beam theory, viscoelastic and inelastic flows and electric circuits.

There is a sizeable literature related to this topic, for instance in the middle of the previous century, the existence of solutions of differential equations was extensively studied by many investigators, see, for example, the papers and books [1]- [9], [11], [12]. During

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the last two decades, there has been an increasing activity in the study of periodic problems of second-order nonlinear neutral differential equations (see [1]- [3], [9], [11], [12] and references therein).

Some mathematicians used transformation in order to reduce the equation into more simple equation or system of equations or used synthetic division, others gave the solution in a series form which converges to the exact solution and some of them dealt with second-order nonlinear neutral differential equations by using some numerical techniques such as Ritz method, finite difference method, finite element method, cubic spline method and multiderivative method. In this paper, these usual methods may seem inefficient to establish the existence of positive periodic solutions of the second-order nonlinear neutral differential equations

$$\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)x(t)=\frac{d^2}{dt^2}Q(t,x(t-\tau(t)))+f(t,h\left(x(t)\right),g\left(x(t-\tau(t))\right)), \quad (1)$$

where p, q are positive continuous real-valued functions. The functions $Q:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$, $h,g:\mathbb{R}\longrightarrow\mathbb{R}$ and $f:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ are continuous with respect to their arguments. Our ideas are inspired by the ones given in the recent papers [1, 3, 9, 11, 12], we will convert the nonlinear neutral differential equation into an integral equation before using the Krasnoselskii's fixed point theorem.

This paper is organized as follows. In the next section, we start by providing some background definitions, lemmas and some preliminary results, then we give Green's function of a second order differential equation and some of their useful properties. We introduce Green's functions of a second order differential equation and we show that the solution of a given equation can be explicitly expressed in terms of Green's function of the corresponding homogeneous equation. Next, we present the inversion of (1) and we assert without proof the well-known Krasnoselskii's fixed point theorem which will be useful in what follows.

Finally, in the last section, we study the neutral functional differential equation (1) and present an existence result for positive periodic solutions for this equation by combining some properties of Green's function together with Krasnoselskii fixed point theorem.

2 Preliminaries

For $T>0$, let P_T be the set of all continuous scalar functions x, periodic in t of period T. Then $(P_T,\|.\|)$ is a Banach space with the supremum norm

$$\|x\|=\sup_{t\in\mathbb{R}}\left|x\left(t\right)\right|=\sup_{t\in[0,T]}\left|x\left(t\right)\right|.$$

Since we are searching for the existence of periodic solutions for equation (1), it is natural to assume that

$$p\left(t+T\right)=p\left(t\right),\ q\left(t+T\right)=q\left(t\right),\ \tau\left(t+T\right)=\tau\left(t\right), \quad (2)$$

with τ being scalar function, continuous, and $\tau\left(t\right)\geq\tau^{*}>0$. Also, we assume

$$\int_0^T p\left(s\right)ds>0,\ \int_0^T q\left(s\right)ds>0. \quad (3)$$

We also assume that the functions $Q\left(t,x\right)$ and $f\left(t,x,y\right)$ are periodic in t with period T, that is,

$$Q\left(t+T,x\right)=Q\left(t,x\right),\ f\left(t+T,x,y\right)=f\left(t,x,y\right). \quad (4)$$

Lemma 2.1 ([9]) *Suppose that (2) and (3) hold and*

$$\frac{R_1\left[\exp\left(\int_0^T p(u)\,du\right)-1\right]}{Q_1 T}\geq 1, \tag{5}$$

where

$$R_1=\max_{t\in[0,T]}\left|\int_t^{t+T}\frac{\exp\left(\int_t^s p(u)\,du\right)}{\exp\left(\int_0^T p(u)\,du\right)-1}q(s)\,ds\right|,\ Q_1=\left(1+\exp\left(\int_0^T p(u)\,du\right)\right)^2 R_1^2.$$

Then there are continuous T-periodic functions a and b such that $b(t)>0$, $\int_0^T a(u)\,du>0$ and

$$a(t)+b(t)=p(t),\ \frac{d}{dt}b(t)+a(t)b(t)=q(t),\ \text{for } t\in\mathbb{R}.$$

Lemma 2.2 ([11]) *Suppose the conditions of Lemma 2.1 hold and $\phi\in P_T$. Then the equation*

$$\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)x(t)=\phi(t)$$

has a T-periodic solution. Moreover, the periodic solution can be expressed by

$$x(t)=\int_t^{t+T}G(t,s)\,\phi(s)\,ds,$$

where

$$G(t,s)=\frac{\int_t^s\exp\left[\int_t^u b(v)\,dv+\int_u^s a(v)\,dv\right]du+\int_s^{t+T}\exp\left[\int_t^u b(v)\,dv+\int_u^{s+T}a(v)\,dv\right]du}{\left[\exp\left(\int_0^T a(u)\,du\right)-1\right]\left[\exp\left(\int_0^T b(u)\,du\right)-1\right]}.$$

Corollary 2.1 ([11]) *Green's function G satisfies the following properties*

$$G(t,t+T)=G(t,t),\ G(t+T,s+T)=G(t,s),$$

$$\frac{\partial}{\partial s}G(t,s)=a(s)G(t,s)-\frac{\exp\left(\int_t^s b(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right)-1},$$

$$\frac{\partial}{\partial t}G(t,s)=-b(t)G(t,s)+\frac{\exp\left(\int_t^s a(v)\,dv\right)}{\exp\left(\int_0^T a(v)\,dv\right)-1},$$

$$\frac{\partial^2}{\partial s^2}G(t,s)=\left(a'(s)+a^2(s)\right)G(t,s)-p(t)\frac{\exp\left(\int_t^s b(v)\,dv\right))}{\exp\left(\int_0^T b(v)\,dv\right))-1}.$$

The following lemma is fundamental to our results.

Lemma 2.3 *Suppose (2)-(4) and (5) hold. If $x\in P_T$, then x is a solution of equation (1) if and only if*

$$\begin{aligned}x(t)&=Q(t,x(t-\tau(t)))-\int_t^{t+T}p(s)E(t,s)Q(s,x(s-\tau(s)))\,ds\\&+\int_t^{t+T}G(t,s)\left[f(s,h(x(s)),g(x(s-\tau(s))))+\left(a'(s)+a^2(s)\right)Q(s,x(s-\tau(s)))\right]ds,\end{aligned} \tag{6}$$

where

$$E\left(t,s\right)=\frac{\exp\left(\int_{t}^{s}b\left(v\right)dv\right)}{\exp\left(\int_{0}^{T}b\left(v\right)dv\right)-1}. \tag{7}$$

Proof. Let $x\in P_T$ be a solution of (1). From Lemma 2.2, we have

$$x\left(t\right)=\int_{t}^{t+T}G\left(t,s\right)\left[\frac{\partial^{2}}{\partial s^{2}}Q\left(s,x\left(s-\tau\left(s\right)\right)\right)+f\left(s,h\left(x\left(s\right)\right),g\left(x\left(s-\tau\left(s\right)\right)\right)\right)\right]ds. \tag{8}$$

Using the integration by parts, we have

$$\begin{aligned}&\int_{t}^{t+T}G\left(t,s\right)\frac{\partial^{2}}{\partial s^{2}}Q\left(s,x\left(s-\tau\left(s\right)\right)\right)ds\\&=\left[G(t,s)\frac{\partial}{\partial s}Q(s,x(s-\tau(s)))\right]_{t}^{t+T}\\&=-\int_{t}^{t+T}\left(\frac{\partial}{\partial s}G(t,s)\right)\left(\frac{\partial}{\partial s}Q(s,x(s-\tau(s)))\right)ds.\end{aligned}$$

But

$$\left[G(t,s)\frac{\partial}{\partial s}Q(s,x(s-\tau(s)))\right]_{t}^{t+T}=0.$$

So

$$\int_{t}^{t+T}G(t,s)\frac{\partial^{2}}{\partial s^{2}}Q(s,x(s-\tau(s)))ds=-\int_{t}^{t+T}\left(\frac{\partial}{\partial s}G(t,s)\right)\left(\frac{\partial}{\partial s}Q(s,x(s-\tau(s)))\right)ds.$$

A second integration by parts gives

$$\begin{aligned}&\int_{t}^{t+T}G(t,s)\frac{\partial^{2}}{\partial s^{2}}Q(s,x(s-\tau(s)))ds\\&=\left[-Q(s,x(s-\tau(s)))\left(\frac{\partial}{\partial s}G(t,s)\right)\right]_{t}^{t+T}\\&+\int_{t}^{t+T}Q(s,x(s-\tau(s)))\frac{\partial^{2}}{\partial s^{2}}G(t,s)ds.\end{aligned}$$

Since

$$\begin{aligned}
&\left[-Q(s,x(s-\tau(s)))\left(\frac{\partial}{\partial s}G(t,s)\right)\right]_t^{t+T}\\
&=\left[-Q(s,x(s-\tau(s)))\left(a(s)G(t,s)-\frac{\exp\left(\int_t^s b(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right)-1}\right)\right]_t^{t+T}\\
&=-Q(t+T,x(t+T-\tau(t+T)))\left(a(t+T)G(t,t+T)-\frac{\exp\left(\int_t^{t+T} b(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right)-1}\right)\\
&+Q(t,x(t-\tau(t)))\left(a(t)G(t,t)-\frac{\exp\left(\int_t^t b(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right)-1}\right)\\
&=-Q(t,x(t-\tau(t)))\left(a(t)G(t,t)-\frac{\exp\left(\int_0^T b(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right)-1}\right)\\
&+Q(t,x(t-\tau(t)))\left(a(t)G(t,t)-\frac{1}{\exp\left(\int_0^T b(v)\,dv\right)-1}\right)\\
&=Q(t,x(t-\tau(t))),
\end{aligned}$$

and

$$\begin{aligned}
&\int_t^{t+T}\left(\frac{\partial^2}{\partial s^2}G(t,s)\right)Q(s,x(s-\tau(s)))ds\\
&=\int_t^{t+T}\left\{\left(a'(s)+a^2(s)\right)Q(s,x(s-\tau(s)))G(t,s)-p(s)\,E(t,s)\,Q(s,x(s-\tau(s)))\right\}ds,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\int_t^{t+T}G(t,s)\frac{\partial^2}{\partial s^2}Q(s,x(s-\tau(s)))ds\\
&=Q(t,x(t-\tau(t)))\\
&+\int_t^{t+T}\left\{\left(a'(s)+a^2(s)\right)Q(s,x(s-\tau(s)))G(t,s)-p(s)\,E(t,s)\,Q(s,x(s-\tau(s)))\right\}ds,
\end{aligned}\tag{9}$$

where E is given by (7). Then substituting (9) in (8) completes the proof.

Lemma 2.4 ([11]) *Let* $A=\int_0^T p(u)\,du$, $B=T^2\exp\left(\frac{1}{T}\int_0^T \ln(q(u))\,du\right)$. *If*

$$A^2\geq 4B, \tag{10}$$

then we have

$$\min\left\{\int_0^T a(u)\,du, \int_0^T b(u)\,du\right\} \geq \frac{1}{2}\left(A-\sqrt{A^2-4B}\right) := l,$$
$$\max\left\{\int_0^T a(u)\,du, \int_0^T b(u)\,du\right\} \leq \frac{1}{2}\left(A+\sqrt{A^2-4B}\right) := m.$$

Corollary 2.2 ([11]) *Functions G and E satisfy*

$$\frac{T}{(e^m-1)^2} \leq G(t,s) \leq \frac{T\exp\left(\int_0^T p(u)\,du\right)}{(e^l-1)^2},\ E(t,s) \leq \frac{e^m}{e^l-1}.$$

Lastly in this section, we state Krasnoselskii's fixed point theorem which enables us to prove the existence of positive periodic solutions to (1). For its proof we refer the reader to [10].

Theorem 2.1 (Krasnoselskii) *Let $\mathbb{D}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{D}$ into $\mathbb{B}$ such that*
(i) $x, y \in \mathbb{D}$, implies $\mathcal{A}x + \mathcal{B}y \in \mathbb{D}$,
(ii) $\mathcal{A}$ is compact and continuous,
(iii) $\mathcal{B}$ is a contraction mapping.
Then there exists $z \in \mathbb{D}$ with $z = \mathcal{A}z + \mathcal{B}z$.

3 Existence of Positive Periodic Solutions

To apply Theorem 2.1, we need to define a Banach space $\mathbb{B}$, a closed convex subset $\mathbb{D}$ of $\mathbb{B}$ and construct two mappings, one is a contraction and the other is compact. So, we let $(\mathbb{B}, \|.\|) = (P_T, \|.\|)$ and $\mathbb{D} = \{\varphi \in \mathbb{B} : K \leq \varphi \leq L\}$, where K is non-negative constant and L is positive constant. We express equation (6) as

$$\varphi(t) = (\mathcal{B}\varphi)(t) + (\mathcal{A}\varphi)(t) := (H\varphi)(t),$$

where $\mathcal{A}, \mathcal{B} : \mathbb{D} \to \mathbb{B}$ are defined by

$$\begin{aligned}&(\mathcal{A}\varphi)(t)\\ &= \int_t^{t+T} G(t,s)\left[f(s, h(\varphi(s)), g(\varphi(s-\tau(s)))) + \left(a'(s)+a^2(s)\right)Q(s, \varphi(s-\tau(s)))\right] ds,\end{aligned} \tag{11}$$

and

$$(\mathcal{B}\varphi)(t) = Q(t, \varphi(t-\tau(t))) - \int_t^{t+T} p(s)\,E(t,s)\,Q(s, \varphi(s-\tau(s)))\,ds. \tag{12}$$

To simplify notations, we introduce the following constants

$$\begin{aligned}&\alpha = \frac{T\exp\left(\int_0^T p(u)\,du\right)}{(e^l-1)^2},\ \beta = \frac{e^m}{e^l-1},\ \gamma = \frac{T}{(e^m-1)^2},\\ &\theta = \max_{t\in[0,T]}\{b(t)\},\ \mu = \min_{t\in[0,T]}\{p(t)\},\ \lambda = \max_{t\in[0,T]}\{p(t)\}.\end{aligned} \tag{13}$$

In this section we obtain the existence of a positive periodic solution of (1) by considering the two cases: $Q(t,x) \geq 0$ and $Q(t,x) \leq 0$ for all $t \in \mathbb{R}$, $x \in \mathbb{D}$. We assume that function $Q(t,x)$ is locally Lipschitz continuous in x. That is, there exists a positive constant k such that

$$|Q(t,x) - Q(t,y)| \leq k\|x-y\|, \text{ for all } t \in [0,T], \ x \in \mathbb{D}. \tag{14}$$

In the case $Q(t,x) \geq 0$, we assume that there exist a non-negative constant k_1 and positive constants k_2, σ such that

$$E(t,s) > \sigma, \text{ for all } (t,s) \in [0,T] \times [0,T], \tag{15}$$

$$k_1 x \leq Q(t,x) \leq k_2 x, \text{ for all } t \in [0,T], \ x \in \mathbb{D}, \tag{16}$$

$$k_2 < 1, \tag{17}$$

and for all $s \in [0,T]$, $x, y \in \mathbb{D}$

$$\frac{(1-k_1)K + \lambda\beta k_2 TL}{\gamma T} \leq f(s,h(x),g(y)) + \left(a'(s)+a^2(s)\right)Q(s,y) \leq \frac{(1-k_2)L+\mu\sigma k_1 TK}{\alpha T}. \tag{18}$$

Lemma 3.1 *Suppose that the conditions (2)-(5), (10) and (15)-(18) hold. Then $\mathcal{A} : \mathbb{D} \to \mathbb{B}$ is compact.*

Proof. Let $\mathcal{A}$ be defined by (11). Obviously, $\mathcal{A}\varphi$ is continuous and it is easy to show that $(\mathcal{A}\varphi)(t+T) = (\mathcal{A}\varphi)(t)$. For $t \in [0,T]$ and for $\varphi \in \mathbb{D}$, we have

$$\begin{aligned} |(\mathcal{A}\varphi)(t)| &\leq \left| \int_t^{t+T} G(t,s)\left[f(s,h(\varphi(s)),g(\varphi(s-\tau(s)))) - a(s)Q(s,\varphi(s-\tau(s)))\right] ds \right| \\ &\leq \alpha T \frac{(1-k_2)L + \mu\sigma k_1 TK}{\alpha T} = (1-k_2)L + \mu\sigma k_1 TK. \end{aligned}$$

Thus from the estimation of $|(\mathcal{A}\varphi)(t)|$ we have

$$\|\mathcal{A}\varphi\| \leq (1-k_2)L + \mu\sigma k_1 TK.$$

This shows that $\mathcal{A}(\mathbb{D})$ is uniformly bounded.

Let us that $\mathcal{A}(\mathbb{D})$ is equicontinuous. Let $\varphi_n \in \mathbb{D}$, where n is a positive integer. Next we calculate $\frac{d}{dt}(\mathcal{A}\varphi_n)(t)$ and show that it is uniformly bounded. By making use of (2), (3) and (4) we obtain by taking the derivative in (11) that

$$\begin{aligned} \frac{d}{dt}(\mathcal{A}\varphi_n)(t) = \int_t^{t+T} &\left[-b(t)G(t,s) + \frac{\exp\left(\int_t^s a(v)\,dv\right)}{\exp\left(\int_0^T a(v)\,dv\right) - 1} \right] \\ &\times \left[f(s,h(\varphi_n(s)),g(\varphi_n(s-\tau(s)))) - a(s)Q(s,\varphi_n(s-\tau(s)))\right] ds. \end{aligned}$$

Consequently, by invoking (13) and (18), we obtain

$$\left| \frac{d}{dt}(\mathcal{A}\varphi_n)(t) \right| \leq T(\theta\alpha + \beta)\frac{(1-k_2)L + \mu\sigma k_1 TK}{\alpha T} \leq D,$$

for some positive constant D. Hence the sequence $(\mathcal{A}\varphi_n)$ is equicontinuous. The Ascoli-Arzela theorem implies that a subsequence $(\mathcal{A}\varphi_{n_k})$ of $(\mathcal{A}\varphi_n)$ converges uniformly to a continuous T-periodic function. Thus $\mathcal{A}$ is continuous and $\mathcal{A}(\mathbb{D})$ is contained in a compact subset of $\mathbb{B}$.

Lemma 3.2 *Suppose that (14) holds. If $\mathcal{B}$ is given by (12) with*

$$k\left(1+\lambda\beta T\right)<1, \tag{19}$$

then $\mathcal{B}:\mathbb{D}\to\mathbb{B}$ is a contraction.

Proof. Let $\mathcal{B}$ be defined by (12). Obviously, $\mathcal{B}\varphi$ is continuous and it is easy to show that $(\mathcal{B}\varphi)(t+T)=(\mathcal{B}\varphi)(t)$. So, for any $\varphi,\psi\in\mathbb{D}$, we have

$$\begin{aligned}
&\left|(\mathcal{B}\varphi)(t)-(\mathcal{B}\psi)(t)\right|\\
&\le\left|Q\left(t,\varphi\left(t-\tau(t)\right)\right)-Q\left(t,\psi\left(t-\tau(t)\right)\right)\right|\\
&+\int_t^{t+T}p(s)\,E(t,s)\left|Q\left(s,\varphi\left(s-\tau(s)\right)\right)-Q\left(s,\psi\left(s-\tau(s)\right)\right)\right|ds\\
&\le k\left(1+\lambda\beta T\right)\left\|\varphi-\psi\right\|.
\end{aligned}$$

Then $\|\mathcal{B}\varphi-\mathcal{B}\psi\|\le k\left(1+\lambda\beta T\right)\|\varphi-\psi\|$. Thus $\mathcal{B}:\mathbb{D}\to\mathbb{B}$ is a contraction by (19).

Theorem 3.1 *Suppose (2)-(5), (10) and (14)-(19) hold. Then equation (1) has a positive T-periodic solution x in the subset $\mathbb{D}$.*

Proof. By Lemma 3.1, the operator $\mathcal{A}:\mathbb{D}\to\mathbb{B}$ is compact and continuous. Also, from Lemma 3.2, the operator $\mathcal{B}:\mathbb{D}\to\mathbb{B}$ is a contraction. Moreover, if $\varphi,\psi\in\mathbb{D}$, we see that

$$\begin{aligned}
&(\mathcal{B}\psi)(t)+(\mathcal{A}\varphi)(t)\\
&=Q\left(t,\psi\left(t-\tau(t)\right)\right)-\int_t^{t+T}p(s)\,E(t,s)\,Q\left(s,\psi\left(s-\tau(s)\right)\right)ds\\
&+\int_t^{t+T}G(t,s)\left[f\left(s,h\left(\varphi(s)\right),g\left(\varphi\left(s-\tau(s)\right)\right)\right)+\left(a'(s)+a^2(s)\right)Q\left(s,\varphi\left(s-\tau(s)\right)\right)\right]ds\\
&\le k_2L-\mu\sigma\int_t^{t+T}Q\left(s,\psi\left(s-\tau(s)\right)\right)ds\\
&+\alpha\int_t^{t+T}\left[f\left(s,h\left(\varphi(s)\right),g\left(\varphi\left(s-\tau(s)\right)\right)\right)+\left(a'(s)+a^2(s)\right)Q\left(s,\varphi\left(s-\tau(s)\right)\right)\right]ds\\
&\le k_2L-\mu\sigma k_1TK+\alpha T\frac{\left(1-k_2\right)L+\mu\sigma k_1TK}{\alpha T}=L,
\end{aligned}$$

and

$$\begin{aligned}
&(\mathcal{B}\psi)(t)+(\mathcal{A}\varphi)(t)\\
&=Q\left(t,\psi\left(t-\tau(t)\right)\right)-\int_t^{t+T}p(s)\,E(t,s)\,Q\left(s,\psi\left(s-\tau(s)\right)\right)ds\\
&+\int_t^{t+T}G(t,s)\left[f\left(s,h\left(\varphi(s)\right),g\left(\varphi\left(s-\tau(s)\right)\right)\right)-a(s)\,Q\left(s,\varphi\left(s-\tau(s)\right)\right)\right]ds\\
&\ge k_1K-\lambda\beta\int_t^{t+T}Q\left(s,\psi\left(s-\tau(s)\right)\right)ds\\
&+\gamma\int_t^{t+T}\left[f\left(s,h\left(\varphi(s)\right),g\left(\varphi\left(s-\tau(s)\right)\right)\right)-a(s)\,Q\left(s,\varphi\left(s-\tau(s)\right)\right)\right]ds\\
&\ge k_1K-\lambda\beta k_2TL+\gamma T\frac{\left(1-k_1\right)K+\lambda\beta k_2TL}{\gamma T}=K.
\end{aligned}$$

Clearly, all the hypotheses of the Krasnoselskii theorem are satisfied. Thus there exists a fixed point $x \in \mathbb{D}$ such that $x = \mathcal{A}x + \mathcal{B}x$. By Lemma 2.3 this fixed point is a solution of (1) and the proof is complete.

In the case $Q(t,x) \leq 0$, we substitute conditions (16)-(18) with the following conditions respectively. We assume that there exist a negative constant k_3 and a non-positive constant k_4 such that

$$k_3 x \leq Q(t,x) \leq k_4 x, \text{ for all } t \in [0,T], \ x \in \mathbb{D}, \tag{20}$$

$$-k_3 \lambda \beta T < 1, \tag{21}$$

and for all $s \in [0,T]$, $x, y \in \mathbb{D}$

$$\frac{K(1+k_4\mu\sigma T)-k_3L}{\gamma T} \leq f(s,h(x),g(y))+\left(a'(s)+a^2(s)\right)Q(s,y) \leq \frac{L(1+k_3\lambda\beta T)-k_4K}{\alpha T}. \tag{22}$$

Theorem 3.2 *Suppose (2)-(5), (10), (14), (15) and (19)-(22) hold. Then equation (1) has a positive T-periodic solution x in the subset $\mathbb{D}$.*

The proof follows along the lines of Theorem 3.2, and hence we omit it.

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Maximal Regularity of Non-autonomous Forms with Bounded Variation

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Abstract: We are concerned with the non-autonomous evolutionary problem

$$(P)\begin{cases} \dot{u}(t) + A(t)u(t) = f(t), & t \in [0, \eta], \\ u(0) = u_0. \end{cases}$$

Each operator $A(t)$ is associated with a symmetric sesquilinear form $\mathfrak{a}(t; ., .)$ on a Hilbert separable space $(H, \|\cdot\|)$. We show that the approximation method considered in [13] to redemonstrate the maximal regularity in H, is still valid to prove this property if the sesquilinear form is symmetric and time bounded variation. This result was already established in [5].

Keywords: *sesquilinear forms; non-autonomous evolution equations; maximal regularity.*

Mathematics Subject Classification (2010): 35K90, 35K50, 35K45, 47D06.

1 Introduction

Let $(H, \|\cdot\|)$ and $(V, \|\cdot\|_V)$ be Hilbert separable spaces such that V is continuously and densely embedded in H, $V \underset{d}{\hookrightarrow} H$. Let V' be the anti-dual of V and denote by $(.|.)$ the scalar product of H and by $\langle .; . \rangle$ the duality pairing $V' \times V$. By the standard identification of H with H' we obtain the continuous and dense embedding

$$V \underset{d}{\hookrightarrow} H \simeq H' \underset{d}{\hookrightarrow} V'.$$

Moreover, it is shown in [4] that there exists a constant c_H such that

$$\begin{aligned} &\|u\| \leqslant c_H \|u\|_V \quad \text{for all} \quad u \in V \\ \text{and} \quad &\|f\|_{V'} \leqslant c_H \|f\| \quad \text{for all} \quad f \in H. \end{aligned}$$

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Note that a result on existence, uniqueness and asymptotic behaviour was established in [11] for the problem (P) on a Banach space and with $t \in [0, \infty[$.

Let $\mathfrak{a}(.; u, v) : [0, \eta] \to \mathbb{C}$ be a measurable function for all $u, v \in V$. For each $t \in [0, \eta]$ the operator $A(t)$ is associated with a sesquilinear form $\mathfrak{a}(t; ., .) : V \times V \longrightarrow \mathbb{C}$ which satisfies:
$[H1]$ $D(\mathfrak{a}(t; ., .)) = V$.
$[H2]$ There exists $M > 0$ such that for all $t \in [0, \eta]$ and $u, v \in V$, we have $|\mathfrak{a}(t; u, v)| \leqslant M\|u\|_V \|v\|_V$, (V-boundedness).
$[H3]$ There exist $\alpha > 0, \delta \in \mathbb{R}$ such that for all $t \in [0, \eta]$ and all $u, v \in V$ we have $\alpha\|u\|_V^2 \leqslant Re\mathfrak{a}(t; u, u) + \delta\|u\|_H^2$, (quasi-coerciveness).

Let $t \in [0, \eta]$. For each fixed $u \in V$, the operator $\mathfrak{a}(t, u; .)$ defines a continuous anti-linear functional on V, then it induces a linear operator $\mathcal{A}(t) \in \mathcal{L}(V, V^{'})$ such that $a(t; u, v) = \langle \mathcal{A}(t)u, v\rangle$ for all $u, v \in V$. In this case, $-\mathcal{A}(t)$ generates a strongly continuous holomorphic semigroup $(e^{-s\mathcal{A}(t)})_{s\geqslant 0}$ on V'. When the problem (P) is considered in the spaces V and H, the form $\mathfrak{a}(t, \cdot; \cdot)$ is associated with $A(t)$ which is a part of $\mathcal{A}(t)$ in H. Therefore the operator $A(t) : D(A(t)) \subset V \to H$ is defined as

$$D(A(t)) = \{u \in V, \ \mathcal{A}(t)u \in H\}, \qquad A(t)u = \mathcal{A}(t)u.$$

Moreover, $-A(t)$ generates a strongly continuous holomorphic semigroup $(e^{-sA(t)})_{s\geqslant 0}$ with $(e^{-.A(t)}) := (e^{-.\mathcal{A}(t)})_{|H}$. Note that all the above results can be found in [18, Chapter 2] or in [15].

Recall that, if the problem (P) is considered in V' we have the following powerful result.

Theorem 1.1 (Lions' theorem) *For each $(f, u_0) \in L^2(0, \eta; V') \times H$ there is a unique solution $u \in MR(V, V') := L^2(0, \eta; V) \cap H^1(0, \eta; V')$ of the Cauchy problem*

$$\dot{u}(t) + \mathcal{A}(t)u(t) = f(t), \quad t \in (0, \eta), \quad u(0) = u_0. \tag{1}$$

We refer to [17, p. 112], [6, XVIII Chapter 3, p. 513] for the proof of this result. It is noteworthy to state that although Lions' theorem proves well-posedness of the Cauchy problem (P) with maximal regularity in V', the result remains unsatisfying in concrete applications to elliptic boundary value problems for which one needs solutions taking values in H. For this type of problems, only the part $A(t)$ of $\mathcal{A}(t)$ in H does really satisfy the boundary conditions. Hence, the central problem is whether maximal regularity in H is valid in the following sense:

Problem 1.1 For $(f, u_0) \in L^2(0, \eta; H) \times V$, does the solution u of (P) belong to $MR(V, H) := L^2(0, \eta; V) \cap H^1(0, \eta; H)$?

We will treat this question in three steps.
Step 1: $t \longmapsto A(t) := A$ for all $t \in [0, \eta]$. For this autonomous case, Problem 1.1 has been treated intensively, and has a positive answer.
Step 2: $t \longmapsto A(t)$ is a step function. This case was studied in [13] in a more general context and the authors have obtained a positive answer.
Step 3: The general case. The measurability condition assumed in Lions' theorem is not sufficient to establish the H-maximal regularity [5]. Extra conditions should be imposed on the regularity of $(\mathfrak{a}(t; ., .))_{0\leqslant t\leqslant \eta}$ with respect to t, or (and) on the space containing u_0. It is proved in [12] that u_0 has to be in a specified interpolation space. In the literature there are various conditions that ensure the H-maximal regularity. In the

works of Lions we distinguish two cases. For $u_0 = 0$, he assumed that $\mathfrak{a}$ is symmetric and $\mathfrak{a}(., u, v) \in C^1[0, \eta]$ for all $u, v \in V$ [14, page 65]. For $u_0 \in D(A(0))$ he obtained a positive answer if $\mathfrak{a}(., u, v) \in C^2[0, \eta]$ [14, page 95], or if the forms are symmetric and $\mathfrak{a}(., u, v) \in C^1[0, \eta]$ (a combination of [14, Theorem 1.1, p. 129] and [14, Theorem 5.1, p. 138]). However, Bardos assumed that the domains of both $A(t)^{1/2}$ and $A(t)^{*1/2}$ coincide with V as spaces and topologically with constants independent of t, and that $\mathcal{A}(.)^{1/2}$ is continuously differentiable with values in $\mathcal{L}(V, V')$ [3]. The results of Bardos were extended in Arendt et al. [2] by assuming the piecewise continuity of $\mathfrak{a}$ instead of continuous differentiability. As Bardos in [3], Arendt et al. [2] assumed the same square property of the domains of $A(.)^{\frac{1}{2}}$ and $A(.)^{*\frac{1}{2}}$. Dier [5] improved the result of Arendt et al. by considering symmetric and bounded variation form: for all $u, v \in V$ and $t, s \in [0, \eta]$ the form satisfies $\mathfrak{a}(t; u, v) = \overline{\mathfrak{a}(t; v, u)}$ and

$$|\mathfrak{a}(t; u, v) - \mathfrak{a}(s; u, v)| \leqslant |g(t) - g(s)| \|u\|_V \|v\|_V, \tag{2}$$

where $g : [0, \eta] \to \mathbb{R}^+$ is a nondecreasing function. Ouhabaz and Spina followed another way in [16] when $u(0) = 0$ and $\mathfrak{a}$ is α-Holder continuous for some $\alpha > \frac{1}{2}$. This result was improved in [10] where the authors imposed that $\mathfrak{a}$ satisfies some Dini-type condition, which is a generalisation of the Holder continuity. Laasri and Sani in [13] gave another approach by approximating the problem (P) and using the frozen coefficient method developed in [7,8]. The authors gave explicitly an approximate solution $u_\Lambda \in MR(V, H)$ which converges to the solution u of the problem (P) in $MR(V, H)$ if the form $\mathfrak{a}$ is symmetric and time Lipschitz continuous. In this work we develop the last approach to re-demonstrate the result of [5]. In fact, Theorem 2.2 shows that the approximate solution converges weakly in $MR(V, H)$ to the solution u of (P).

2 Main Results

Let us recall some known results for the autonomous case that we use in the proof. In the following the constant $c > 0$ varies but does not depend on the variable to be estimated. Let $[a, b]$ be an arbitrary subinterval of $[0, \eta]$ and let $(f, u_0) \in L^2(a, b; V') \times H$. Lions theorem ensures the existence of a unique solution $u \in MR(a, b; V, V') := L^2(a, b; V) \cap H^1(a, b; V')$ of the autonomous problem

$$\dot{u}(t) + Au(t) = f(t), \quad \text{t. a.e. on } (a, b) \subset [0, \eta], \qquad u(a) = u_0. \tag{P_0}$$

It is shown in [17, Chapter III, Proposition 1.2] and in [18, Lemma 5.5.1] that if $u \in MR(a, b; V, V')$, then $\|u(.)\|^2$ is absolutely continuous on $[a, b]$ and

$$\frac{d}{dt}\|u(.)\|^2 = 2Re\langle \dot{u}; u\rangle. \tag{3}$$

For $(f, u_0) \in L^2(a, b; H) \times V$ the solution u of (P_0) belongs to the maximal regularity space $MR(a, b; D(A), H) := L^2(a, b; D(A)) \cap H^1(a, b; H)$ which is continuously embedded into $C([a, b], V)$, see [6, Example 1, page 577]. In addition, if the form $\mathfrak{a}$ is symmetric, W. Arendt and R. Chill proved in [1] the following results.

Proposition 2.1 *Let $\mathfrak{a}$ be a continuous symmetric sesquilinear form satisfying hypotheses $[H1] - [H3]$. Let $(f, u_0) \in L^2(a, b; H) \times V$ and $u \in MR(a, b; D(A), H)$. Then*

the following results hold:
i) The function $\mathfrak{a}(u(.)) \in W^{1,1}(a,b)$. *Moreover, the following product formula holds*

$$\frac{d}{dt}\mathfrak{a}(u(t)) = 2(Au(t)|\dot{u}) \quad \textit{for a.e.} \quad t \in [a,b]. \tag{4}$$

In this case we infer the following estimate

$$\frac{d}{dt}\mathfrak{a}(u(t)) \leqslant \|f(t)\|^2 \quad \textit{for a.e.} \quad t \in [a,b]. \tag{5}$$

ii) If the function u *satisfies* (P_0), *then there exists a constant* $c(M,\alpha,\delta,\eta) > 0$ *independent of* f, u_0 *and* $[a,b] \subset [0,\eta]$ *for which*

$$\sup_{s\in[a,b]} \|u(s)\|_V^2 \leqslant c\left[\|u(a)\|_V^2 + \|f\|^2_{L^2(a,b;H)}\right]. \tag{6}$$

The method considered in [13] consists in the approximation of $\mathfrak{a}$ and $\mathcal{A}$ by step function. Let $\Lambda = (0 = \lambda_0 < \lambda_1 < ... < \lambda_{n+1} = \eta)$ be a subdivision of $[0,\eta]$. Let

$$\mathfrak{a}_k : V \times V \to \mathbb{C} \quad \text{for } k = 0,1,...,n$$

be a finite family of continuous and H-elliptic forms. The associated operators are denoted by $A_k \in \mathcal{L}(V,V')$. The function $\mathfrak{a}$ is approximated by $\mathfrak{a}_\Lambda : [0,\eta] \times V \times V \to \mathbb{C}$ for each $k = 0,1,...,n$ and $\lambda_k \leq t < \lambda_{k+1}$

$$\begin{cases} \mathfrak{a}_\Lambda(t;u,v) := \mathfrak{a}_k(u,v) = \dfrac{1}{\lambda_{k+1}-\lambda_k}\displaystyle\int_{\lambda_k}^{\lambda_{k+1}} \mathfrak{a}(r;u,v)dr, \\ \mathfrak{a}_\Lambda(\eta;u,v) := \mathfrak{a}_n(u,v). \end{cases}$$

Thus, the approximate $\mathcal{A}_\Lambda : [0,\eta] \to \mathcal{L}(V;V')$ of $\mathcal{A}$ is given by

$$\begin{cases} \mathcal{A}_\Lambda(t) := \mathcal{A}_k = \dfrac{1}{\lambda_{k+1}-\lambda_k}\displaystyle\int_{\lambda_k}^{\lambda_{k+1}} \mathcal{A}(r)u\mathrm{d}r \quad \text{for } \lambda_k \leqslant t < \lambda_{k+1},\ k = 0,1,...,n, \\ \mathcal{A}_\Lambda(\eta) := \mathcal{A}_n. \end{cases}$$

For $u_0 \in H$ and $f \in L^2(0,T;V')$ there exists a unique $u_\Lambda \in MR(V,V')$ such that

$$(P_\Lambda)\begin{cases} \dot{u}_\Lambda(t) + \mathcal{A}_\Lambda(t)u_\Lambda(t) = f(t), \quad \text{for a.e } t \in [0,\eta], \\ u_\Lambda(0) = u_0. \end{cases}$$

Note that on each interval $[\lambda_k, \lambda_{k+1}[$ the solution u_Λ coincides with the solution of the autonomous Cauchy problem

$$(P_k)\begin{cases} \dot{u}_k(t) + A_k u_k(t) = f(t) \ \ t\text{−a.e.} \quad \text{on } (\lambda_k,\lambda_{k+1}), \\ u_k(\lambda_k) = u_{k-1}(\lambda_k) \in V, \end{cases} \tag{7}$$

which belongs to $MR(\lambda_k,\lambda_{k+1};D(A_k),H)$.

The problem (P) is invariant under shifting the operator by a scalar multiplication. Then, for the sake of simplicity, we may assume without loss of generality that $\delta = 0$.

Proposition 2.2 *[13, Theorem 3.2]. Let $(f, u_0) \in L^2(a, b; V') \times H$. Let u and u_Λ be the solutions of (P) and (P_Λ) respectively. Then*
i) There exists a constant $c > 0$ witch is independent of $\{f, u_0, \Lambda\}$ such that

$$\int_0^t \|u_\Lambda(s)\|_V^2 ds \leqslant c \left[\int_0^t \|f(s)\|_{V'}^2 ds + \|u_0\|^2 \right] \quad \text{for a.e.} \quad t \in [0, \eta], \tag{8}$$

ii) The solution u_Λ converges weakly to u in $MR(V, V')$ as $|\Lambda| \to 0$.

If the conditions $(f, u_0) \in L^2(0, \eta; H) \times V$ are fulfilled, then the solution u_Λ of (P_Λ) belongs to the maximal regularity space $MR(V, H)$ which is continuously embedded into $C([0, \eta], V)$. In this case, the same estimate as in 8 is provided with the following theorem.

Theorem 2.1 *Let $g : [0, \eta] \to \mathbb{R}^+$ be a nondecreasing function. Let $(f, u_0) \in L^2(0, \eta; H) \times V$. Let $\mathfrak{a}$ be a symmetric sesquilinear form satisfying $[H1] - [H3]$ and*

$$|\mathfrak{a}(t; u, v) - \mathfrak{a}(s; u, v)| \leqslant (g(t) - g(s))\|u\|_V \|v\|_V \qquad (t, s \in [0; \eta], s \leqslant t).$$

If u_Λ is the solution of (P_Λ), then there exists a constant $c(\alpha, c_H, M, \eta, g)$ such that

$$\|u_\Lambda(t)\|_V^2 \leqslant c \left[\|u_0\|_V^2 + \|f\|_{L^2(0,\eta;H)}^2 \right], \qquad \forall t \in [0, \eta]. \tag{9}$$

Proof. Let $t \in [0, \eta]$, then there exists $k \in \{0, 1, 2, ..., n\}$ such that $t \in [\lambda_k, \lambda_{k+1}[\subset [0, \eta]$. Since the solution u_Λ coincides with the solution u_k of the autonomous problem (P_k) on each interval $[\lambda_k, \lambda_{k+1}[$, then the coercivity property and (5) yield

$$\begin{aligned}
\alpha\|u_\Lambda(t)\|_V^2 &\leqslant \mathfrak{a}_k(u_\Lambda(t)) \\
&= [\mathfrak{a}_k(u_k(t)) - \mathfrak{a}_k(u_k(\lambda_k))] + \sum_{i=0}^{i=k-1} \mathfrak{a}_i(u_i(\lambda_{i+1})) - \mathfrak{a}_i(u_i(\lambda_i)) \\
&+ \sum_{i=1}^{i=k} \mathfrak{a}_i(u_i(\lambda_i)) - \mathfrak{a}_{i-1}(u_i(\lambda_i)) + \mathfrak{a}_0(u_0(\lambda_0)) \\
&= \int_{\lambda_k}^t \frac{d}{ds}\mathfrak{a}_k(u_k(s))ds + \sum_{i=0}^{i=k-1} \int_{\lambda_i}^{\lambda_{i+1}} \frac{d}{ds}\mathfrak{a}_i(u_i(s))ds \\
&+ \sum_{i=1}^{i=k} \mathfrak{a}_i(u_i(\lambda_i)) - \mathfrak{a}_{i-1}(u_i(\lambda_i)) + \mathfrak{a}_0(u_0(\lambda_0)) \\
&\leqslant \int_0^t \|f(s)\|^2 ds + M\|u_0\|_V^2 + \sum_{i=1}^{i=k} \mathfrak{a}_i(u_i(\lambda_i)) - \mathfrak{a}_{i-1}(u_i(\lambda_i)).
\end{aligned}$$

First, we give for each $k = 0, 1, 2, .., n-2$ an estimate of

$$|\mathfrak{a}_k(u_\Lambda(\lambda_{k+1}) - \mathfrak{a}_{k+1}(u_\Lambda(\lambda_{k+1}))| \, .$$

Obviously, the function g is of bounded variation. And since $\|u_\Lambda(.)\|^2$ is continuous on $[0, \eta]$, it is Riemann-Stieltjes integrable with respect to g.

For each $k = 0,1,2,..,n$ and for each arbitrary $t_k \in [\lambda_k, \lambda_{k+1}[$ there exists, by the inequality (6), a constant $c \geqslant 0$ depending only on M, δ, α, c_H, and η such that $u_{\Lambda|_{[t_k,\lambda_{k+1}[}} \in MR(t_k, \lambda_{k+1}; D(A_k), H)$ and

$$\|u_\Lambda(\lambda_{k+1})\|_V^2 \leqslant c\left[\|u(t_k)\|_V^2 + \|f\|^2_{L^2(\lambda_k,\lambda_{k+1};H)}\right]. \tag{10}$$

By the mean value theorem, the t_k is chosen such that

$$(g(\lambda_{k+1}) - g(\lambda_k)\|u_\Lambda(t_k)\|^2) = \int_{\lambda_k}^{\lambda_{k+1}} \|u_\Lambda(t)\|^2 d(g(t)). \tag{11}$$

Thus, the estimates (2), (10) and (11) yield

$$\begin{aligned}
&|\mathfrak{a}_k(u_\Lambda(\lambda_{k+1}) - \mathfrak{a}_{k+1}(u_\Lambda(\lambda_{k+1}))| \\
&\qquad \leqslant (g(\lambda_{k+1}) - g(\lambda_k))\|u_\Lambda(\lambda_{k+1})\|^2 \\
&\qquad \leqslant c(g(\lambda_{k+1}) - g(\lambda_k))\left[\|u_\Lambda(t_k)\|^2 + \|f\|^2_{L^2(0,\eta;H)}\right] \\
&\qquad \leqslant c\int_{\lambda_k}^{\lambda_{k+1}} \|u_\Lambda(t)\|_V^2 d(g(s)) + c\left((g(\lambda_{k+1}) - g(\lambda_k))\right)\|f\|^2_{L^2(0,\eta;H)}.
\end{aligned} \tag{12}$$

Thus,

$$\begin{aligned}
&\sum_{i=1}^{i=k} |\mathfrak{a}_i(u_i(\lambda_i)) - \mathfrak{a}_{i-1}(u_i(\lambda_i))| \\
&\qquad \leqslant \sum_{i=1}^{i=k} c\int_{\lambda_{i-1}}^{\lambda_i} \|u_\Lambda(t)\|_V^2 d(g(s)) + \sum_{i=1}^{i=k} c\left((g(\lambda_i) - g(\lambda_{i-1}))\right)\|f\|^2_{L^2(0,\eta;H)} \\
&\qquad \leqslant c\int_0^t \|u_\Lambda(t)\|_V^2 d(g(s)) + c\left((g(\eta) - g(0))\right)\|f\|^2_{L^2(0,\eta;H)}.
\end{aligned} \tag{13}$$

Consequently,

$$\alpha\|u_\Lambda(t)\|_V^2 \leqslant c\left[\|f\|^2_{L^2(0,\eta;H)}) + \|u_0\|_V^2\right] + c\int_0^t \|u_\Lambda(s)\|_V^2 d(g(s)).$$

By Gronwall's inequality, see [9, Theorem 5.1, page 498], we obtain that

$$\|u_\Lambda(t)\|_V^2 \leqslant c\left[\|f\|^2_{L^2(0,\eta;H)}) + \|u_0\|_V^2\right]. \tag{14}$$

□

The following theorem shows that the solution u_Λ converges weakly in $MR(V,H)$ to the solution u of (P) which belongs to the maximal regularity space $MR(V,H)$.

Theorem 2.2 *Let* $(f, u_0) \in L^2(a,b;V') \times H$. *We suppose that the forms* $(a(t;.,.))_{0\leqslant t\leqslant\eta}$ *satisfy the standing hypotheses* $[H1]$-$[H3]$ *and the regularity condition*

$$|\mathfrak{a}(t;u,v) - \mathfrak{a}(s;u,v)| \leqslant (g(t) - g(s))\|u\|_V\|v\|_V \qquad (0 \leqslant s \leqslant t \leqslant \eta), \tag{15}$$

where $g : [0,\eta] \to [0,\infty)$ *is a non-decreasing function. Then the solution* u_Λ *of* (P_Λ) *converges weakly in* $MR(V,H)$ *as* $|\Lambda| \to 0$ *to the solution* u *of* (P). *Moreover*

$$\|u\|_{MR(V,H)} \leqslant c\left[\|u_0\|_V^2 + \|f\|^2_{L^2(0,\eta;H)}\right],$$

the constant c *depends only on* α, c_H, M, η *and* g.

Proof. Let $(f, u_0) \in L^2(0, \eta; H) \times V$. Let $u_\Lambda \in MR(V, H)$ be the solution of (P_Λ). Taking into account the weak convergence of the function u_Λ to u in the space $MR(V, V')$, it is enough to show that u_Λ is bounded in $MR(V, H)$. Theorem 2.1 assures the boundedness of u_Λ in $L^2(0, \eta; V)$, so it remains to prove this property for the derivative in $L^2(0, \eta; H)$.

$$\begin{aligned}
\int_0^\eta \|\dot{u}_\Lambda(t)\|^2 dt &= \int_0^\eta Re(-\mathcal{A}_\Lambda u_\Lambda(t)|\dot{u}_\Lambda(t))dt + \int_0^\eta Re(f(t); \dot{u}_\Lambda(t))_H dt \\
&= -\sum_{k=0}^{n-1} \int_{\lambda_k}^{\lambda_{k+1}} \frac{d}{dt}\frac{1}{2}\mathfrak{a}_k(u_\Lambda(t))dt + \int_0^\eta Re(f(t)|\dot{u}_\Lambda(t))dt \\
&= -\sum_{k=0}^{n-1} (\mathfrak{a}_k(u_\Lambda(\lambda_{k+1}) - \mathfrak{a}_k(u_\Lambda(\lambda_k)) + \int_0^\eta Re(f(t)|\dot{u}_\Lambda(t))dt \\
&= -\sum_{k=0}^{n-2} \mathfrak{a}_k(u_\Lambda(\lambda_{k+1})) - \mathfrak{a}_{k+1}(u_\Lambda(\lambda_{k+1})) + \int_0^\eta Re(f(t)|\dot{u}_\Lambda(t))dt \\
&\quad [-\mathfrak{a}_{n-1}(u_\Lambda(\lambda_n)) + \mathfrak{a}_0(u_\Lambda(0))]. \qquad (16)
\end{aligned}$$

For the first term on the right-hand side of the equality (16) the inequality (13) yields

$$\begin{aligned}
|\sum_{k=0}^{n-2} (\mathfrak{a}_k(u_\Lambda(\lambda_{k+1}) - \mathfrak{a}_{k+1}(u_\Lambda(\lambda_{k+1}))| &\leqslant *c \int_0^\eta \|u_\Lambda(t)\|_V^2 d(g(t)) + c[g(\eta) - g(0)]\|f\|^2_{L^2(0,\eta;H)} \\
&\leqslant c\left[\|f\|^2_{L^2(0,\eta;H)}) + \|u_0\|_V^2\right].
\end{aligned}$$

By the Cauchy-Schwarz and the Young inequalities

$$\begin{aligned}
\int_0^\eta \|\dot{u}_\Lambda(t)\|^2 dt &\leqslant c\left[\|f\|^2_{L^2(0,\eta;H)}) + \|u_0\|_V^2\right] + \int_0^\eta (f(t)|\dot{u}_\Lambda(t))dt \\
&\leqslant c\left[\|f\|^2_{L^2(0,\eta;H)}) + \|u_0\|_V^2\right] + \frac{1}{2}\int_0^\eta \|f(t)\|^2 dt + \frac{1}{2}\int_0^\eta \|\dot{u}_\Lambda(t)\|^2 dt.
\end{aligned}$$

Thus, by the inequality (9), there exists a constant $c > 0$ depending on $(c_H, M, \alpha, g(\eta), g(0))$ such that

$$\int_0^\eta \|\dot{u}_\Lambda(t)\|^2 dt + \int_0^\eta \|u_\Lambda(t)\|_V^2 dt \leqslant c\left[\|f\|^2_{L^2(0,\eta;H)} + \|u_0\|_V^2\right].$$

□

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A NASC for Equicontinuous Maps for Integral Equations

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Abstract: We offer necessary and sufficient conditions for a mapping of the form

$$(P\phi)(t) = p(t) - \int_0^t C(t,s)g(s,\phi(s))ds$$

to send sets of bounded continuous functions on $[0,\infty)$ into equicontinuous sets. When that equicontinuity holds then one may study the problem of obtaining a bounded solution of the integral equation by means of a Schauder-type fixed point theorem. When the mapped sets are equicontinuous then we use Schaefer's fixed point theorem to show that we can obtain a bounded positive solution provided that we know that the resolvent kernel, $R(t,s)$, of C is non-negative and that

$$p(t) - \int_0^t R(t,s)p(s)ds$$

is bounded and positive, while $g(t,x)$ does not grow too fast near $x = 0$. The known literature shows that there are wide classes of important problems from applied mathematics and fractional equations for which these conditions hold. For those classes, the problem of obtaining a positive solution is largely solved when equicontinuity, characterized by our theorem, holds.

Keywords: *integral equations; compact maps; positive kernels; positive solutions; Schaefer's fixed point theorem.*

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1 Introduction

In the study of integral equations by fixed point methods of the Schauder type (see [11, pp. 25-34], for example) there commonly occurs an integral of the form

$$\int_0^t C(t,s)g(s,x(s))ds. \tag{1.1}$$

A main part of the investigation involves using that integral to map a set of bounded continuous functions into an equicontinuous set. Our objective here is to establish a necessary and sufficient condition on C and g to ensure that this will happen. The conditions needed are (1.2), (1.4), and sometimes (1.3).

The function $C : (0,\infty) \times (0,\infty) \to (0,\infty)$ is measurable and for any finite interval $J \subset [0,\infty)$ the integral $\int_J C(t,s)ds$ exists for each $t \in J$ with

$$\sup_{t\in J} \int_0^t C(t,s)ds < \infty. \tag{1.2}$$

The function $g : [0,\infty) \times \Re \to \Re$ is continuous and bounded when x is bounded, while

$$x > 0 \implies g(t,x) > 0. \tag{1.3}$$

The function C is of fading memory type by which we mean that

$$0 < s < t_2 < t_1 \implies C(t_2,s) \geq C(t_1,s). \tag{1.4}$$

There is a more elementary formulation in place of (1.2) which is worth noting. Instead of asking (1.2) and deriving a subsequent (2.2), in all the statements of Theorem 2.2 and Corollary 2.1 replace (1.2) with: Let $C : (0,\infty) \times (0,\infty) \to (0,\infty)$ be continuous for $0 < s < t$ and suppose that for each $\epsilon > 0$ there is a $\delta > 0$ such that

$$0 \leq t_2 \leq t_1, \quad t_1 - t_2 < \delta \implies \int_{t_2}^{t_1} C(t_1,s)ds < \epsilon.$$

Remark 1. These conditions are the only ones needed to prove the necessary and sufficient condition. In fact, stopping at the end of that proof makes a complete note which can stand alone. But it turns out that this result can be used in Schaefer's fixed point theorem to give a very simple solution to the classical problem of finding a positive solution of the integral equation

$$x(t) = p(t) - \int_0^t C(t,s)g(s,x(s))ds, \tag{1.5}$$

where $p : [0,\infty) \to (0,\infty)$ is continuous. By using our theorem, it turns out that all we need prove is a certain *a priori* bound on solutions of a related equation. This work offers a way of attacking Riemann-Liouville fractional equations after suitable transformations given in [4]. Using a transformation introduced in [5] that bound will be automatic if we use two results which have been well studied in the theory of integral equations. We need to know that the resolvent, $R(t,s)$, for $C(t,s)$ is non-negative and that

$$p(t) - \int_0^t R(t,s)p(s)ds > 0. \tag{1.6}$$

In the appendix we summarize the literature giving affirmative answers to both properties.

It is convenient to first deal with the requirement that when M is a set of bounded continuous functions then the mapping Q defined by $\phi \in M$ implies that

$$(Q\phi)(t) = \int_0^t C(t,s)\phi(s)ds \tag{1.7}$$

maps M into an equicontinuous subset of the Banach space $(\mathcal{B}, \|\cdot\|)$ of bounded continuous functions $\phi : [0,\infty) \to \Re$ with the supremum norm.

Problem. Let $\mathcal{M}$ be a collection of bounded sets M of continuous functions. Find necessary and sufficient conditions on C to ensure that QM is equicontinuous for each M in $\mathcal{M}$.

All scalar fractional differential equations of both Caputo and Riemann-Liouville type, as well as many problems from applied mathematics, typically including heat transfer problems, have kernels $C(t,s) = (t-s)^{q-1}$, $0 < q < 1$, which satisfy (1.2) and (1.4).

A short section is added to show that the basic idea also works for $g(t,s,x(s))$.

2 Equicontinuity

In our work on equicontinuity we always discuss properties on the entire interval $[0,\infty)$. But we see in Theorem 6.1 that we are working on an arbitrary interval $[0,E]$. The reason for this is that compactness of the map is then a consequence of the equicontinuity alone. If uniqueness holds then solutions on such finite intervals can be parlayed into solutions on $[0,\infty)$ without usual difficulties with compactness of such mappings. The results here can be restricted to finite intervals and the theorems will hold without further change. Uniqueness results are also given.

We begin with (1.7)

$$(Q\phi)(t) = \int_0^t C(t,s)\phi(s)ds$$

for which we have the following result. Theorems 2.1 and 2.2 will be put together in Corollary 2.1 to yield the promised necessary and sufficient condition for equicontinuity.

Theorem 2.1 *The mapping Q defined in (1.7) with condition (1.2) holding will map every bounded set M in $(B, \|\cdot\|)$ into an equicontinuous set only if*

$$\int_0^t C(t,s)ds \text{ is uniformly continuous} \tag{2.1}$$

on any finite interval $J \subset [0,\infty)$.

The result is obvious since we could choose M as a set of constant functions. The same result is true with $\phi(s)$ replaced by $g(\phi(s))$ if there is a constant c with $g(c) \neq 0$ since we could again take M to be a set of functions containing c. Notice that the idea fails when we replace $g(\phi(s))$ by $g(s,\phi(s))$ unless there is a constant c with $g(s,c)$ being a nonzero constant.

The convolution case is especially simple and it is the case most often encountered in applied mathematics. We then have

$$\int_0^t C(t-s)ds = \int_0^t C(s)ds$$

which we require to be uniformly continuous on any finite interval of $[0,\infty)$ when C is locally integrable on $(0,\infty)$, which is the case for the very large class of functions $A(t)$ discussed in Section 8 in which (A1) holds. That integral will always be uniformly continuous on any closed bounded interval. Moreover, one of the primary applications has $C(t) = t^{q-1}$ where $0 < q < 1$ so we always have uniform continuity on $[0,\infty)$. All of this brings in an important property of the integral of C; that is, the integral as a measure is absolutely continuous with respect to the Lebesgue measure. In fact, for a given $\epsilon > 0$ and $t_1, t_2 \geq 0$, we can certainly find a $\delta > 0$ so that

$$\left|\int_{t_2}^{t_1} C(t_1 - s)ds\right| = \left|\int_0^{t_1-t_2} C(u)du\right| < \epsilon,$$

if $|t_1 - t_2| < \delta$ because C is in $L^1(0,1)$. In general, the δ depends on t_1. If (1.2) holds with t_1 fixed, then $|t_1 - t_2| < \delta$ implies

$$\left|\int_{t_2}^{t_1} C(t_1, s)ds\right| < \epsilon. \tag{2.2}$$

The reader may wish to refer also to Section 8 in which we discuss (A1)-(A3) and, in particular, (A1) which requires the kernel $A \in L^1(0,1)$ giving us exactly that same property.

The following result shows that the uniform continuity is also sufficient for equicontinuity. The combined form for a necessary and sufficient condition will be given in Corollary 2.1.

Theorem 2.2 *Let M be a bounded subset of $(\mathcal{B}, \|\cdot\|)$. Suppose that $\int_0^t C(t,s)$ is uniformly continuous on any finite interval $J \subset [0,\infty)$. If Q is defined by (1.7) with condition (1.2) holding then QM is an equicontinuous set.*

Proof. Let f be a typical element of M, let $\epsilon > 0$ be given, and let $t_1 \geq 0$ be fixed. Let J be a finite interval of $[0,\infty)$ with $t_1 \in J$. Because of the assumed uniform continuity there is a $\delta > 0$ such that $|t_1 - t_2| < \delta$ with $t_2 \in J$ implies that

$$\left|\int_0^{t_2} C(t_2, s)ds - \int_0^{t_1} C(t_1, s)ds\right| < \epsilon \quad \text{so} \quad \left|\int_0^{t_2} [C(t_2, s) - C(t_1, s)]ds - \int_{t_2}^{t_1} C(t_1, s)ds\right| < \epsilon.$$

Since M is bounded, we find $K > 0$ so that $\|f\| \leq K$ for all $f \in M$. Because of (1.2), it follows from (2.2) that a δ can be chosen so that $0 \leq t_1 - t_2 < \delta$ implies

$$\left|\int_{t_2}^{t_1} C(t_1, s)f(s)ds\right| \leq K\left|\int_{t_2}^{t_1} C(t_1, s)ds\right| < \epsilon.$$

We now have

$$\int_0^{t_2} |C(t_2,s) - C(t_1,s)|ds = \int_0^{t_2} [C(t_2,s) - C(t_1,s)]ds$$

$$\leq \left| \int_0^{t_2} C(t_2,s)ds - \int_0^{t_1} C(t_1,s)ds \right| + \left| \int_{t_2}^{t_1} C(t_1,s)ds \right| < 2\epsilon. \tag{2.3}$$

Now from Q, checking equicontinuity we have

$$\left| \int_0^{t_2} C(t_2,s)f(s)ds - \int_0^{t_1} C(t_1,s)f(s)ds \right|$$

$$= \left| \int_0^{t_2} C(t_2,s)f(s)ds - \int_0^{t_2} C(t_1,s)f(s)ds - \int_{t_2}^{t_1} C(t_1,s)f(s)ds \right|$$

$$\leq \|f\| \left| \int_0^{t_2} |C(t_2,s) - C(t_1,s)|ds \right| + \|f\| \left| \int_{t_2}^{t_1} C(t_1,s)ds \right|$$

$$\leq \|f\| \left[\int_0^{t_2} |C(t_2,s) - C(t_1,s)|ds + \epsilon \right] \leq \|f\| 3\epsilon.$$

The same inequality holds if $0 \leq t_2 - t_1 < \delta$. □

We will be illustrating the results by invoking Schaefer's fixed point theorem which requires a compact mapping of a collection of sets. It is this which motivates our collection $\mathcal{M}$, below.

The last two results will now be combined into a necessary and sufficient condition. Let $\mathcal{M}$ be the class of all sets M for which there are positive constants L_M with the property that if $\phi \in M$ then $\phi : [0,\infty) \to \Re$ is continuous and $\|\phi\| \leq L_M$. For $M \in \mathcal{M}$ let W be the mapping defined by $\phi \in M$ which implies that

$$(W\phi)(t) = \int_0^t C(t,s)g(s,\phi(s))ds. \tag{2.4}$$

The next results need to be stated in the two parts because of the "only if" statement. We would need to ask that there is a constant c so that $g(t,c)$ is a nonzero constant function.

Corollary 2.1 *Let (1.2) and (1.4) hold.*

(i) Let g be continuous and independent of t and suppose there is a constant c with $g(c) \neq 0$. The mapping W defined in (2.4) will map every set $M \in \mathcal{M}$ into an equicontinuous subset of $(B, \|\cdot\|)$ if and only if (2.1) holds.

(ii) If (2.1) holds and if $g(t,x)$ is continuous and bounded for x bounded then the mapping W defined in (2.4) will map every set $M \in \mathcal{M}$ into an equicontinuous subset of $(B, \|\cdot\|)$.

Proof. For a given set $M \in \mathcal{M}$ with the constant function $c \in M$ construct a new set M^* defined by $\phi^* \in M^*$ which implies that $\phi^*(t) = g(\phi(t))$ for $\phi \in M$. The new set M^* is also in $\mathcal{M}$. If ϕ is the constant c then ϕ^* is a nonzero constant and $(W\phi)(t) = (Q\phi^*)(t)$

will reside in an equicontinuous set only if (2.1) holds, where Q is defined in (1.7). On the other hand, if (2.1) holds, by Theorem 2.2 we see that QM^* is an equicontinuous set. By the definition of M^*, we have $WM = QM^*$ so that WM is an equicontinuous set. This proves (i).

To prove (ii), proceed as in the proof of (i) with $g(t,x)$ and construct M^* again for a given set M so that W will map M into an equicontinuous set exactly as in part (i) above. □

3 Dependence on t

Frequently the mapping takes the form

$$(H\phi)(t) = \int_0^t C(t,s)v(t,s,\phi(s))ds, \tag{3.1}$$

where $v : [0,\infty) \times [0,\infty) \times \Re \to \Re$ is continuous and for each $L > 0$ there exists $D > 0$ so that for $0 \le s \le t$,

$$|x| \le L \implies |v(t,s,x)| \le D. \tag{3.2}$$

A treatment of this mapping may allow us to reduce (1.4) by asking for a $\beta : [0,\infty) \to (0,\infty)$ with the property that

$$C^*(t,s) =: C(t,s)/\beta(t) \tag{3.3}$$

satisfies (1.4). Then

$$(H\phi)(t) = \int_0^t C^*(t,s)\beta(t)\phi(s)ds \tag{3.4}$$

will have the form of (3.1). While we would expect $\beta(t)$ to tend to infinity, this will be no problem in our theorem below so long as we work on finite intervals $[0,E]$.

Theorem 3.1 *Let (1.2), (1.4), (2.1), and (3.2) hold for (3.1). For M as in Theorem 2.2 and $M \in \mathcal{M}$ then (3.1) maps M into an equicontinuous set on $[0,E]$.*

Proof. Notice first that for each $E > 0$ there is a $K > 0$ such that $0 < t \le E$ implies that

$$\int_0^t C(t,s)ds \le K.$$

For a given $M \in \mathcal{M}$, $\epsilon_1 > 0$, and $t_1 \in [0,E]$, we seek $\delta > 0$ so that $\phi \in M$ and $0 \le t_2 < t_1$ and $t_1 - t_2 < \delta$ implies that

$$|(H\phi)(t_2) - (H\phi)(t_1)| < \epsilon_1. \tag{3.5}$$

Let L be the bound for this M and D be defined in (3.2). We have

$$\begin{aligned}
&(H\phi)(t_2) - (H\phi)(t_1)\\
&= \int_0^{t_2} C(t_2,s)v(t_2,s,\phi(s))ds - \int_0^{t_1} C(t_1,s)v(t_1,s,\phi(s))ds\\
&= \int_0^{t_2} [C(t_2,s)v(t_2,s,\phi(s)) - C(t_2,s)v(t_1,s,\phi(s))]ds\\
&+ \int_0^{t_2} [C(t_2,s)v(t_1,s,\phi(s)) - C(t_1,s)v(t_1,s,\phi(s))]ds\\
&- \int_{t_2}^{t_1} C(t_1,s)v(t_1,s,\phi(s))ds.
\end{aligned}$$

Now, v is uniformly continuous on M for $0 \le t \le E$ so for a given $\epsilon > 0$ there is a $\delta > 0$ such that $\phi \in M$, $|t_1 - t_2| < \delta$ implies that $|v(t_1, s, \phi(s)) - v(t_2, s, \phi(s))| < \epsilon$. At the same time, let δ be so small that for this ϵ then (2.3) holds. Thus

$$\begin{aligned}
&|(H\phi)(t_2) - (H\phi)(t_1)| \\
&\le \int_0^{t_2} C(t_2, s)\epsilon ds + \int_0^{t_2} [C(t_2, s) - C(t_1, s)]|v(t_1, s, \phi(s))|ds \\
&+ \int_{t_2}^{t_1} C(t_1, s)|v(t_1, s, \phi(s))|ds \\
&\le \epsilon \int_0^{t_2} C(t_2, s)ds + D\int_0^{t_2} [C(t_2, s) - C(t_1, s)]ds + D\int_{t_2}^{t_1} C(t_1, s)ds \\
&< \epsilon \int_0^{t_2} C(t_2, s)ds + 2\epsilon D + D\epsilon \\
&< \epsilon K + 2D\epsilon + D\epsilon < \epsilon_1 \ \text{ if } \ \epsilon < \epsilon_1/(K + 3D).
\end{aligned}$$

Similarly, we can show that (3.5) holds if $0 \le t_1 < t_2$ and $t_2 - t_1 < \delta$. □

4 Schauder's Theorem and Measures of Noncompactness

Study of the literature shows that investigators using fixed point theory frequently pursue either a contraction, a Schauder type fixed point theorem based on compactness of the mapping, or Darbo's fixed point theorem based on measures of noncompactness. If a contraction is present, it is usually the most elementary, but if it is not available then a compactness type result is usually far more elementary than Darbo's theorem . Theorem 2.2 can be a definite asset in determining if the compactness path is feasible. Darbo's path can require far less structure in the kernel. See, for example the lengthy expository paper of Appell [3, p. 195] for discussions of measures of non-compactness related to the present discussion.

The point of the second half of this paper is to show that in the choice of theorems in that compactness path, Schaefer's theorem can be so very natural, simple, and direct. To see this we start with Schauder's theorem [11, p. 25] and then in the next section compare it with Schaefer's for this class of problems.

Theorem 4.1 (Schauder) *Let M be a non-empty convex subset of a normed space $(\mathcal{B}, \|\cdot\|)$. Let P be a continuous mapping of M into a compact set $K \subset M$. Then P has a fixed point.*

To apply the theorem we see that:
1. We must find a self-mapping set as described.
2. The natural mapping defined by (1.5) needs to be continuous and into a compact set.

The next section will show that Schaefer's theorem can get us past both of these in a very smooth way.

5 Schaefer's Fixed Point Theorem

In this and the following sections we work our way up to application of Schaefer's theorem.

The object of this section is to point out two requirements for a positive solution. We need $R(t,s) \geq 0$ and $p(t) - \int_0^t R(t,s)p(s)ds > 0$. There is large literature detailed in the appendix giving sufficient conditions for them to hold.

We now show how Schaefer's fixed point theorem [11, p. 29] gets us past the Items 1 and 2 discussed in the previous section. We place this discussion in the context of the search for a positive solution of (6.1). Much has been written about such existence, as may be seen, for example, in the books ([1], [2]). In many problems, such as population studies, only a positive solution has any meaning.

Theorem 5.1 (Schaefer) *Let $(\mathcal{B}, \|\cdot\|)$ be a normed space, P be a continuous mapping of $\mathcal{B}$ into $\mathcal{B}$ which is compact on each bounded subset X of $\mathcal{B}$. Then either*

(i) *the equation $x = \lambda Px$ has a solution for $\lambda = 1$, or*

(ii) *the set of all such solutions x, for $0 < \lambda < 1$, is unbounded.*

Notice the difference between this and Schauder's theorem in the search for a positive solution of (1.5).

1. The most challenging part of application of Schauder's theorem is locating a self mapping set. Schaefer's theorem does not require it.

2. We will restrict our mapping to an arbitrary interval $[0, E]$ which will later be extended to $[0, \infty)$. The mapping P will be the natural mapping defined by (1.5). Our Theorem 2.2 will take care of the requirements that $P : \mathcal{B} \to \mathcal{B}$ and that the equicontinuous mapping is compact on bounded sets.

3. Because we are working on a bounded interval, pointwise continuity of $g(t,x)$ will take care of continuity of the map.

4. We only have to show that there is an *a priori* bound on all possible solutions of our (1.5) when we insert a parameter. This is a two step process.

a. With $p(t) > 0$ and $g(t,x) > 0$ for $x > 0$ it is clear from (1.5) that a solution begins positive and is bounded above by $p(t)$ so long as it remains positive. Thus, we need to provide a non-negative lower bound for the solution.

b. To obtain a lower bound, in Section 6 we transform (1.5) (with a parameter λ) into an equation, later designated as (6.10)

$$x(t) = \lambda\left[p(t) - \int_0^t R(t,s)p(s)ds\right] + \int_0^t R(t,s)\left[x(s) - \frac{g(s,x(s))}{J}\right]ds.$$

Here, we repeat some classical theory of integral equations [10, pp. 189-193, 207-213] with detail in Section 6. The function R is a resolvent satisfying

$$R(t,s) = \lambda JC(t,s) - \int_s^t \lambda JC(t,u)R(u,s)du \tag{5.1}$$

for $0 < s < t < \infty$ with

$$\int_0^t R(t,s)\phi(s)ds \tag{5.2}$$

continuous for any continuous function $\phi : [0, \infty) \to \Re$.

We observe that the resolvent R is also a function of λ. For brevity in notation, we will suppress the λ in the expression of R here. But this will cause us no trouble in Theorem 6.1 below since we ask that (5.2)-(5.4) hold for each λ, $0 < \lambda \leq 1$.

Now, in order to ensure that $x(t)$ remains positive we require **three** things. The first two are

$$R(t,s) \geq 0 \tag{5.3}$$

and

$$p(t) - \int_0^t R(t,s)p(s)ds > 0, \quad t \geq 0 \tag{5.4}$$

for each λ, $0 < \lambda \leq 1$. These two properties have been studied for decades in the standard integral equation theory and several sufficient conditions are offered in the appendix which cover major areas in applied mathematics and fractional differential equations. A prominent example satisfying (5.3), (5.4), and Theorem 2.2 is $C(t,s) = (t-s)^{q-1}, 0 < q < 1$ and $p(t)$ non-decreasing. For such problems, all conditions of Schaefer's theorem will immediately hold except for the *a priori* bound. And the third requirement is treated next and can be absolutely elementary.

c) Main note. We used the negative integral in (1.5) to get an upper bound on all possible solutions. When (5.3) and (5.4) hold and when we can find a $J > 0$ with $0 < g(t,x)/(Jx) < 1$ when $0 < x \leq p(t)$ and $0 \leq t \leq E$, then a transformation will change (6.1) into an equivalent equation with positive right-hand side, thereby making $x = 0$ a lower bound. This will satisfy Schaefer's theorem and we will have a positive solution on an arbitrary interval $[0, E]$. This is the content of Theorem 6.1 and we notice that the inequality $0 < g(t,x)/(Jx) < 1$ must fail for such functions as $g(t,x) = x^{1/3}$, but Theorem 6.2 will pick up just such cases.

6 The Resolvent and a Transformation

In the previous section we gave an outline indicating that the conditions $\int_0^t C(t,s)ds$ is continuous, $R(t,s) \geq 0$, $p(t) - \int_0^t R(t,s)p(s)ds > 0$, and $0 < g(t,x)/(Jx) < 1$ "locally" imply the existence of a positive solution on any interval $[0, E]$. These offer a major contrast to conditions required in Schauder's theorem. Here are the details of the promised transformation.

Schaefer's fixed point theorem requires the introduction of a parameter $\lambda \in (0,1]$ into the integral equation. We return to (1.2), (1.3), and (1.6) which we restate here with the parameter for reference as

$$x(t) = \lambda\left[p(t) - \int_0^t C(t,s)g(s,x(s))ds\right], \tag{6.1}$$

where $0 < \lambda \leq 1$,

$$C : (0,\infty) \times (0,\infty) \to (0,\infty) \tag{6.2}$$

satisfies (1.2), g and p are continuous where

$$g : [0,\infty) \times \Re \to \Re, \ x > 0 \implies g(t,x) > 0 \tag{6.3}$$

and

$$p : [0,\infty) \to (0,\infty). \tag{6.4}$$

Let J be an arbitrary positive constant and write (6.1) as

$$\begin{aligned}x(t) &= \lambda p(t) + \int_0^t C(t,s)[-\lambda J x(s) + \lambda J x(s) - \lambda g(s,x(s))]ds \\ &= \lambda p(t) - \lambda J \int_0^t C(t,s)x(s)ds + \lambda J \int_0^t C(t,s)\left[x(s) - \frac{g(s,x(s))}{J}\right]ds. \qquad (6.5)\end{aligned}$$

Define

$$D(t,s) = \lambda J C(t,s) \qquad (6.6)$$

and write the linear part as

$$z(t) = \lambda p(t) - \int_0^t D(t,s)z(s)ds \qquad (6.7)$$

with resolvent equation

$$R(t,s) = D(t,s) - \int_s^t D(t,u)R(u,s)du \qquad (6.8)$$

so that by the linear variation of parameters formula

$$z(t) = \lambda p(t) - \int_0^t R(t,s)\lambda p(s)ds. \qquad (6.9)$$

We have again suppressed the λ in the expressions of R and D for brevity here.

The nonlinear variation of parameters formula [10, pp. 190-193] yields

$$x(t) = z(t) + \int_0^t R(t,s)\left[x(s) - \frac{g(s,x(s))}{J}\right]ds. \qquad (6.10)$$

Main note. We emphasize that this equation is used in Theorems 6.1 and 6.2 only for establishing a lower bound on the solution. It is never the mapping equation. It is used in Theorem 7.1 to show uniqueness.

The reader will note that (iii) in Theorem 6.1 below requires a near Lipschitz condition centered on the $t-$axis. This eliminates such functions as $g(t,x) = x^{1/3}$. But those functions can be included with a simple translation device which we show in Theorem 6.2.

There is so much to be gained by working on an arbitrary finite interval $[0,E]$. In that case an equicontinuous map becomes a compact map. Moreover, if $p(t)$ is unbounded, then we obtain a solution which may be unbounded, but the unboundedness of p then causes us no trouble with the compactness arguments. When we examine proofs of continuity of the mapping, if we were working directly on the whole interval $[0,\infty)$ we would be needing some severe uniform continuity conditions on $g(t,x)$. The introduction of $\beta(t)$ in (3.3) can completely save a problem for this program. Since $\beta(t)\phi(s)$ in (3.4) could be unbounded if $\beta(t) \to \infty$ as $t \to \infty$ the finite interval avoids any problem with that property.

When we argue that by uniqueness we can continue a solution to all of $(0,\infty)$, notice that we are only using uniform convergence on compact sets to obtain that solution. By contrast, if we examine the end of Miller's proof [10, pp. 210–212] of Theorem 6.1 we will see that he does not have uniqueness and obtains a solution on a finite interval, stating then in the last line of the proof that the solution can be extended by continuation methods. However, that requires another significant argument. As noted in [9, p. 42], this type of argument is not elementary and relies on Zorn's lemma.

Finally, there is another significant advantage of work on $[0, E]$ and it is something of a surprise. In Theorem 2.2 if we had asked for uniform continuity of $\int_0^t C(t,s)ds$ on $[0,\infty)$, we would have restricted the growth rate of $g(t,x)$ in x to essentially linear growth. But when we work on $[0, E]$ we need only ask for continuity, leaving growth rate completely unrestricted.

Theorem 6.1 *Suppose that:*

(i) Conditions (1.2), (1.3), and (1.4) hold.

(ii) $\int_0^t C(t,s)ds$ *is continuous on any interval* $[0, E]$.

(iii) For each $E > 0$ *there are* $K < 1$, $J > 0$, *and* $L = \sup_{0 \le t \le E} p(t)$ *with* $0 < \frac{g(t,x)}{Jx} < K$ *if* $0 < x \le L$ *and* $0 \le t \le E$.

(iv) The unique solution $R(t,s)$ *of (6.8) is non-negative for* $0 < s < t < \infty$ *and (5.4) holds.*

Then for $\lambda = 1$ *(6.1) has a positive solution on* $[0, E]$ *for any* $E > 0$. *If solutions are uniquely determined by the initial condition then the solution exists on* $[0,\infty)$.

Proof. While $\mathcal{B}$ is defined for functions on $[0,\infty)$, in this theorem all of the functions are restricted to the interval $[0, E]$ until we come to (iv) when we then develop the solution on $[0,\infty)$. Define a mapping P from (6.1) so that for $\phi \in \mathcal{B}$

$$(P\phi)(t) = p(t) - \int_0^t C(t,s)g(s,\phi(s))ds \tag{6.11}$$

and item (i) of Schaefer's theorem will be

$$x(t) = \lambda\left[p(t) - \int_0^t C(t,s)g(s,x(s))ds\right]. \tag{6.12}$$

I. First notice that in (6.12) if there is a solution, then $x(0) = \lambda p(0) > 0$ and, so long as $x(t) > 0$, the integrand in (6.12) is positive. Therefore

$$0 < x(t) \le \lambda p(t) \le L. \tag{6.13}$$

Notice also that the upper bound on $x(t)$ is independent of λ. If we can show that $x(t)$ remains positive then (6.13) will represent an *a priori* bound on all possible solutions of (6.12) and (ii) in Schaefer's theorem will be excluded.

II. We now show that $P : \mathcal{B} \to \mathcal{B}$. First, p is uniformly continuous on $[0, E]$. Next, if ϕ is in $\mathcal{B}$ then $g(t,\phi(t))$ is continuous and bounded on $[0, E]$ so by Theorem 2.2 $g(t,\phi(t))$ will be mapped by (6.11) into an equicontinuous set, from which we conclude that $P\phi \in \mathcal{B}$.

III. Continuity of the mapping P follows from that of $g(t,x)$ and the fact that $\int_0^t C(t,s)ds$ is uniformly continuous. Details are very similar to those in [6].

IV. Now we must show that *zero* is a lower bound on all possible solutions of (6.12). For this we go to the equivalent transformed equation (6.10). That transformation is reversible so a bound on solutions of (6.10) means a bound on the solutions of (6.12). It is easy to see that if 0 is a lower bound of solutions of (6.10), then it is independent of the λ, just as in the case of the upper bound. Notice that (iii) implies that if a solution is positive on an interval $[0, t_1)$, then the integrand in (6.10) is positive. This means that the solution can not vanish at t_1 because $z(t) > 0$. Hence, if a solution exists on $[0, E]$ then it is positive. We have established that any solution satisfies (6.13) on $[0, E]$.

V. Concerning the compact map, note that while we have two forms for our equation, it is only (6.11) which is the mapping equation. Equation (6.10) is only used to establish the lower bound on solutions, although in the next section it is also used for uniqueness. In order to show that P is a compact map, we only need to show that P maps bounded sets into equicontinuous sets since we are working on $[0, E]$. We noted in II that p is uniformly continuous. Let M be any bounded set in $\mathcal{B}$ on $[0, E]$ and determine $H > 0$ so that $\phi \in M$ implies $\|\phi\| \leq H$. This means that $g(t, x)$ is bounded for $|x| \leq H, 0 \leq t \leq E$. By Corollary 2.1 the set M will be mapped by P in (6.11) into an equicontinuous set. Adding in the uniformly continuous function p shows that P maps the given bounded set into an equicontinuous set.

The conditions of Schaefer's theorem are satisfied and P has a fixed point satisfying (6.12) for $\lambda = 1$.

In the event that solutions are unique then a solution, $x_n(t)$, is uniquely determined on any interval $[0, n]$ for n an arbitrary positive integer. Notice that the x_{n+k} agrees with x_n on $[0, n]$. Extend each $x_n(t)$ to a function $y_n(t)$ which is continuous on $[0, \infty)$ and agreeing with x_n on $[0, n]$. The sequence $y_n(t)$ converges uniformly on compact sets to a single function $x(t)$ on $[0, \infty)$ which does satisfy (6.1) with $\lambda = 1$ at every point on $[0, \infty)$. $\square$

We now outline the changes needed in order to accommodate functions like $g(t, x) = x^{1/3}$. All of the conditions of Theorem 6.1 will be retained except (iii). The fact is that, unlike the classical result of Miller [10, p. 210] in which he obtains a non-negative solution which is not necessarily unique, our result will be a strictly positive solution on any interval $[0, E]$. Because the solution is positive, for each E and $\lambda \in (0, 1]$ if we can find $D_\lambda > 0$ and construct a line $x = \lambda D_\lambda > 0$ above which the solution lies, then condition (iii) holds, again above the line, so long as we are working on the fixed interval $[0, E]$. In that region functions like $x^{1/3}$ will allow the dominance displayed in (iii).

The big change here from Theorem 6.1 is that the region in (6.16) depends on each fixed λ, but we always keep the solution in the region $0 < \lambda D_\lambda \leq x(t) \leq L$, $0 \leq t \leq E$, yielding the *a priori* bound of L for every λ. When we invoke Schaefer's theorem then we obtain a solution in that region for $\lambda = 1$. Non-uniqueness problems for $g(t, x) = x^{1/3}$ will vanish since any pair of solutions must reside in that region, and that is the topic of the next section.

Now consider the region $0 < \lambda D_\lambda \leq x(t) \leq L$. Notice that it gets closer to the $x-$axis as $\lambda \to 0$ if D_λ is bounded. For $g(t, x) = x^{1/3}$ the region in (iii) of Theorem 6.1 of $0 < x \leq L$ can not possibly yield the inequality $0 < g(t, x)/(Jx) < K < 1$. For each λ we must go back to our transformation discussed in (6.5)-(6.10) and pick a larger value of J. This is no cause for concern because the transformation is reversible and the solution of (6.10) with each such J will correspond to the solution of (6.12).

Up to this point we have been suppressing the λ in the expression of $R(t, s)$ for brevity. We now reinsert the λ in the notation, say $R_\lambda(t, s)$, to emphasize that $R_\lambda(t, s)$ is the unique solution of (5.1) for the fixed λ. Unlike the case in Theorem 6.1 where J is an arbitrary positive constant, in Theorem 6.2 we choose J as a function of λ so we write

$J = J_\lambda$. We now restate (5.1) here with the new parameters as

$$R_\lambda(t,s) = \lambda J_\lambda C(t,s) - \int_s^t \lambda J_\lambda C(t,u) R_\lambda(u,s)du \tag{6.14}$$

for $0 < s < t$ and $\lambda \in (0,1]$. Here J_λ is to be determined.

In Theorem 6.2 below we ask that (5.4) hold and for a given $\gamma > 1$, define

$$D_\lambda = \min_{0\le t\le E}\left[p(t) - \int_0^t R_\lambda(t,s)p(s)ds\right]/\gamma \tag{6.15}$$

for each λ, $0 < \lambda \le 1$.

In particular, the solution obtained by Schaefer's theorem satisfies $x(t) \ge D$ with $D = D_1$. And this D will be critical in proving the uniqueness result in the next section, enabling us to continue our solution on an arbitrarily large interval $[0,E]$ to a positive solution on $[0,\infty)$.

Theorem 6.2 *Let the conditions of Theorem 6.1 hold except for (iii). Let* $0 < K < 1$, $E > 0$ *be given and let* $L = \sup_{0\le t\le E} p(t) > 0$. *Suppose that for each* $\lambda \in (0,1]$, *there exists* $J_\lambda > 0$ *so that*

$$\lambda D_\lambda \le x \le L,\ \ t\in[0,E] \implies 0 < \frac{g(t,x)}{J_\lambda x} \le K. \tag{6.16}$$

Then for $\lambda = 1$ *(6.12) has a positive solution on* $[0,E]$ *and any solution of (6.12) for* $\lambda \in (0,1]$ *satisfies* $0 < \lambda D_\lambda \le x(t) \le L$.

Proof. In the second to last sentence of the theorem recall that (6.10) and (6.12) share solutions. We explain the second sentence of Theorem 6.2 as follows. Recall from (6.9) that

$$z(t) = \lambda\left[p(t) - \int_0^t R_\lambda(t,s)p(s)ds\right]. \tag{6.17}$$

From this and (6.15) it follows that $z(t) > \lambda D_\lambda$ on $[0,E]$. All of the work on continuity and compactness needed for Schaefer's theorem was given in the proof of Theorem 6.1.

The only thing left to be proved here is the *a priori* bound. The upper bound of L still holds. We need only to show a lower bound. Suppose a solution $x(t)$ exists on $[0,E]$ and, since $\lambda p(0) > \lambda D_\lambda > 0$, we can assume $x(t) > \lambda D_\lambda$ on an interval $[0,t_1)$. It follows from (6.16) that the integrand in (6.10) is positive on $[0,t_1]$. We now investigate the possibility that $x(t_1) = \lambda D_\lambda$. Note that on $[0,t_1)$ we have

$$x(t) = z(t) + \int_0^t R_\lambda(t,s)x(s)\left[1 - \frac{g(s,x(s))}{J_\lambda x(s)}\right]ds$$

and this integral is positive on $[0,t_1]$ and $x(t) > \lambda D_\lambda$ on $[0,t_1)$. Because the integral is positive we have $x(t) \ge z(t)$ at t_1. However, at t_1 we have $x(t_1) \ge z(t_1) > \lambda D_\lambda$, a contradiction to our assumption that $x(t_1) = \lambda D_\lambda$. We conclude that the solution remains above λD_λ on $[0,E]$. □

In Example 6.1 below, we will outline some basic steps that can be taken for determining the number J_λ. To this end, we examine (5.4) and (6.15). First we see that, in

(6.14), if $C(t,s) = A(t-s)$, and if A satisfies conditions (A1)-(A3) in Section 8, then for each $\lambda \in (0,1]$, $R_\lambda(t,s) = R_\lambda(t-s)$, $R_\lambda(t) > 0$, and

$$\int_0^\infty R_\lambda(t)dt = \lambda J_\lambda A^*(1+\lambda J_\lambda A^*)^{-1}, \tag{6.18}$$

if $A^* = \int_0^\infty A(s)ds < \infty$ (see Section 8 and Miller [10, pp. 212–213]). If the function A is defined on $[0,E]$, we can easily extend its domain to $[0,\infty)$ with A satisfying (A1)-(A3) on $[0,\infty)$ and $A \in L^1[0,\infty)$.

Next, we observe that if $p(t)$ is non-decreasing, $R_\lambda(t,s) \geq 0$, and $\int_0^t R_\lambda(t,s)ds < 1$ for $t \in [0,E]$, then

$$p(t) - \int_0^t R_\lambda(t,s)p(s)ds \geq p(t)\left[1 - \int_0^t R_\lambda(t,s)ds\right] > 0. \tag{6.19}$$

This implies that (5.4) holds. The lower bound of $\left(1 - \int_0^t R_\lambda(t,s)ds\right)$ for $t \in [0,E]$ is essential for determining J_λ. For the convolution case, we have from (6.18) that

$$1 - \int_0^t R_\lambda(t-s)ds = (1+\lambda J_\lambda A^*)^{-1} + \int_t^\infty R_\lambda(u)du. \tag{6.20}$$

For the non-covolution case, in Example 6.1 we ask that the resolvent $R_\lambda(t,s)$ in (6.14) satisfy

$$1 - \int_0^t R_\lambda(t,s)ds \geq (1+\lambda J_\lambda N)^{-1} \tag{6.21}$$

for all $t \in [0,E]$ and a fixed positive number N. Condition (6.21) holds for a general class of convex kernels. We will not go into the details here and refer the readers to Section 8 for reference.

We now consider the equation

$$x(t) = p(t) - \int_0^t C(t,s)x^{1/3}(s)ds \quad \text{for} \quad t \in [0,E]. \tag{6.22}$$

Example 6.1 *Let $E > 0$ be given, let $p(t)$ be continuous, positive, and non-decreasing for $t \in [0,E]$, and let $L = \sup_{0\leq t\leq E} p(t)$. Suppose that*

(i) Conditions (1.2), (1.4), (5.2), and (5.3) hold.

(ii) $\int_0^t C(t,s)ds$ is continuous on any interval $[0,E]$.

If (6.21) holds, then (6.22) has a positive solution on $[0,E]$.

Proof. Let $g(t,x) = x^{1/3}$ and $0 < K < 1$ be given. We first observe that (6.21) implies (5.4). Thus, to apply Theorem 6.2 we only need to verify that (6.16) holds. For each $\lambda \in (0,1]$, to determine J_λ in (6.16) we need to find a lower bound of D_λ in terms of J_λ. To this end, we apply (6.21) and we proceed as follows.

$$\begin{aligned} D_\lambda &= \min_{0\leq t\leq E}\left[p(t) - \int_0^t R_\lambda(t,s)p(s)ds\right]/\gamma \\ &\geq \min_{0\leq t\leq E} p(t)\left[1 - \int_0^t R_\lambda(t,s)ds\right]/\gamma \geq p_0\,(1+\lambda J_\lambda N)^{-1}/\gamma, \end{aligned} \tag{6.23}$$

where $p_0 = p(0)$. We now define

$$G_\lambda = \max\left\{\frac{g(t,x)}{x},\ 0 \le t \le E,\ \lambda D_\lambda \le x \le L\right\}$$
$$= \max\left\{\frac{x^{1/3}}{x},\ 0 \le t \le E,\ \lambda D_\lambda \le x \le L\right\} = \frac{1}{(\lambda D_\lambda)^{2/3}}. \tag{6.24}$$

Thus, to show that (6.16) holds, it suffices to solve the inequality

$$0 < \frac{g(t,x)}{J_\lambda x} \le \frac{G_\lambda}{J_\lambda} \le K \tag{6.25}$$

on $\lambda D_\lambda \le x \le H,\ t \in [0,E]$ for a positive J_λ. It follows from (6.23) and (6.24) that

$$\frac{G_\lambda}{J_\lambda} = \frac{1}{J_\lambda(\lambda D_\lambda)^{2/3}} \le \frac{1}{J_\lambda \lambda^{2/3}\left[p_0(1+\lambda J_\lambda N)^{-1}\gamma^{-1}\right]^{2/3}} = \frac{(1+\lambda J_\lambda N)^{2/3}\gamma^{2/3}}{J_\lambda\,(\lambda p_0)^{2/3}} \le K. \tag{6.26}$$

It is easy to see that (6.26) has a positive solution J_λ (infinitely many). Thus, (6.16) is satified and (6.22) has a positive solution on $[0,E]$ by Theorem 6.2. $\square$

7 Uniqueness

We will begin with an example in which Theorem 6.2 already shows that there is a positive solution and for positive x we will write $g(t,x) = x^{1/2}$ which is very instructive for several reasons. First, early in our study of differential equations we find that $x' = -x^{1/2}$ for $x \ge 0$ can generate non-uniqueness so we are immediately on guard. It is especially effective here in that there are no inequalities; everything is absolutely exact and we can see at each step exactly what is promoting uniqueness and where uniqueness is likely to fail.

Example 7.1 *Let $g(t,x) = x^{1/2}$ for $x \ge 0$. If the other conditions of Theorem 6.2 hold, then there is a unique positive solution of (6.1) on $[0,\infty)$ for $\lambda = 1$.*

Proof. According to the proof of Theorem 6.2 for any $E > 0$ any solution of (6.1) for $\lambda = 1$ on $[0,E]$ satisfies $0 < D \le x(t) \le L$ for a pair of fixed numbers D and L with $D = D_1$. Then $x(t)$ satisfies (6.10) for any positive number J. By way of contradiction to uniqueness, if x_1 and x_2 are two solutions on some interval $[0,E]$ then we will rationalize the subsequent numerator and have

$$x_1(t) - x_2(t) = \int_0^t R(t,s)\left[x_1(s) - \frac{x_1^{1/2}(s)}{J} - x_2(s) + \frac{x_2^{1/2}(s)}{J}\right]ds$$
$$= \int_0^t R(t,s)\left[x_1(s) - x_2(s) - \frac{x_1^{1/2}(s) - x_2^{1/2}(s)}{J}\right]ds$$
$$= \int_0^t R(t,s)\left[x_1(s) - x_2(s) - \frac{(x_1(s) - x_2(s))}{J(x_1^{1/2}(s) + x_2^{1/2}(s))}\right]ds$$
$$= \int_0^t R(t,s)(x_1(s) - x_2(s))\left[1 - \frac{1}{J(x_1^{1/2}(s) + x_2^{1/2}(s))}\right]ds. \tag{7.1}$$

We proceed to estimate the right-hand side of (7.1) by choosing a sufficiently large J in (6.10) so that $\beta := \dfrac{1}{2JL^{1/2}} \le \dfrac{1}{2JD^{1/2}} < 1$. It follows from (7.1) that

$$\begin{aligned} |x_1(t) - x_2(t)| &\le \int_0^t R(t,s)|x_1(s) - x_2(s)|\left[1 - \frac{1}{J(L^{1/2} + L^{1/2})}\right] ds \\ &= \int_0^t R(t,s)|x_1(s) - x_2(s)| ds \, (1 - \beta). \end{aligned}$$

Thus, taking the supremum of both sides

$$\|x_1 - x_2\| \le (1 - \beta) \sup_{t \in [0,E]} \int_0^t R(t,s) ds \, \|x_1 - x_2\|, \tag{7.2}$$

we now show that

$$\int_0^t R(t,s) ds \le 1 \quad \text{for} \quad t \in [0, E]. \tag{7.3}$$

In fact, since $R(t,s) \ge 0$ and $C(t,s)$ is non-increasing in t, we have for $t > s$

$$\begin{aligned} \int_s^t R(t,u) du &\le \int_s^t R(t,u) C(u,s) du / C(t,s) \\ &= [C(t,s) - R(t,s)]/C(t,s) = 1 - R(t,s)/C(t,s) \le 1. \end{aligned}$$

Thus, (7.3) holds and (7.2) would yield a contradiction. □

Notice that the denominator in the last term of (7.1) is a type of average value of the derivative of $g(x) = x^{1/2}$. We will see this in the general case with a very explicit application of the mean value theorem for derivatives.

Theorem 7.1 *Let the conditions of Theorem 6.2 hold, let $E > 0$ be given so that D and L are known with $D = D_1$, and let $g(t,x) = g(x)$ with $(d/dx)g(x) > 0$ and continuous for $D \le x \le L$. If $C(t,s)$ is non-increasing in t for $t > s$, then the positive solution of Theorem 6.2 is unique.*

Proof. By way of contradiction, assume that x_1 and x_2 are two positive solutions on $[0, E]$ so that by Theorem 6.2 they must both satisfy $D \le x \le L$. Pick $J > (d/dx)g(x)$ for $D \le x \le L$.

By the mean value theorem for derivatives, for each $s \in [0, E]$ either $x_1(s) = x_2(s)$ or there is an ξ between x_1 and x_2 with

$$g(x_1) - g(x_2) = \frac{dg(\xi)}{dx}(x_1 - x_2).$$

Thus, we have

$$\begin{aligned} x_1(t) - x_2(t) &= \int_0^t R(t,s)\left[x_1(s) - x_2(s) - \frac{g(x_1(s)) - g(x_2(s))}{J}\right] ds \\ &= \int_0^t R(t,s)\left[(x_1(s) - x_2(s)) - \left[\frac{\frac{dg(\xi(s))}{dx}[x_1(s) - x_2(s)]}{J}\right]\right] ds \\ &= \int_0^t R(t,s)[x_1(s) - x_2(s)]\left[1 - \frac{\frac{dg(\xi(s))}{dx}}{J}\right] ds. \end{aligned}$$

Now let $\alpha = \inf\left\{\frac{dg(x)}{dx} : D \le x \le L\right\}$. Then $\alpha > 0$. Note that $\alpha < J$. Therefore,

$$\|x_1 - x_2\| \le \|x_1 - x_2\|(1 - \alpha/J) \sup_{t\in[0,E]} \int_0^t R(t,s)ds < \|x_1 - x_2\|,$$

is a contradiction.

8 Appendix: Survey of Non-negative Resolvents

Note carefully that we always ask that $C(t,s) > 0$ and then notice in (5.1) that if there is a (t,s) at which the integral in (5.1) is negative, then $R(t,s)$ is positive at that (t,s). This is extremely important; the resolvent can never be a negative function for all (t,s). Everything we do here will depend on the resolvent being always non-negative. Thus we survey some of the main conditions known to ensure that property.

The convolution case

The first, and certainly the main, result is given by Miller [10, p. 209] and it concerns the case of

$$C(t,s) = A(t-s) \tag{8.1}$$

with the resolvent equation now reducing to

$$R(t) = A(t) - \int_0^t A(t-u)R(u)du. \tag{8.2}$$

The conditions on A are:
(A1) A is continuous on $(0,\infty)$ and is in $L^1(0,1)$.
(A2) $A(t)$ is positive and non-increasing for $t > 0$.
(A3) For each $T > 0$ the function $A(t)/A(t+T)$ is non-increasing in t for $0 < t < \infty$.
The classical example is $A(t) = t^{q-1}, 0 < q < 1$ and that is the kernel in all fractional differential equations of both Caputo and Riemann-Liouville type, many problems in heat transfer, and in a virtually endless list of other prominent problems from applied mathematics.
When A satisfies those conditions then Miller [10, pp. 212–213] establishes that
a) $R(t)$ is continuous on $(0,\infty)$.

b) $0 \le R(t) \le A(t)$ for all $t > 0$.

c) If $\int_0^\infty A(s)ds = \infty$ then $\int_0^\infty R(s)ds = 1$.

d) If $\int_0^\infty A(s)ds = A^* < \infty$ then $\int_0^\infty R(s)ds = A^*\left(1 + A^*\right)^{-1}$.

e) It is also true that if $A(t)$ is completely monotone on $(0,\infty)$ with $A(t) \not\equiv 0$, so is $R(t)$ with $R(t) > 0$ for all $t > 0$ [10, p. 224], and $A(t)$ satisfies (A1) - (A3) [10, p. 221].

Gripenberg [7, p.381] improves b) obtaining
f) $0 < R(t) \le A(t)/(1 + \int_0^t A(s)ds)$.
This is a result giving us the non-negativity of $R(t-s)$. We will now give two extreme examples for the companion result that

$$z(t) = p(t) - \int_0^t A(t-s)z(s)ds = p(t) - \int_0^t R(t-s)p(s)ds > 0.$$

The point of this theorem is that $\int_0^t R(s)ds < 1$. Thus, $1 - \int_0^t R(s)ds > 0$, a property which is critical.

Proposition 8.1 *Let $A(t)$ satisfy $(A1)-(A3)$, and let $p(t)$ be continuous, positive, and non-decreasing for $t \geq 0$. Then $R(t) > 0$ for all $t > 0$, $\int_0^T R(s)ds < 1$ for each finite $T > 0$, and $z(t) > 0$ for all $t \geq 0$.*

Proof. The assertion that $R(t) > 0$ is from Item f), above. It is clear from c) and d) that $\int_0^T R(s)ds < 1$ for each finite $T > 0$ since $R(t) > 0$ for all $t > 0$. The last assertion now follows from

$$z(t) = p(t) - \int_0^t R(t-s)p(s)ds \geq p(t)\left[1 - \int_0^t R(s)ds\right] > 0 \tag{8.3}$$

since $p(t)$ is positive and non-decreasing. This completes the proof.

Until we get to Item e), above, we know little about the behavior of $R(t)$. But with e) things change radically. $A(t)$ is monotone decreasing and so is $R(t)$. Moreover, it is true that $R(t) > 0$ if $A(t) \not\equiv 0$ so that the assertions of Proposition 8.1 will follow from Item e).

Proposition 8.2 *Let $A(t)$ be completely monotone on $(0, \infty)$, and let $p(t)$ be continuous, positive, and non-decreasing for $t \geq 0$. If $A(t) \not\equiv 0$, then $R(t) > 0$ for all $t > 0$, $\int_0^T R(s)ds < 1$ for each finite $T > 0$, and $z(t) > 0$ for all $t \geq 0$.*

The non-convolution case

The first step toward the non-convolution case is found in Miller [10, p. 217] where it is shown that if $A(t)$ satisfies (A1) - (A3) and if $B(s)$ is bounded, non-negative, and continuous with $\beta = \sup\{B(s) \colon 0 \leq s < \infty\}$, then the resolvent function $R(t,s)$ associated with the equation

$$R(t,s) = A(t-s)B(s) - \int_s^t A(t-u)B(u)R(u,s)du \tag{8.4}$$

exists, is measurable in (t,s) and satisfies

$$0 \leq R(t,s) \leq \beta A(t-s). \tag{8.5}$$

We want to establish a result that is parallel to that of Proposition 8.1. This requires a repetition of Proposition 8.1 and further analysis on $R(t,s)$.

Proposition 8.3 *Let $A(t)$ satisfy $(A1)-(A3)$, let $B(t)$ be bounded, continuous, and non-negative for $t \geq 0$, and let $p(t)$ be continuous, positive, and non-decreasing for $t \geq 0$. Then*

(i) $R(t,s) \geq 0$ *for* $t > s \geq 0$,

(ii) $\int_0^t R(t,s)ds < 1$ *for* $t \geq 0$,

(iii) $p(t) - \int_0^t R(t,s)p(s)ds > 0 \quad for\ t \geq 0.$

Proof. Note that (i) follows from (8.5) and the proof of (iii) is exactly the same as that for (8.3) with $R(t,s)$ in the place of $R(t-s)$ if (ii) holds. To prove (ii), we set $C(t,s) = A(t-s)B(s)$ and observe that

$$C(t,u) \leq C(v,u) \ \text{ if } \ u \leq v \leq t \tag{8.6}$$

since A is positive and non-increasing for $t > 0$ by (A2). It then follows from Theorem 8.7 of Gripenberg et al. [8, p. 263, lines 11-13 from the bottom] with $f(t) \equiv 1$ that the solution $Z(t)$ of

$$Z(t) = 1 - \int_0^t C(t,s)Z(s)ds$$

is positive yielding

$$Z(t) = 1 - \int_0^t R(t,s)ds > 0.$$

Thus, (ii) holds. The proof is complete. □

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Nonlinear Dynamics and Systems Theory, **17** (3) (2017) 266–278

A New Hyper Chaotic System and Study of Hybrid Projective Synchronization Behavior

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Abstract: In this paper, hybrid projective synchronization (HPS) of two identical new hyper chaotic systems is defined and scheme of HPS is developed by using tracking control method. A new hyper chaotic system has been constructed and then response system. Numerical simulations verify the effectiveness of this scheme, which has been performed by mathematica.

Keywords: *hybrid projective synchronization; chaotic systems and hyper chaos; tracking control method.*

Mathematics Subject Classification (2010): 34D06.

1 Introduction

Chaos is a dynamical regime in which a system becomes extremely sensitive to initial conditions and reveals an unpredictable and random-like behavior, even though the underlying model of a system exhibiting chaos can be deterministic and very simple. Small differences in initial conditions yield widely diverging outcomes for chaotic systems, rendering long term prediction impossible in general. Chaotic behavior can be observed in many natural phenomenon such as weather etc. Pecora and Carroll introduced a paper entitled *Synchronization in Chaotic Systems* in 1990. By that time, if there was a system challenging the capability of synchronizing that was a chaotic one. They demonstrated that chaotic synchronization could be achieved by driving or replacing one of the variables of a chaotic system with a variable of another similar chaotic device. Chaotic synchronization did not attract much attention until Pecora and Carroll [8] introduced a method to synchronize two identical chaotic systems

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with different initial conditions. From then on, enormous studies have been done by researchers on the synchronization of dynamical systems [5–7,26]. In the last two decades considerable research has been done in non-linear dynamical systems and their various properties. One of the most important properties is synchronization. Synchronization techniques have been improved in recent years and many different methods are applied theoretically as well as experimentally to synchronize the chaotic-systems including adaptive control [9–11], back stepping design [12–14], active control [15–17], nonlinear control [18,19] and observer based control method [20]. Using these methods, numerous synchronization problem of well-known chaotic systems such as Lorenz, Chen, Lü and Rössler system have been worked on by many researchers.

Also, several types of chaos synchronization are well known, which include complete synchronization (CS), antisynchronization (AS), phase synchronization, generalized synchronization (GS), projective synchronization (PS), and modified projective synchronization (MPS). Among all type of synchronization, Projective synchronization (PS) [21, 24, 25] has been extensively considered because it can obtain faster communication. The drive and response system could be synchronized up to a scaling factor in projective synchronization. In this continuation of study, in order to increase the degree of secrecy for secure communications, in hybrid projective synchronization same scaling factor can be replaced by vector function factor. In this paper, we have constructed a new hyper chaotic system and verified the chaotic behavior of this system by time series analysis and drawing chaotic attractors via mathematica. Hyperchaotic behavior of this system is discovered within some system parameter range, which has not yet been reported previously. Since hyperchaotic systems have the characteristics of high capacity, high security and high efficiency, it has been studied with increasing interest in recent years [23, 24] in the fields of non-linear circuits, secure communications, lasers, control, synchronization, and so on. So we have studied Hybrid Projective Synchronization behavior for this new hyper chaotic systems, which is ofcourse more effective and useful in secure communication as HPS is more useful in secure communication as compare to others because of its unpredictability. Here we have used tracking control scheme for HPS. Numerical simulations have been done by using Mathematica.

2 Preliminaries

In this section, we mention some definitions and scheme of the main work.

Definition 2.1 Hybrid Projective Synchronization(HPS) between two chaotic system achieved if there exist an $n \times n$ matrix A such that $\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|Ay - x\| = 0$, where $\|\cdot\|$ is the Euclidean norm.

2.1 Methodology for HPS

In this section, we put a glimpse of methodology and problem formulation for hybrid projective synchronization for identical hyperchaotic systems via tracking control. Consider the following n-dimensional hyperchaotic system as drive (master) system

$$\tfrac{dx}{dt} = f(x), \tag{2.1}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a differentiable function. Now construct the following identical response system

$$\frac{dy}{dt} = g(y) + \Psi(y, x), \tag{2.2}$$

where $y \in \mathbb{R}^n$ and $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a differentiable function and $\Psi(y, x)$ is vector controller to be designed via tracking control method.

In order to achieve the hybrid projective synchronization between two hyperchaotic systems, we choose the system (2.1) as a drive system and construct a response system as follows:

$$\frac{dy}{dt} = A^{-1}[f(Ay) + \Psi(y, x)], \tag{2.3}$$

where A^{-1} is the inverse matrix of the invertible matrix A and $y \in \mathbb{R}^n$ are state vector of the response system(2.2) and $\Psi(y, x)$ is controller which will be designed. Now define the HPS errors between two given systems (2.1)and (2.3) as

$$e(t) = Ay - x,$$

where e=$(e_1, e_2 ... e_n)^T$, and $A = \begin{pmatrix} a_{11} & a_{12} & ...a_{1n} \\ a_{21} & a_{22} & ...a_{2n} \\ . & . & . \\ a_{n1} & a_{n2} & ...a_{nn} \end{pmatrix}$.

So,

$$e_i = \left(\sum_{j=1}^{n} a_{ij} y_j \right) - x_i, (i, j = 1, 2, ...n). \tag{2.4}$$

Let

$$f(Ay) - f(x) = F(x, e). \tag{2.5}$$

Now, we assume that the error vectors e can be divided into $\overline{e_k}$=$(e_1, e_2 ... e_k)^T$ and $\overline{e_{k+1}}$=$(e_{k+1}, e_{k+2} ... e_n)^T$ such that $F(x, e)$ has the following form

$$F(x, e) = \begin{pmatrix} B_k \overline{e_k} + h_1(x, \overline{e_k}, \overline{e_{k+1}}) \\ B_{k+1} \overline{e_{k+1}} + h_{21}(x, \overline{e_k}, \overline{e_{k+1}}) + h_{22}(x, \overline{e_k}, \overline{e_{k+1}}) \end{pmatrix}, \tag{2.6}$$

where $h_1(x, \overline{e_k}, \overline{e_{k+1}}) \in \mathbb{R}^k$, $h_{21}(x, \overline{e_k}, \overline{e_{k+1}}) \in \mathbb{R}^{n-k}$, $h_{22}(x, \overline{e_k}, \overline{e_{k+1}}) \in \mathbb{R}^{n-k}$ and $\lim\limits_{\overline{e_k} \to 0} h_{21}(x, \overline{e_k}, \overline{e_{k+1}}) = 0$, respectively and $B_k \in \mathbb{R}^{k \times k}$, $B_{k+1} \in \mathbb{R}^{n-k \times n-k}$ are real constant matrix. Now, following theorem is based on the Lyapunov stability theory, which gives the final destination of the problem formulation.

Theorem 2.1 *If controller $\Psi(y, x)$ in response system (2.3) is*

$$\Psi(y, x) = \begin{pmatrix} \Psi_k(x, y) \\ \Psi_{k+1}(x, y) \end{pmatrix} = \begin{pmatrix} \Lambda_k \overline{e_k} - h_1(x, \overline{e_k}, \overline{e_{k+1}}) \\ \Lambda_{k+1} \overline{e_{k+1}} - h_{22}(x, \overline{e_k}, \overline{e_{k+1}}) \end{pmatrix}, \tag{2.7}$$

where $\Lambda_k \in \mathbb{R}^{k \times k}$ and $\Lambda_{k+1} \in \mathbb{R}^{n-k \times n-k}$ are suitable chosen constant matrices. If all eigenvalues of $B_k + \Lambda_k$ and $B_{k+1} + \Lambda_{k+1}$ have negative real parts, then hybrid projective synchronization between drive and response systems can be achieved.

3 System Description

3.1 Hyper chaotic Rabinovich-Fabrikant system

The Rabinovich-Fabrikant chaotic system is a set of three coupled ordinary differential equations exhibiting chaotic behavior for certain values of parameters. They are named after Mikhail Rabinovich and Anatoly Fabrikant, who described them in 1979 [22]. The equations of system are :

$$\left.\begin{aligned} \dot{x_1} &= x_2(x_3 - 1 + x_1^2) + \gamma x_1, \\ \dot{x_2} &= x_1(3x_3 + 1 - x_1^2) + \gamma x_2, \\ \dot{x_3} &= -2x_3(x_1 x_2 + \alpha), \end{aligned}\right\} \tag{3.1}$$

where α and γ are constant parameters that control the evolution of the system. For some values of α and γ, the system is chaotic but for other it tends to a stable periodic orbit. Now, we construct a new hyper chaotic system by introducing one more differential equation with a new parameter δ in the above system as follow:

$$\left.\begin{aligned} \dot{x_1} &= x_2(x_3 - 1 + x_1^2) + \gamma x_1, \\ \dot{x_2} &= x_1(3x_3 + 1 - x_1^2) + \gamma x_2, \\ \dot{x_3} &= -2x_3(x_1 x_2 + \alpha), \\ \dot{x_4} &= -3x_3(x_2 x_4 + \delta) + x_4^2. \end{aligned}\right\} \tag{3.2}$$

This new system shows hyper chaotic behavior with some values of parameters and tend to stable periodic orbits with other values of parameters. We have investigated system's behavior for different values of δ. Figures are given below:

4 Results and Discussions

In this section, we perform hybrid projective synchronization for hyper chaotic Rabinovich Fabrikant system. If we take this system as a drive system, then according to methodology, response system is

$$\tfrac{dy}{dt} = A^{-1}[f(Ay) + \Psi(y, x)], \tag{4.1}$$

which leads to response system as follows

$$\left(\frac{dy_1}{dt}, \frac{dy_2}{dt}, \frac{dy_3}{dt}, \frac{dy_4}{dt}\right)^T = A^{-1}[f(Ay) + \Psi(x, y)], \tag{4.2}$$

yields

$$A \begin{pmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \\ \frac{dy_3}{dt} \\ \frac{dy_4}{dt} \end{pmatrix} =$$

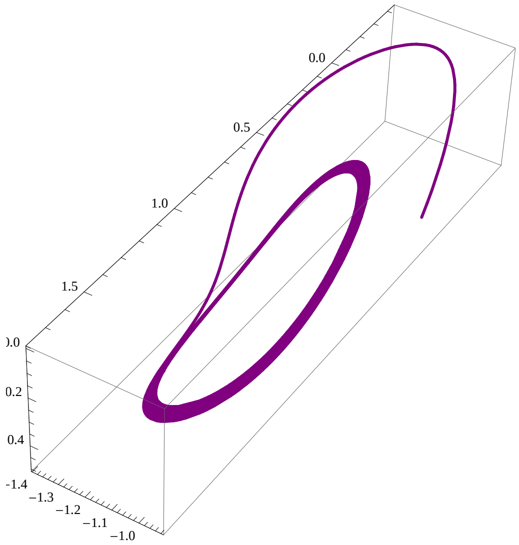

Figure 1: Chaotic behavior of the system (3.2) with $\alpha = 0.14, \gamma = 1.1$ and $-0.01 \leq \delta \leq 7650$ tending to stable periodic orbits.

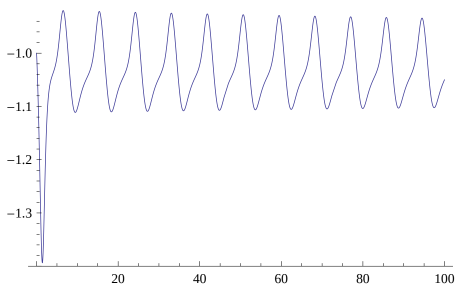

Figure 2: Time series analysis of x1[t] with $\alpha = 0.14, \gamma = 1.1$ and $-0.01 \leq \delta \leq 7650$.

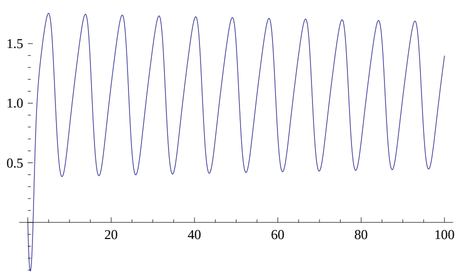

Figure 3: Time series analysis of x2[t] with $\alpha = 0.14, \gamma = 1.1$ and $-0.01 \leq \delta \leq 7650$.

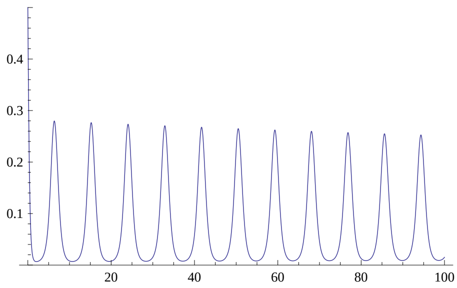

Figure 4: Time series analysis of x3[t] with $\alpha = 0.14$,$\gamma = 1.1$ and $-0.01 \leq \delta \leq 7650$.

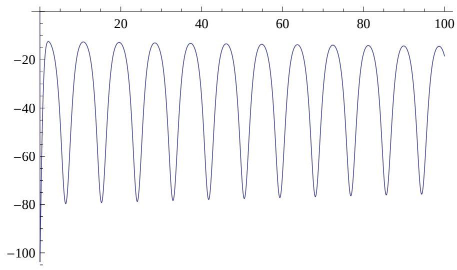

Figure 5: Time series analysis of x4[t] with $\alpha = 0.14$,$\gamma = 1.1$ and $-0.01 \leq \delta \leq 7650$.

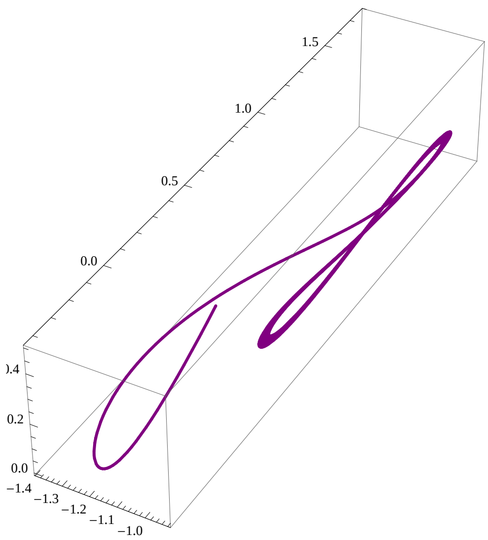

Figure 6: Chaotic Behavior of the system (3.2) with $\alpha = 0.87$,$\gamma = 1.1$ and $\delta = 1890$.

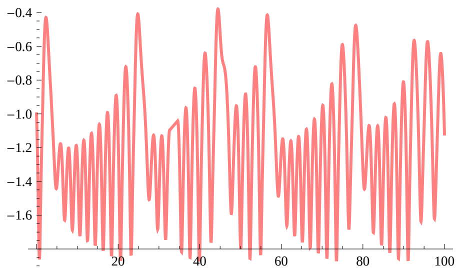

Figure 7: Time series analysis of x1[t] with $\alpha = 0.87$,$\gamma = 1.1$ and $\delta = 1890$.

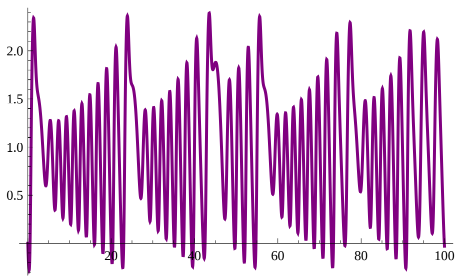

Figure 8: Time series analysis of x2[t] with $\alpha = 0.87$,$\gamma = 1.1$ and $\delta = 1890$.

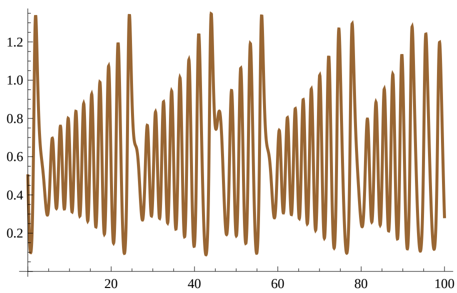

Figure 9: Time series analysis of x3[t] with $\alpha = 0.87$,$\gamma = 1.1$ and $\delta = 1890$.

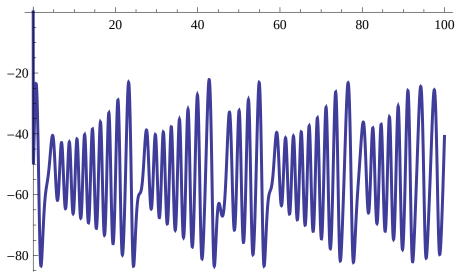

Figure 10: Time series analysis of x4[t] with $\alpha = 0.87$,$\gamma = 1.1$ and $\delta = 1890$.

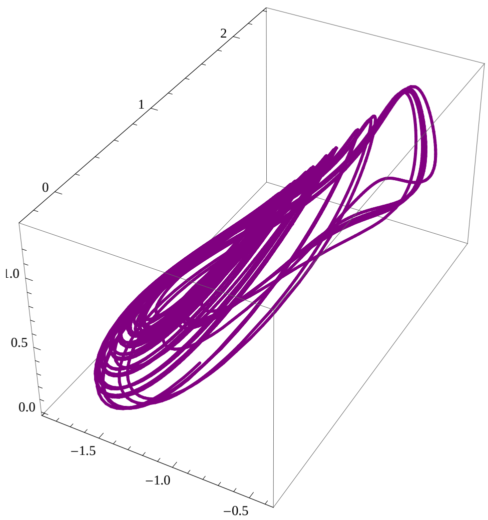

Figure 11: Chaotic Behavior of the system (3.2) with $\alpha = 0.87, \gamma = 1.1$ and $\delta = -0.2$.

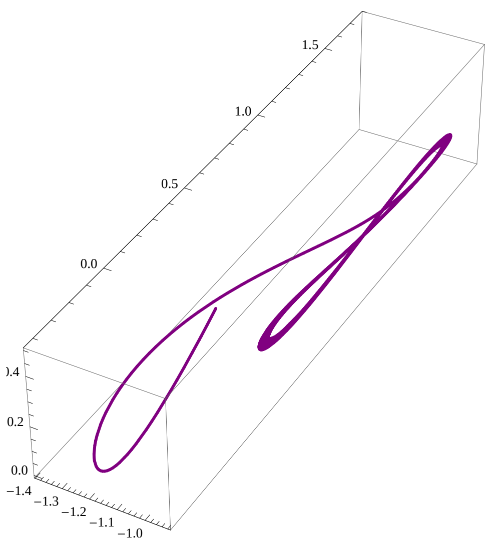

Figure 12: Time series analysis of x1[t] with $\alpha = 0.87, \gamma = 1.1$ and $\delta = -0.2$.

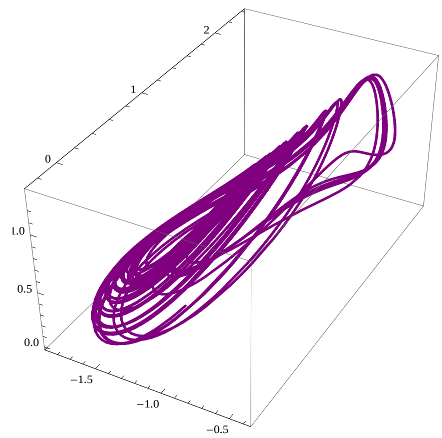

Figure 13: Time series analysis of x2[t] with $\alpha = 0.87$,$\gamma = 1.1$ and $\delta = -0.2$.

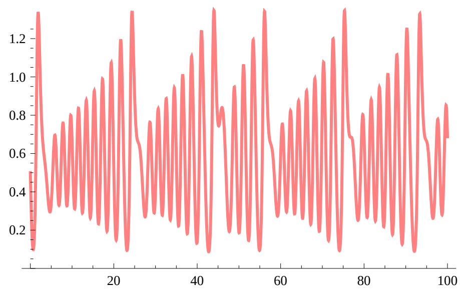

Figure 14: Time series analysis of x3[t] with $\alpha = 0.87$,$\gamma = 1.1$ and $\delta = -0.2$.

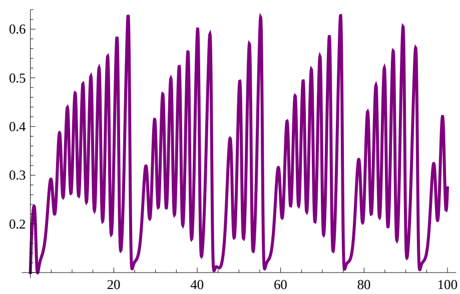

Figure 15: Time series analysis of x4[t] with $\alpha = 0.87$,$\gamma = 1.1$ and $\delta = -0.2$.

$$\begin{pmatrix} \sum_{j=1}^4 a_{2j}y_j \sum_{j=1}^4 a_{3j}y_j - \sum_{j=1}^4 a_{2j}y_j + \sum_{j=1}^4 a_{2j}y_j \sum_{j=1}^4 a_{1j}^2 y_j^2 + \gamma \sum_{j=1}^4 a_{1j}y_j \\ 3\sum_{j=1}^4 a_{1j}y_j \sum_{j=1}^4 a_{3j}y_j + \sum_{j=1}^4 a_{1j}y_j - \sum_{j=1}^4 a_{1j}^3 y_j^3 + \gamma \sum_{j=1}^4 a_{2j}y_j \\ -2\sum_{j=1}^4 a_{3j}y_j \sum_{j=1}^4 a_{2j}y_j \sum_{j=1}^4 a_{3j}y_j - 2\alpha \sum_{j=1}^4 a_{3j}y_j \\ -3\sum_{j=1}^4 a_{3j}y_j \sum_{j=1}^4 a_{2j}y_j \sum_{j=1}^4 a_{4j}y_j - 3\delta \sum_{j=1}^4 a_{3j}y_j + \sum_{j=1}^4 a_{4j}^2 y_j^2 \end{pmatrix} + \Psi(x,y). \tag{4.3}$$

Now, according to definition of HPS error dynamics we have,

$$\frac{de}{dt} = A\frac{dy}{dt} - \frac{dx}{dt} = f(Ay) - f(x) + \Psi(x,y). \tag{4.4}$$

Let

$$f(Ay) - f(x) = F(x,e). \tag{4.5}$$

From equation (4.4) and (4.5), we have following

$$\frac{de}{dt} = F(x,e) + \Psi(x,y). \tag{4.6}$$

Our goal is to find $F(x,e)$ and to design controller $\Psi(x,y)$ to achieve the HPS.

Equation (4.5) gives $F(x,e) =$

$$\begin{pmatrix} \sum_{j=1}^4 a_{2j}y_j \sum_{j=1}^4 a_{3j}y_j - \sum_{j=1}^4 a_{2j}y_j + \sum_{j=1}^4 a_{2j}y_j \sum_{j=1}^4 a_{1j}^2 y_j^2 + \gamma \sum_{j=1}^4 a_{1j}y_j \\ 3\sum_{j=1}^4 a_{1j}y_j \sum_{j=1}^4 a_{3j}y_j + \sum_{j=1}^4 a_{1j}y_j - \sum_{j=1}^4 a_{1j}^3 y_j^3 + \gamma \sum_{j=1}^4 a_{2j}y_j \\ -2\sum_{j=1}^4 a_{3j}y_j \sum_{j=1}^4 a_{2j}y_j \sum_{j=1}^4 a_{3j}y_j - 2\alpha \sum_{j=1}^4 a_{3j}y_j \\ -3\sum_{j=1}^4 a_{3j}y_j \sum_{j=1}^4 a_{2j}y_j \sum_{j=1}^4 a_{4j}y_j - 3\delta \sum_{j=1}^4 a_{3j}y_j + \sum_{j=1}^4 a_{4j}^2 y_j^2 \end{pmatrix}$$

$$- \begin{pmatrix} x_2x_3 - x_2 + x_2x_1^2 + \gamma x_1 \\ 3x_1x_3 + x_1 - x_1^3 + \gamma x_2 \\ -2x_1x_2x_3 - 2\alpha x_3 \\ -3x_2x_3x_4 - 3\delta x_3 + x_4^2 \end{pmatrix}$$

which yields, $F(x,e)$

$$= \begin{pmatrix} e_2e_3 + e_2e_1^2 + e_2x_3 + e_3x_2 - e_2 + e_2x_1^2 + 2e_1e_2x_1 + x_2e_1^2 + 2e_1x_1x_2 + \gamma e_1 \\ e_1 - e_1^3 + 3e_1e_3 + 3x_3e_1 + 3x_1e_3 - 3x_1^2e_1 - 3x_1e_1^2 + \gamma e_2 \\ -2x_1x_3e_2 - 2x_2x_3e_1 - 2x_3e_1e_2 - 2x_1x_2e_3 - 2x_1e_2e_3 - 2e_1e_3x_2 - 2e_1e_2e_3 - 2e_3\alpha \\ -3x_4x_3e_2 - 3x_2x_3e_4 - 3x_3e_2e_4 - 3x_2x_4e_3 - 3x_2e_4e_3 - 3e_2e_3x_4 - 3e_2e_3e_4 - 3e_3\delta + e_4^2 + 2e_4x_4 \end{pmatrix}.$$

So, after putting all above values, we have

$$F(x,e) = \begin{pmatrix} B_1\overline{e_1} + h_1(x,\overline{e_1},\overline{e_2}) \\ B_2\overline{e_2} + h_{21}(x,\overline{e_1},\overline{e_2}) + h_{22}(x,\overline{e_1},\overline{e_2}) \end{pmatrix}. \tag{4.7}$$

Obviously, $\lim_{e_1 \to 0} h_{21}(x, \overline{e_1}, \overline{e_2}) = 0$. Now, according to theorem (2.1), we define feedback controller $\Psi(x, y)$ as,

$$\Psi(y,x) = \begin{pmatrix} \Psi_1(x,y) \\ \Psi_2(x,y) \end{pmatrix} = \begin{pmatrix} \Lambda_1 \overline{e_1} - h_1(x, \overline{e_1}, \overline{e_2}) \\ \Lambda_2 \overline{e_2} - h_{22}(x, \overline{e_2}, \overline{e_2}) \end{pmatrix}. \tag{4.8}$$

So from equations (4.7) and (4.8) error dynamical system (4.6) can be rewritten as,

$$\left. \begin{array}{l} \frac{d\overline{e_1}}{dt} = (B_1 + \Lambda_1)\overline{e_1}, \\ \frac{d\overline{e_2}}{dt} = (B_2 + \Lambda_2)\overline{e_2} + h_{21}(x, \overline{e_1}, \overline{e_2}). \end{array} \right\} \tag{4.9}$$

So we choose now suitable $B_1 + \Lambda_1 \in \mathbb{R}^1$ and $B_2 + \Lambda_2 \in \mathbb{R}^{3\times 3}$, for which eigen values are negative. As Eq.(4.9) is asymptotically stable with equilibrium point $e_1 = 0$ and $\overline{e_2} = 0$. Obviously $\lim_{t\to\infty} \|e_1\| = 0$ and $\lim_{e_1 \to 0} h_{21}(x, \overline{e_1}, \overline{e_2}) = 0$, then the hybrid projective synchronization between response system and master system can be achieved.

5 Numerical Simulations

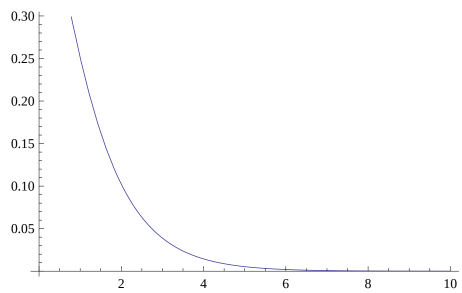

Figure 16: Convergence of error e_1, $t \in [0, 10]$.

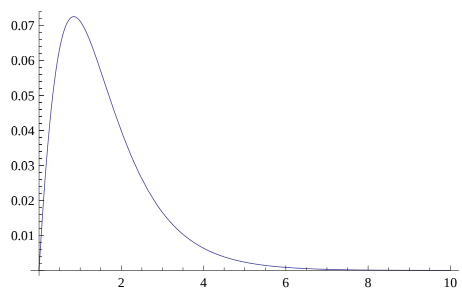

Figure 17: Convergence error of e_2, $t \in [0, 10]$.

Parameters of the system are $-0.01 \leq \delta \leq 7650$ with $\alpha = 0.14$, $\gamma = 1.1$ and $-0.2 \leq \delta \leq 1890$ with $\alpha = 0.87$,$\gamma = 1.1$ for which the systems are chaotic. In (4.9),

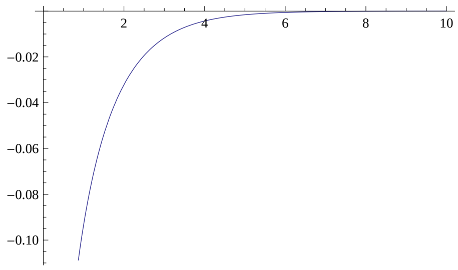

Figure 18: Convergence of error e_3, $t \in [0, 10]$.

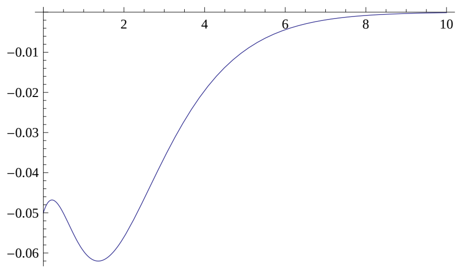

Figure 19: Convergence of error e_4, $t \in [0, 10]$.

we have chosen $\Lambda_1 = (-2)$ and $\Lambda_2 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 3 & -1 \end{pmatrix}$, which leads to stability conditions as eigenvalues of $B_1 + \Lambda_1$ and $B_2 + \Lambda_2$ are negative. The initial conditions for master and slave systems $[x_1(0), x_2(0), x_3(0), x_4(0)] = [8, 3, 1, 4]$ and $[y_(0), y_2(0), y_3(0), y_4(0)] = [0.1, 0.41, 0.31, 0.51]$, respectively, and $A = \begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & -2 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 1 \end{pmatrix}$. Then for $[e_1(0), e_2(0), e_3(0), e_4(0)] = [-8.41, -4.03, -0.51, -2.15]$ diagrams of convergence of errors are the witness of achieving hybrid projective synchronization between master and slave system.

6 Conclusion

In this paper, we have investigated hybrid projective synchronization behavior of a new hyper chaotic Rabinovich-Fabrikant system. The results are validated by numerical simulations using mathematica. It has more advantage over other synchronization to enhance security of communication as hybrid projective synchronization is more unpredictable and moreover it is performed for hyperchaotic system, which makes it more useful.

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Different Types of Synchronization Between Different Fractional Order Chaotic Systems

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Abstract: In this paper complete synchronization, anti-synchronization and projective synchronization are achieved between two different fractional order chaotic systems, fractional order Lotka Volterra system and fractional order Lu system, via active control method. Numerical simulations have been done in Matlab by using Grunwald Letnikov method. Numerical results demonstrate the effectiveness and feasibility of the proposed control techniques.

Keywords: *synchronization; anti-synchronization; projective synchronization; fractional order chaotic systems; active control*

Mathematics Subject Classification (2010): 37B25, 37D45, 37N35, 70K99.

1 Introduction

A chaotic dynamical system is defined as the system which satisfies the properties of boundedness, infinite recurrence and sensitive dependence on initial conditions [2]. Chaos theory investigates the unstable behavior in deterministic nonlinear dynamical systems which cause 'chaos'. Sometimes chaotic behavior of a dynamical system is found useful like in secure communications [21, 37]. First time in 1963, Lorenz discovered a three dimensional chaotic system while studying weather model for atmospheric convection. After a decade, Rossler discovered a three dimensional chaotic system, which was constructed during the study of a chemical reaction. Synchronization is an important and famous phenomenon which can be understood within the unifying framework of the nonlinear sciences. Due to its potential applications in the field of nonlinear dynamics it has been hot topic of research. Since the pioneer work of Pecora and Carroll [26] it has been

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an active area for researchers in applied sciences. Synchronization in the language of the nonlinear dynamics is defined as an adjustment of rhythms of oscillating objects due to their weak interaction. So many types of synchronization have been achieved: generalized synchronization [3], phase synchronization and anti-phase synchronization [4, 9, 34], lag synchronization [5], Q-S synchronization [24], etc. So many techniques have been developed for synchronization like adaptive control [14], feedback control [22], fuzzy control [7], nonlinear control [25], active backstepping [23], adaptive sliding mode control [38] etc. Chaos synchronization has so many applications in different fields like power systems [18], physics [39], chemistry [19], medicine [20], diffusion [12] etc.

Recently, fractional differential equations have been used to study different dynamical systems and chaos have been analyzed in different fractional order systems. So many numerical methods have been developed for the solution of fractional differential equation [29–31]. Also, fractional calculus is a 300 years old subject which can be traced back to Leibniz, Riemann, Grunwald and Letnikov. So many systems have been found in real life which can be represented more accurately by fractional order systems. But for the last few decades fractional calculus with chaos has been an attractive field and so many works have been done on synchronization and control of chaos in fractional order systems like Lorenz system [10], Rossler system [17], Volta System [28], Chua system [11], Chen system [16] etc. Recently so many new chaotic and hyperchaotic systems [8, 27, 32, 33, 35] have also been developed and analyzed by the researchers.

Synchronization has so many applications in which secure communication is very important. Synchronization between integer order and fractional order system via tracking control [13] and sliding mode control [6], synchronization of fractional order systems with different dimensions [36] and hybrid projective synchronization [15] of fractional order chaotic systems between order (1,2) have also been obtained in recent years. The aim of this paper is to achieve different synchronizations between different fractional order systems which is important for secure communication. Amongst different types of chaos synchronization, projective synchronization has been found to be more secure because of its unpredictable scaling factor and this is why it has received so much attention in the last few years.

In this paper we achieve three types of synchronization between two chaotic systems, fractional order Lotka Volterra system (master system) and fractional order Lu system (slave system) via active control. The achieved synchronizations are complete synchronization, anti-synchronization and projective synchronization. For numerical simulations Matlab software has been used and to solve fractional differential equation Grunwald Letnikov method has been used.

2 Fractional Order Derivatives [29]

Fractional calculus generalizes differentiation and integration to non integer order fundamental operator ${}_aD_t^\alpha$ where a and t are the limits of the operator and α is the non integer order. This operator is defined as

$$
{}_aD_t^\alpha = \begin{cases} \frac{d^\alpha}{dt^\alpha}, & : \alpha > 0 \\ 0, & : \alpha = 0 \\ \int\limits_a^t (d\tau)^\alpha, & : \alpha < 0. \end{cases}
$$

The most known three definitions for fractional integro differential operator are Riemann-Liouville definition, Grunwald-Letnikov definition and Caputo's definition.

The Riemann-Liouville definition is given as

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int\limits_a^t (t-\tau)^{n-\alpha-1} f(\tau) d(\tau), \quad n-1 < \alpha < n. $$

The Caputo's fractional derivative is defined as

$$ {}_0D_t^\alpha = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^n(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \quad : n-1 < \alpha < n, $$

where $\Gamma(.)$ is the Gamma function, and the Grunwald Letnikov definition is

$$ {}_aD_t^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{j=0}^{[\frac{(t-\alpha)}{h}]} (-1)^j \left(\begin{array}{c} \alpha \\ j \end{array} \right) f(t-jh), $$

where [.] denotes the greatest integer part.

3 Stability of Fractional Order System [13]

Theorem: *We consider the following linear fractional order system*

$$ D^\alpha x = Ax, \quad x(0) = x_0. \tag{1} $$

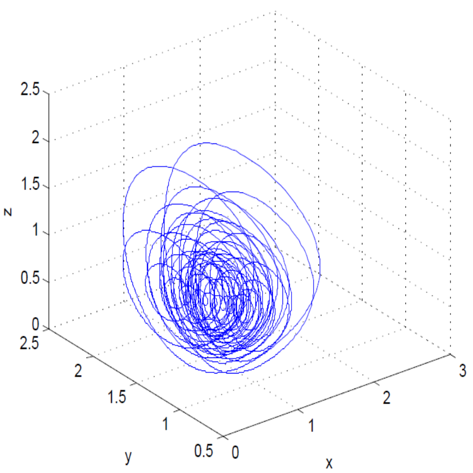
Here, $A \in R^{n\times n}$, and $\alpha = (\alpha_1, \alpha_2, ., \alpha_n), (0 < \alpha_i \leq 1)$. System (1) is asymptotically stable if and only if $\mid arg(\lambda_i) \mid > \alpha\pi/2$ is satisfied for all eigenvalues λ_i of the matrix A. Furthermore, this system is stable if and only if $|arg(\lambda_i)| \geq \alpha\pi/2$ is satisfied for all eigenvalues λ_i of the matrix A and those critical eigenvalues that satisfy the condition $|arg(\lambda_i)| = \alpha\pi/2$ and have geometric multiplicity one. The geometric multiplicity of an eigenvalue is defined as the dimension of the associated eigenspace, i.e., the number of linearly independent eigenvectors with that eigenvalue.

Master and Slave systems. The fractional order Lotka-Volterra system [1] is considered as master system

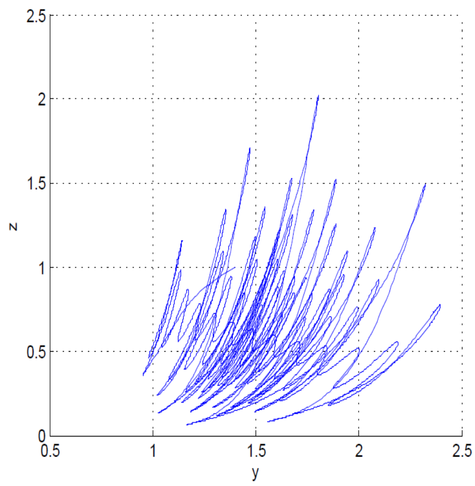
$$ \begin{aligned} \frac{d^{q_1}x_1}{dt^{q_1}} &= ax_1 - bx_1y_1 + ex_1^2 - sz_1x_1^2, \\ \frac{d^{q_2}y_1}{dt^{q_2}} &= dx_1y_1 - cy_1, \\ \frac{d^{q_3}z_1}{dt^{q_3}} &= sz_1x_1^2 - pz_1. \end{aligned} \tag{2} $$

This system exhibits chaotic behavior for parameter values $a = b = c = d = 1, e = 2, p = 3, s = 2.7$ and order $q_1 = q_2 = q_3 = 0.95$ for these values behavior of the system (2) is shown in Figure 1. Consider the following fractional order Lu chaotic system [29] as slave system

$$ \begin{aligned} \frac{d^{q_1}x_2}{dt^{q_1}} &= \alpha(y_2 - x_2), \\ \frac{d^{q_2}y_2}{dt^{q_2}} &= \gamma y_2 - x_2z_2, \\ \frac{d^{q_3}z_2}{dt^{q_3}} &= x_2y_2 - \beta z_2, \end{aligned} \tag{3} $$



(a)



(b)

Figure 1: (a) Chaotic attractor of Lotka Volterra system in $x - y - z$ space for order $\alpha = 0.95$. (b) $y - z$ view of the Lotka-Volterra system for order $\alpha = 0.95$.

which exhibits chaotic behavior for parameters $\alpha = 36, \beta = 3, \gamma = 20$, and order $q_1 = q_2 = q_3 = 0.95$, as shown in Figure 2.

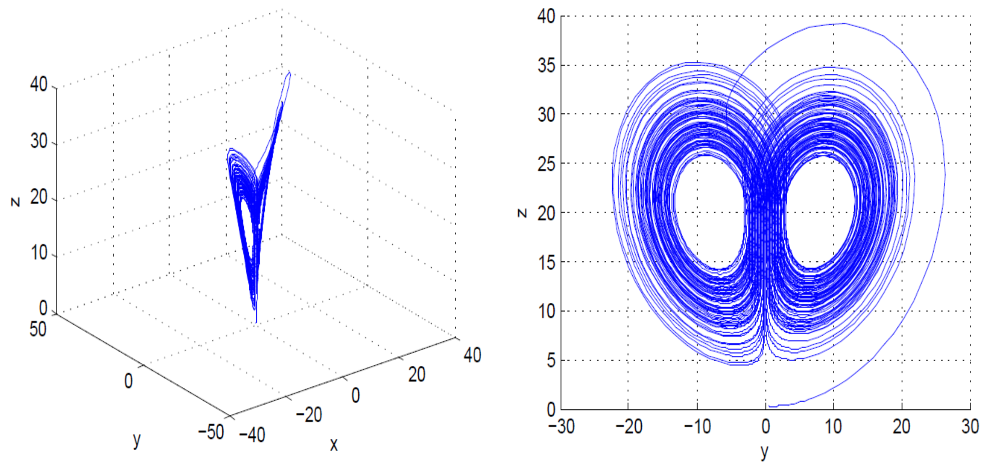
4 Synchronization Methodology and Numerical Simulations

To achieve different synchronizations between the considered two chaotic systems via active control method the error is defined as $e = y - \chi x$. For complete synchronization we take $\chi = 1$, for anti-synchronization $\chi = -1$, for projective synchronization arbitrary value of χ may be chosen. In this paper we took $\chi = 2$. Our aim is to design an effective controller $u(t)$ so that error e converges to zero. The master system is described by

$$\begin{aligned} \frac{d^{q_1}x_1}{dt^{q_1}} &= ax_1 - bx_1y_1 + ex_1^2 - sz_1x_1^2, \\ \frac{d^{q_2}y_1}{dt^{q_2}} &= dx_1y_1 - cy_1, \\ \frac{d^{q_3}z_1}{dt^{q_3}} &= sz_1x_1^2 - pz_1. \end{aligned} \tag{4}$$

The slave system with controllers is described by

$$\begin{aligned} \frac{d^{q_1}x_2}{dt^{q_1}} &= \alpha(y_2 - x_2) + u_1(t), \\ \frac{d^{q_2}y_2}{dt^{q_2}} &= \gamma y_2 - x_2z_2 + u_2(t), \\ \frac{d^{q_3}z_2}{dt^{q_3}} &= x_2y_2 - \beta z_2 + u_3(t), \end{aligned} \tag{5}$$



(a) (b)

Figure 2: (a) Chaotic attractor of Lu system in $x - y - z$ space for $\alpha = 0.95$. (b) $y - z$ view of the Lu system for order $\alpha = 0.95$.

where $u_1(t), u_2(t), u_3(t)$ are three control functions. For complete synchronization, the error dynamical system is

$$\begin{aligned}
\frac{d^{q_1}e_1}{dt^{q_1}} &= \alpha(e_2 - e_1) - (\alpha + a)x_1 + \alpha y_1 + bx_1y_1 - ex_1^2 + sz_1x_1^2 + u_1(t),\\
\frac{d^{q_2}e_2}{dt^{q_2}} &= \gamma e_2 - dx_1y_1 + (\gamma + c)y_1 - x_2z_2 + u_2(t),\\
\frac{d^{q_3}e_3}{dt^{q_3}} &= -\beta e_3 + x_2y_2 - sz_1x_1^2 + (p - \beta)z_1 + u_3(t).
\end{aligned} \tag{6}$$

Define the active control functions $u_1(t), u_2(t), u_3(t)$ as

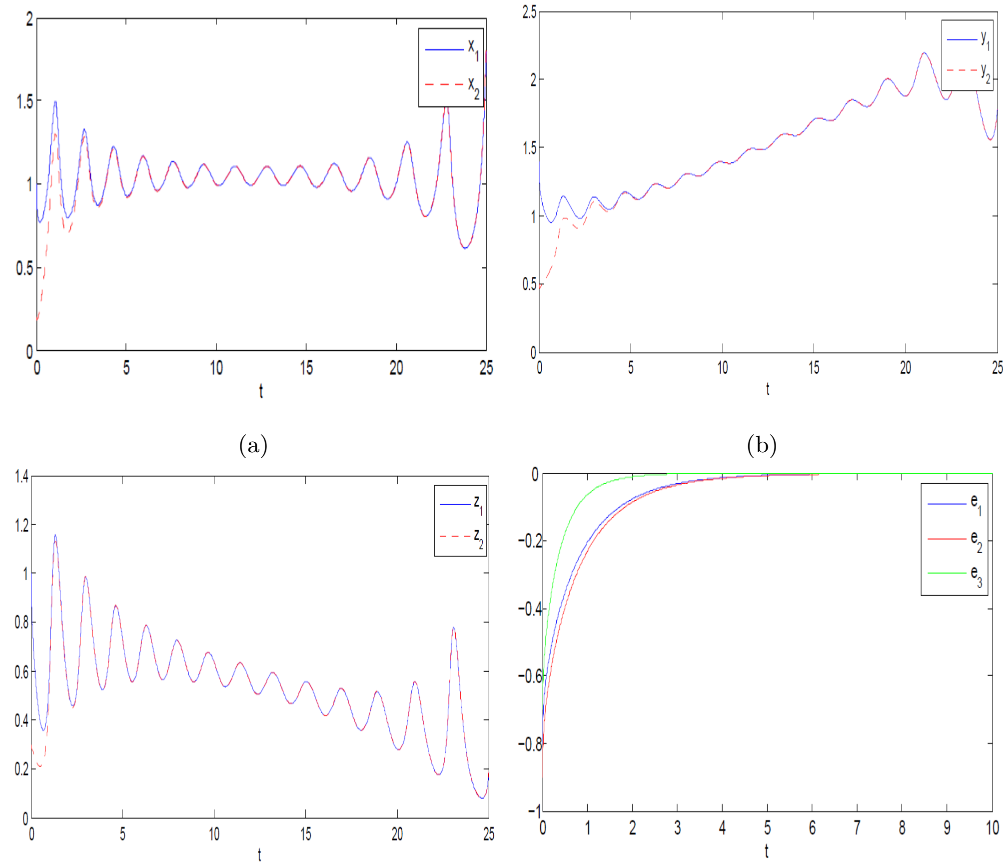
$$\begin{aligned}
u_1(t) &= (\alpha + a)x_1 - \alpha y_1 - bx_1y_1 + ex_1^2 - sz_1x_1^2 + v_1(t),\\
u_2(t) &= dx_1y_1 - (\gamma + c)y_1 + x_2z_2 + v_2(t),\\
u_3(t) &= -x_2y_2 + sz_1x_1^2 - (p - \beta)z_1 + v_3(t).
\end{aligned} \tag{7}$$

With these controllers the error dynamical system becomes

$$\begin{aligned}
\frac{d^{q_1}e_1}{dt^{q_1}} &= \alpha(e_2 - e_1) + v_1(t),\\
\frac{d^{q_2}e_2}{dt^{q_2}} &= \gamma e_2 + v_2(t),\\
\frac{d^{q_3}e_3}{dt^{q_3}} &= -\beta e_3 + v_3(t).
\end{aligned} \tag{8}$$

Define the suitable control inputs $v_1(t), v_2(t), v_3(t)$ to obtain the stable error dynamical system. Choosing $v_1(t), v_2(t), v_3(t)$ as

$$\begin{pmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{pmatrix} = C \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{9}$$



(a) (b)

(c) (d)

Figure 3: (a) Complete synchronization between state variables x_1, x_2. (b) Complete synchronization between state variables y_1, y_2. (c) Complete synchronization between state variables z_1, z_2. (d) Complete synchronization errors e_1, e_2, e_3 converging to zero.

where C is a 3×3 constant matrix, we choose C as

$$\begin{pmatrix} (\alpha - 1) & -\alpha & 0 \\ 0 & -(\gamma + 1) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the error dynamical system is

$$\begin{aligned} \frac{d^{q_1} e_1}{dt^{q_1}} &= -e_1, \\ \frac{d^{q_2} e_2}{dt^{q_2}} &= -e_2, \\ \frac{d^{q_3} e_3}{dt^{q_3}} &= -2e_3. \end{aligned} \tag{10}$$

Eigenvalues for this error dynamical system are $-1, -1, -2$ which satisfies stability condition. Figures 3(a),(b),(c) show complete synchronization between state variables of master system and slave system, and 3(d) shows errors of complete synchronization converging to zero at the initial conditions $(-0.8, -0.9, -0.7)$. For anti-synchronization, we choose $\chi = -1$, and the error dynamical system for the same master and slave systems becomes

$$\begin{aligned}
\frac{d^{q_1}e_1}{dt^{q_1}} &= \alpha(e_2 - e_1) + (\alpha + a)x_1 - \alpha y_1 - bx_1y_1 + ex_1^2 - sz_1x_1^2 + u_1(t),\\
\frac{d^{q_2}e_2}{dt^{q_2}} &= \gamma e_2 + dx_1y_1 - (\gamma + c)y_1 - x_2z_2 + u_2(t),\\
\frac{d^{q_3}e_3}{dt^{q_3}} &= -\beta e_3 + x_2y_2 + sz_1x_1^2 - (p - \beta)z_1 + u_3(t).
\end{aligned} \tag{11}$$

The active control functions $u_1(t), u_2(t), u_3(t)$ are defined as

$$\begin{aligned}
u_1(t) &= -(\alpha + a)x_1 + \alpha y_1 + bx_1y_1 - ex_1^2 + sz_1x_1^2 + v_1(t),\\
u_2(t) &= -dx_1y_1 + (\gamma + c)y_1 + x_2z_2 + v_2(t),\\
u_3(t) &= -x_2y_2 - sz_1x_1^2 + (p - \beta)z_1 + v_3(t),
\end{aligned} \tag{12}$$

the error dynamical system becomes

$$\begin{aligned}
\frac{d^{q_1}e_1}{dt^{q_1}} &= \alpha(e_2 - e_1) + v_1(t),\\
\frac{d^{q_2}e_2}{dt^{q_2}} &= \gamma e_2 + v_2(t),\\
\frac{d^{q_3}e_3}{dt^{q_3}} &= -\beta e_3 + v_3(t).
\end{aligned} \tag{13}$$

Now we have to choose suitable control inputs $v_1(t), v_2(t), v_3(t)$ so that the stable error dynamical system could be obtained. Choosing $v_1(t), v_2(t), v_3(t)$ as

$$\begin{pmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{pmatrix} = C \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix}, \tag{14}$$

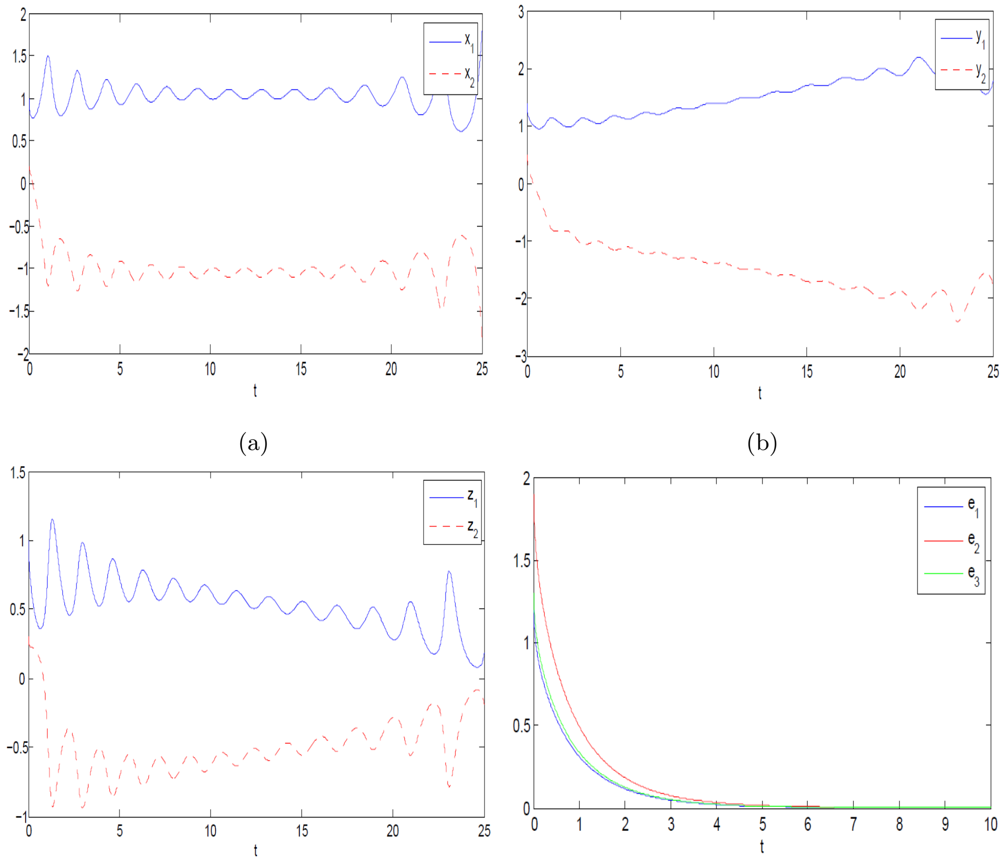
where C is a constant matrix and C is

$$\begin{pmatrix} (\alpha - 1) & -\alpha & 0 \\ 0 & -(\gamma + 1) & 0 \\ 0 & 0 & (\beta - 1) \end{pmatrix}.$$

Then the error dynamical system becomes

$$\begin{aligned}
\frac{d^{q_1}e_1}{dt^{q_1}} &= -e_1,\\
\frac{d^{q_2}e_2}{dt^{q_2}} &= -e_2,\\
\frac{d^{q_3}e_3}{dt^{q_3}} &= -e_3.
\end{aligned} \tag{15}$$

Eigenvalues for this error dynamical system are $-1, -1, -1$ which satisfies stability condition and initial conditions are $(1.2, 1.9, 1.3)$.



(a) (b) (c) (d)

Figure 4: (a) Anti-synchronization between state variables x_1, x_2. (b) Anti-synchronization between state variables y_1, y_2. (c) Anti-synchronization between state variables z_1, z_2. (d) Anti-synchronization errors e_1, e_2, e_3 converging to zero.

Figure 4(a),(b) shows anti-synchronization between state variables x_1, x_2 and y_1, y_2, and Figure 4(c),(d) exhibits anti-synchronization between state variables z_1, z_2 and anti-synchronization errors tending to zero.

For projective synchronization the choice of the scalar χ is arbitrary, but we choose $\chi = 2$ and the error dynamical system with the same master and slave system becomes

$$\begin{aligned} \frac{d^{q_1}e_1}{dt^{q_1}} &= \alpha(e_2 - e_1) - 2(\alpha + a)x_1 + 2\alpha y_1 + 2bx_1y_1 - 2ex_1^2 + 2sz_1x_1^2 + u_1(t), \\ \frac{d^{q_2}e_2}{dt^{q_2}} &= \gamma e_2 - 2dx_1y_1 + 2(\gamma + c)y_1 - x_2z_2 + u_2(t), \\ \frac{d^{q_3}e_3}{dt^{q_3}} &= -\beta e_3 + x_2y_2 - 2sz_1x_1^2 + u_3(t). \end{aligned} \tag{16}$$

The active control functions $u_1(t), u_2(t), u_3(t)$ are defined as

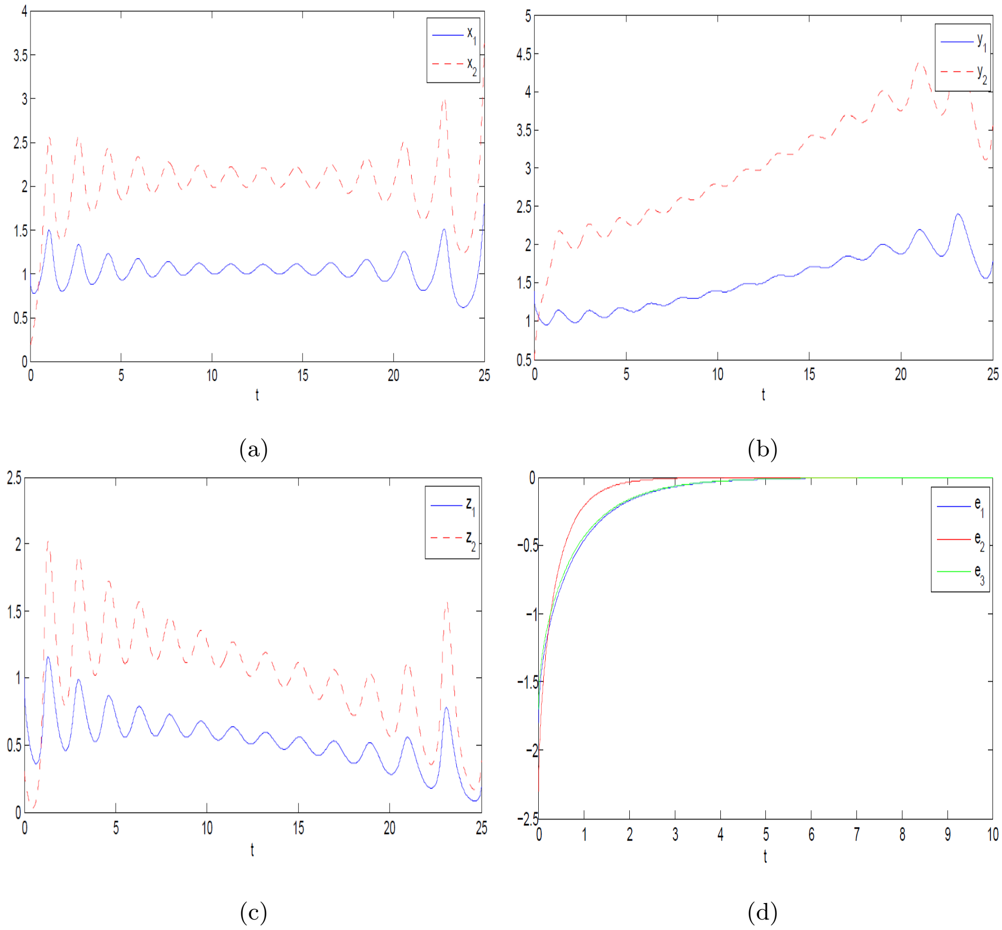


Figure 5: (a) Projective synchronization for state variables x_1, x_2. (b) Projective synchronization for state variables y_1, y_2. (c) Projective synchronization for state variables z_1, z_2. (d) Projective synchronization errors e_1, e_2, e_3 converging to zero.

$$\begin{aligned} u_1(t) &= 2(\alpha + a)x_1 - 2\alpha y_1 - 2bx_1y_1 + 2ex_1^2 - 2sz_1x_1^2 + v_1(t), \\ u_2(t) &= 2dx_1y_1 - 2(\gamma + c)y_1 + x_2z_2 + v_2(t), \\ u_3(t) &= -x_2y_2 + 2sz_1x_1^2 + v_3(t). \end{aligned} \tag{17}$$

Then the error dynamical system becomes

$$\begin{aligned} \frac{d^{q_1}e_1}{dt^{q_1}} &= \alpha(e_2 - e_1) + v_1(t), \\ \frac{d^{q_2}e_2}{dt^{q_2}} &= \gamma e_2 + v_2(t), \\ \frac{d^{q_3}e_3}{dt^{q_3}} &= -\beta e_3 + v_3(t). \end{aligned} \tag{18}$$

Now we have to choose suitable control inputs $v_1(t), v_2(t), v_3(t)$ so that the stable error dynamical system could be obtained. We choose $v_1(t), v_2(t), v_3(t)$ as follows

$$\begin{pmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{pmatrix} = C \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{19}$$

where C is a 3×3 constant matrix

$$\begin{pmatrix} (\alpha - 1) & -\alpha & 0 \\ 0 & -(\gamma + 2) & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then the error dynamical system becomes

$$\begin{aligned} \frac{d^{q_1} e_1}{dt^{q_1}} &= -e_1, \\ \frac{d^{q_2} e_2}{dt^{q_2}} &= -2e_2, \\ \frac{d^{q_3} e_3}{dt^{q_3}} &= -e_3. \end{aligned} \tag{20}$$

Eigenvalues for this error dynamical system are $-1, -2, -1$ which satisfies stability condition and initial conditions for this error dynamical system are $(-1.8, -2.3, -1.7)$. Figure 5 (a),(b),(c) shows projective synchronization between state variables of master system and slave system and Figure 5(d) exhibits projective synchronization errors converging to zero .

5 Conclusion

Different types of synchronizations have been achieved between two different fractional order chaotic systems Lotka-Volterra system and Lu system by using active control method. The method is easy to apply. The results are validated by numerical simulations using Matlab. Numerical simulations are a witness to achievement of desired goal between these two systems. The analytical and computational results are in an excellent agreement.

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Controllability of Neutral Functional Differential Equations Driven by Fractional Brownian Motion with Infinite Delay

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Abstract: This paper is concerned with the controllability of neutral stochastic functional differential equations with infinite delay driven by fractional Brownian motion in a real separable Hilbert space. The controllability results are obtained using stochastic analysis and a fixed-point strategy.

Keywords: *Controllability; neutral functional differential equations; fractional powers of closed operators; infinite delay; fractional Brownian motion.*

Mathematics Subject Classification (2010): 35R10, 93B05, 60G22, 60H20.

1 Introduction

For the practical applications in the areas such as biology, medicine, physics, finance, electrical engineering, telecommunication networks, and so on, the theory of stochastic evolution equations has attracted research's great interest. For more details, one can see Da Prato and Zabczyk [5], and Ren and Sun [14] and the references therein. In many areas of science, there has been an increasing interest in the investigation of the systems incorporating memory or aftereffect, i.e., there is the effect of delay on state equations. Therefore, there is a real need to discuss stochastic evolution systems with delay. In many mathematical models the claims often display long-range memories, possibly due to extreme weather, natural disasters, in some cases, many stochastic dynamical systems depend not only on present and past states, but also contain the derivatives with delays. Neutral functional differential equations are often used to describe such systems.

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Moreover, control theory is an area of application-oriented mathematics which deals with basic principles underlying the analysis and design of control systems. Roughly speaking, controllability generally means that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. Controllability plays a crucial role in a lot of control problems, such as the case of stabilization of unstable systems by feedback or optimal control [8, 9]. Conceived by Kalman, the controllability concept has been studied extensively in the fields of finite-dimensional systems, infinite-dimensional systems, hybrid systems, and behavioral systems. If a system cannot be controlled completely then different types of controllability can be defined such as approximate, null, local null and local approximate null controllability. For more details the reader may refer to [16–19] and the references therein. In this paper, we study the controllability of neutral functional stochastic differential equations of the form

$$\begin{cases} d[x(t)-g(t,x_t)] = [Ax(t)+f(t,x_t)+Bu(t)]dt+\sigma(t)dB^H(t),\ t\in[0,T], \\ \\ x(t)=\varphi(t)\in L^2(\Omega,\mathcal{B}_h),\ for\ a.e.\ t\in(-\infty,0]. \end{cases} \tag{1}$$

Here, A is the infinitesimal generator of an analytic semigroup of bounded linear operators, $(S(t))_{t\geq 0}$, in a Hilbert space X; B^H is a fractional Brownian motion with $H>\frac{1}{2}$ on a real and separable Hilbert space Y; and the control function $u(\cdot)$ takes values in $L^2([0,T],U)$, the Hilbert space of admissible control functions for a separable Hilbert space U; and B is a bounded linear operator from U into X.

The history $x_t:(-\infty,0]\to X$, $x_t(\theta)=x(t+\theta)$, belongs to an abstract phase space $\mathcal{B}_h$ defined axiomatically, and $f,g:[0,T]\times\mathcal{B}_h\to X$ are appropriated functions, and $\sigma:[0,T]\to\mathcal{L}_2^0(Y,X)$, are appropriate functions, where $\mathcal{L}_2^0(Y,X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y into X (see section 2 below).

A general theory for the infinite-dimensional stochastic differential equations driven by a fractional Brownian motion (fBm) has begun to receive attention by various researchers, see e.g. [15]. For example, Dung studied the existence and uniqueness of impulsive stochastic Volterra integro-differential equation driven by fBm in [6]. Using the Riemann-Stieltjes integral, Boufoussi et al. [1] proved the existence and uniqueness of a mild solution to a related problem and studied the dependence of the solution on the initial condition in infinite dimensional space. Very recently, Caraballo and Diop [3], Caraballo et al. [4], and Boufoussi and Hajji [2] have discussed the existence, uniqueness and exponential asymptotic behavior of mild solutions by using the Wiener integral.

To the best of the author's knowledge, an investigation concerning the controllability for neutral stochastic differential equations with infinite delay of the form (1) driven by a fractional Brownian motion has not yet been conducted. Thus, we will make the first attempt to study such problem in this paper. Our results are motivated by those in [10, 11] where the controllability of mild solutions to neutral stochastic functional integro-differential equations driven by fractional Brownian motion with finite delays is studied.

The outline of this paper is as follows: In Section 2 we introduce some notations, concepts, and basic results about fractional Brownian motion, the Wiener integral defined in general Hilbert spaces, phase spaces and properties of analytic semigroups and the fractional powers associated to its generator. In Section 3, we derive the controllability of neutral stochastic differential systems driven by a fractional Brownian motion.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. A standard fractional Brownian motion (fBm) $\{\beta^H(t), t \in \mathbb{R}\}$ with Hurst parameter $H \in (0,1)$ is a zero mean Gaussian process with continuous sample paths such that

$$R_H(t,s) = \mathbb{E}[\beta^H(t)\beta^H(s)] = \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big), \qquad s,t \in \mathbb{R}. \tag{2.1}$$

Remark 2.1 In the case $H > \frac{1}{2}$, it follows from [12]that the second partial derivative of the covariance function

$$\frac{\partial R_H}{\partial t \partial s} = \alpha_H |t-s|^{2H-2},$$

where $\alpha_H = H(2H-2)$, is integrable, and we can write

$$R_H(t,s) = \alpha_H \int_0^t \int_0^s |u-v|^{2H-2} du dv. \tag{2.2}$$

The following result is fundamental to prove our result, it can be proved by similar arguments as those used to prove Lemma 2 in [4].

Lemma 2.1 *If $\psi : [0,T] \to \mathcal{L}_2^0(Y,X)$ satisfies $\int_0^T \|\psi(s)\|^2_{\mathcal{L}_2^0} ds < \infty$, we have*

$$\mathbb{E}\|\int_0^t \psi(s) dB^H(s)\|^2 \leq 2Ht^{2H-1} \int_0^t \|\psi(s)\|^2_{\mathcal{L}_2^0} ds.$$

Next, we introduce some notations and basic facts about the theory of semigroups and fractional power operators. Let $A : D(A) \to X$ be the infinitesimal generator of an analytic semigroup, $(S(t))_{t \geq 0}$, of bounded linear operators on X. The theory of strongly continuous is thoroughly discussed in [13] and [7]. It is well-known that there exist $M \geq 1$ and $\lambda \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{\lambda t}$ for every $t \geq 0$. If $(S(t))_{t \geq 0}$ is a uniformly bounded, analytic semigroup such that $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A, then it is possible to define the fractional power $(-A)^\alpha$ for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in X, and the expression

$$\|h\|_\alpha = \|(-A)^\alpha h\|$$

defines a norm in $D(-A)^\alpha$. If X_α represents the space $D(-A)^\alpha$ endowed with the norm $\|.\|_\alpha$, then the following properties hold (see [13], p. 74).

Lemma 2.2 *Suppose that A, X_α, and $(-A)^\alpha$ are as described above.*

(i) For $0 < \alpha \leq 1$, X_α is a Banach space.

(ii) If $0 < \beta \leq \alpha$, then the injection $X_\alpha \hookrightarrow X_\beta$ is continuous.

(iii) For every $0 < \alpha \leq 1$, there exists $M_\alpha > 0$ such that

$$\|(-A)^\alpha S(t)\| \leq M_\alpha t^{-\alpha} e^{-\lambda t}, \quad t > 0, \;\; \lambda > 0.$$

3 Controllability Result

In this section we derive controllability conditions for a class of neutral stochastic functional differential equations with infinite delays driven by a fractional Brownian motion in a real separable Hilbert space. Before starting, we introduce the concepts of a mild solution of the problem (1) and the meaning of controllability of neutral stochastic functional differential equation.

Definition 3.1 *An X-valued process $\{x(t) : t \in (-\infty, T]\}$ is a mild solution of (1) if*

1. *$x(t)$ is continuous on $[0, T]$ almost surely and for each $s \in [0, t)$ the function $AS(t-s)g(s, x_s)$ is integrable,*
2. *for arbitrary $t \in [0, T]$, we have*

$$\begin{aligned} x(t) &= S(t)(\varphi(0) - g(0,\varphi)) + g(t, x_t) \\ &+ \int_0^t AS(t-s)g(s, x_s)ds + \int_0^t S(t-s)f(s, x_s)ds \\ &+ \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)\sigma(s)dB^H(s), \qquad \mathbb{P}-a.s. \end{aligned} \tag{3.1}$$

3. *$x(t) = \varphi(t)$ on $(-\infty, 0]$ satisfying $\|\varphi\|^2_{\mathcal{B}_h} < \infty$.*

Definition 3.2 The neutral stochastic functional differential equation (1) is said to be controllable on the interval $(-\infty, T]$ if for every initial stochastic process φ defined on $(-\infty, 0]$, there exists a stochastic control $u \in L^2([0, T], U)$ such that the mild solution $x(\cdot)$ of (1) satisfies $x(T) = x_1$, where x_1 and T are the preassigned terminal state and time, respectively.

Our main result in this paper is based on the following fixed point theorem.

Theorem 3.1 *(Karasnoselskii's fixed point theorem) Let V be a bounded closed and convex subset of a Banach space X and let Π_1, Π_2 be two operators of V into X satisfying:*

1. *$\Pi_1(x) + \Pi_2(x) \in V$ whenever $x \in V$,*
2. *Π_1 is a contraction mapping, and*
3. *Π_2 is completely continuous.*

Then, there exists a $z \in V$ such that $z = \Pi_1(z) + \Pi_2(z)$.

In order to establish the controllability of (1), we impose the following conditions on the data of the problem:

($\mathcal{H}$.1) The analytic semigroup, $(S(t))_{t\geq 0}$, generated by A is compact for $t > 0$, and there exist constants M, $M_{1-\beta}$ such that

$$\|S(t)\|^2 \leq M \quad \text{and} \quad \|(-A)^{1-\beta}S(t)\| \leq \frac{M_{1-\beta}}{t^{1-\beta}}, \text{ for all } t \in [0, T]$$

(see Lemma 2.2).

($\mathcal{H}$.2) The map $f : [0, T] \times \mathcal{B}_h \to X$ satisfies the following conditions:

(i) The function $t \longmapsto f(t, x)$ is measurable for each $x \in \mathcal{B}_h$, the function $x \longmapsto f(t, x)$ is continuous for almost all $t \in [0, T]$,

(ii) for every positive integer k there exists $p_k \in L^1([0, T], \mathbb{R}^+)$, such that

$$\|f(t, x)\|^2 \leq p_k(t), \text{ for all } \|x\|^2_{\mathcal{B}_h} \leq k \text{ almost surely and for a.e. } t \in [0, T],$$

and

$$\liminf_{k \longrightarrow +\infty} \frac{1}{k} \int_0^T p_k(\tau) d\tau = \gamma < \infty.$$

($\mathcal{H}$.3) The function $g : [0, T] \times \mathcal{B}_h \longrightarrow X$ is continuous and there exist constants $0 < \beta < 1$, $M_g > 0$ and $\nu > 0$ such that the function g is X_β-valued and satisfies

i) $\|(-A)^\beta g(t, x) - (-A)^\beta g(t, y)\|^2 \leq M_g \|x - y\|^2_{\mathcal{B}_h}$, almost surely, for a.e. $t \in [0, T]$, and for all $x, y \in \mathcal{B}_h$ with

$$\nu = 4M_g l^2 (\|(-A)^{-\beta}\|^2 + \frac{(M_{1-\beta} T^\beta)^2}{2\beta - 1}) < 1.$$

ii) $c_1 = \|(-A)^{-\beta}\|$ and $\overline{M}_g = \sup_{t \in [0,T]} \|(-A)^{-\beta} g(t, 0)\|^2$.

($\mathcal{H}$.4) The function $\sigma : [0, \infty) \to \mathcal{L}_2^0(Y, X)$ satisfies

$$\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds < \infty, \quad \forall T > 0.$$

($\mathcal{H}$.5) The linear operator W from U into X defined by

$$Wu = \int_0^T S(T - s) B u(s) ds$$

has an inverse operator W^{-1} that takes values in $L^2([0, T], U) \setminus kerW$, where

$$kerW = \{x \in L^2([0, T], U) : \; Wx = 0\}$$

(see [8]), and there exists finite positive constants M_b, M_w such that $\|B\|^2 \leq M_b$ and $\|W^{-1}\|^2 \leq M_w$.

The main result of this chapter is the following.

Theorem 3.2 *Suppose that* ($\mathcal{H}$.1) – ($\mathcal{H}$.5) *hold. Then, the system (1) is controllable on* $(-\infty, T]$ *provided that*

$$6l^2(1 + 6MM_bM_wT^2)\{8(c_1^2 + \frac{(M_{1-\beta}T^\beta)^2}{2\beta - 1})M_g + 8MT\gamma\} < 1. \tag{3.2}$$

Proof. Transform the problem(1) into a fixed-point problem. To do this, using the hypothesis ($\mathcal{H}$.5) for an arbitrary function $x(\cdot)$, define the control by

$$
\begin{aligned}
u(t) &= W^{-1}\{x_1 - S(T)(\varphi(0) - g(0, x_0)) - g(T, x_T))\\
&- \int_0^T AS(T-s)g(s, x_s)ds - \int_0^T S(T-s)f(s, x_s)ds\\
&- \int_0^T S(T-s)\sigma(s)dB^H(s)\}(t).
\end{aligned}
$$

To formulate the controllability problem in the form suitable for application of the fixed point theorem, put the control $u(.)$ into the stochastic control system (3.1) and obtain a non linear operator Π on $\mathcal{B}_T$ given by

$$
\Pi(x)(t) = \begin{cases}
\varphi(t), \quad \text{if } \ t \in (-\infty, 0],\\
S(t)(\varphi(0) - g(0, \varphi)) + g(t, x_t)\\
+ \int_0^t AS(t-s)g(s, x_s)ds + \int_0^t S(t-s)f(s, x_s)ds\\
+ \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)\sigma(s)dB^H(s), \quad \text{if } \ t \in [0, T].
\end{cases}
$$

Then it is clear that to prove the existence of mild solutions to equation (1) is equivalent to find a fixed point for the operator Π. Clearly, $\Pi x(T) = x_1$, which means that the control u steers the system from the initial state φ to x_1 in time T, provided we can obtain a fixed point of the operator Π which implies that the system in controllable.

Let $y : (-\infty, T] \longrightarrow X$ be the function defined by

$$
y(t) = \begin{cases} \varphi(t), & \text{if } \ t \in (-\infty, 0],\\ S(t)\varphi(0), & \text{if } \ t \in [0, T], \end{cases}
$$

then, $y_0 = \varphi$. For each function $z \in \mathcal{B}_T$, set

$$
x(t) = z(t) + y(t).
$$

It is obvious that x satisfies the stochastic control system (3.1) if and only if z satisfies $z_0 = 0$ and

$$
\begin{aligned}
z(t) = \quad & g(t, z_t + y_t) - S(t)g(0, \varphi) + \int_0^t AS(t-s)g(s, z_s + y_s)ds\\
& + \int_0^t S(t-s)f(s, z_s + y_s)ds + \int_0^t S(t-s)Bu_{z+y}(s)ds\\
& + \int_0^t S(t-s)\sigma(s)dB^H(s),
\end{aligned}
\tag{3.3}
$$

where

$$
\begin{aligned}
u_{z+y}(t) \quad &= W^{-1}\{x_1 - S(T)(\varphi(0) - g(0, z_0 + y_0)) - g(T, z_T + y_T))\\
&- \int_0^T AS(T-s)g(s, z_s + y_s)ds - \int_0^T S(T-s)f(s, z_s + y_s)ds\\
&- \int_0^T S(T-s)\sigma(s)dB^H(s)\}(t).
\end{aligned}
$$

Set

$$
\mathcal{B}_T^0 = \{z \in \mathcal{B}_T : z_0 = 0\};
$$

for any $z \in B_T^0$, we have

$$\|z\|_{\mathcal{B}_T^0} = \|z_0\|_{\mathcal{B}_h} + \sup_{t\in[0,T]} (\mathbb{E}\|z(t)\|^2)^{\frac{1}{2}} = \sup_{t\in[0,T]} (\mathbb{E}\|z(t)\|^2)^{\frac{1}{2}}.$$

Then, $(\mathcal{B}_T^0, \|.\|_{\mathcal{B}_T^0})$ is a Banach space. Define the operator $\widehat{\Pi} : \mathcal{B}_T^0 \longrightarrow \mathcal{B}_T^0$ by

$$(\widehat{\Pi}z)(t) = \begin{cases} 0, \text{ if } t \in (-\infty, 0], \\ g(t, z_t + y_t) - S(t)g(0,\varphi)) + \int_0^t AS(t-s)g(s, z_s + y_s)ds \\ + \int_0^t S(t-s)f(s, z_s + y_s)ds + \int_0^t S(t-s)Bu_{z+y}(s)ds \\ + \int_0^t S(t-s)\sigma(s)dB^H(s), \qquad \text{if } t \in [0,T]. \end{cases} \tag{3.4}$$

Set

$$\mathcal{B}_k = \{z \in \mathcal{B}_T^0 : \|z\|^2_{\mathcal{B}_T^0} \leq k\}, \qquad \text{for some } k \geq 0,$$

then $\mathcal{B}_k \subseteq \mathcal{B}_T^0$ is a bounded closed convex set, and for $z \in \mathcal{B}_k$, we have

$$\begin{aligned} \|z_t + y_t\|^2_{\mathcal{B}_h} &\leq 2(\|z_t\|^2_{\mathcal{B}_h} + \|y_t\|^2_{\mathcal{B}_h}) \\ &\leq 4(l^2 \sup_{0\leq s\leq t} \mathbb{E}\|z(s)\|^2 + \|z_0\|^2_{\mathcal{B}_h} \\ &+ l^2 \sup_{0\leq s\leq t} \mathbb{E}\|y(s)\|^2 + \|y_0\|^2_{\mathcal{B}_h}) \\ &\leq 4l^2(k + M\mathbb{E}\|\varphi(0)\|^2) + 4\|y\|^2_{\mathcal{B}_h} := q'. \end{aligned}$$

From our assumptions, using the fact that $(\sum_{i=1}^n a_i)^2 \leq n \sum_{i=1}^n a_i^2$ for any positive real numbers a_i, $i = 1, 2, ..., n$, we have

$$\begin{aligned} \mathbb{E}\|u_{z+y}\|^2 \leq \quad & 6M_w\{\|x_1\|^2 + M\mathbb{E}\|\varphi(0)\|^2 + 2Mc_1^2[M_g\|y\|^2_{\mathcal{B}_h} + \overline{M}_g] \\ & + 2(c_1^2 + \tfrac{(M_{1-\beta}T^\beta)^2}{2\beta-1})[M_g q' + \overline{M}_g] + MT\int_0^T p_{q'}(s)ds \\ & + 2MT^{2H-1}\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds\} := \mathcal{G}. \end{aligned} \tag{3.5}$$

It is clear that the operator Π has a fixed point if and only if $\widehat{\Pi}$ has one, so it turns to prove that $\widehat{\Pi}$ has a fixed point. To this end, we decompose $\widehat{\Pi}$ as $\widehat{\Pi} = \Pi_1 + \Pi_2$, where Π_1 and Π_2 are defined on $\mathcal{B}_T^0$, respectively by

$$(\Pi_1 z)(t) = \begin{cases} 0, \text{ if } t \in (-\infty, 0], \\ g(t, z_t + y_t) - S(t)g(0,\varphi)) + \int_0^t AS(t-s)g(s, z_s + y_s)ds \\ + \int_0^t S(t-s)\sigma(s)dB^H(s), \quad \text{if } t \in [0,T], \end{cases} \tag{3.6}$$

and

$$(\Pi_2 z)(t) = \begin{cases} 0, \text{ if } t \in (-\infty, 0], \\ \int_0^t S(t-s)f(s, z_s + y_s)ds \\ + \int_0^t S(t-s)Bu_{z+y}(s)ds, \text{ if } t \in [0,T]. \end{cases} \tag{3.7}$$

In order to apply the Karasnoselskii fixed point theorem for the operator $\widehat{\Pi}$, we prove the following assertions:

1. $\Pi_1(x) + \Pi_2(x) \in \mathcal{B}_k$ whenever $x \in \mathcal{B}_k$,
2. Π_1 is a contraction;
3. Π_2 is continuous and compact map.

For the sake of convenience, the proof will be given in several steps.

Step 1. We claim that there exists a positive number k, such that $\Pi_1(x) + \Pi_2(x) \in \mathcal{B}_k$ whenever $x \in \mathcal{B}_k$. If it is not true, then for each positive number k, there is a function $z^k(.) \in \mathcal{B}_k$, but $\Pi_1(z^k) + \Pi_2(z^k) \notin \mathcal{B}_k$, that is $\mathbb{E}\|\Pi_1(z^k)(t) + \Pi_2(z^k)(t)\|^2 > k$ for some $t \in [0, T]$. However, on the other hand, we have

$$
\begin{aligned}
k \quad & < \mathbb{E}\|\Pi_1(z^k)(t) + \Pi_2(z^k)(t)\|^2 \\
& \leq 6\{2Mc_1^2(M_g\|y\|_{\mathcal{B}_h}^2 + \overline{M}_g) + 2(c_1^2 q' + \overline{M}_g) + 2\tfrac{(M_{1-\beta}T^\beta)^2}{2\beta-1}[M_g q' + \overline{M}_g] \\
& +MM_bT^2\mathcal{G} + MT\textstyle\int_0^T p_{q'}(s)ds + 2MT^{2H-1}\int_0^T \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds\} \\
& \leq 6(1+6MM_bM_wT^2)\{2Mc_1^2(M_g\|y\|_{\mathcal{B}_h}^2 + \overline{M}_g) + 2(c_1^2 + \tfrac{(M_{1-\beta}T^\beta)^2}{2\beta-1})[M_g q' + \overline{M}_g]) \\
& +MT\textstyle\int_0^T p_{q'}(s)ds + 2MT^{2H-1}\int_0^T \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds\} \\
& +6MM_bM_wT^2(\|x_1\|^2 + M\mathbb{E}\|\varphi(0)\|^2) \\
& \leq \overline{K} + 6(1+6MM_bM_wT^2)\{2(c_1^2 + \tfrac{(M_{1-\beta}T^\beta)^2}{2\beta-1})M_g q' + 2MT\textstyle\int_0^T p_{q'}(s)ds\},
\end{aligned}
$$

where

$$
\begin{aligned}
\overline{K} \quad & = 6(1+6MM_bM_wT^2)\{2Mc_1^2(M_g\|y\|_{\mathcal{B}_h}^2 + \overline{M}_g) + 2(c_1^2 + \tfrac{(M_{1-\beta}T^\beta)^2}{2\beta-1})\overline{M}_g) \\
& +2MT^{2H-1}\textstyle\int_0^T \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds\} + 6MM_bM_wT^2(\|x_1\|^2 + M\mathbb{E}\|\varphi(0)\|^2)\}.
\end{aligned}
$$

Noting that $\overline{K}$ is independent of k. Dividing both sides by k and taking the lower limit as $k \longrightarrow \infty$, we obtain

$$
q' = 4l^2(k + M\mathbb{E}\|\varphi(0)\|^2) + 4\|y\|_{\mathcal{B}_h} \longrightarrow \infty \ \text{ as } k \longrightarrow \infty,
$$

$$
\liminf_{k\longrightarrow\infty} \frac{\int_0^T p_{q'}(s)ds}{k} = \liminf_{k\longrightarrow\infty} \frac{\int_0^T p_{q'}(s)ds}{q'}.\frac{q'}{k} = 4l^2\gamma.
$$

Thus, we have

$$
6l^2(1+6MM_bM_wT^2)\{8(c_1^2 + \frac{(M_{1-\beta}T^\beta)^2}{2\beta-1})M_g + 8MT\gamma\} \geq 1.
$$

This contradicts (3.2). Hence for some positive k,

$$
(\Pi_1 + \Pi_2)(\mathcal{B}_k) \subseteq \mathcal{B}_k.
$$

Step 2. Π_1 is a contraction. Let $t \in [0,T]$ and $z^1, z^2 \in \mathcal{B}_T^0$

$$
\begin{aligned}
\mathbb{E}\|(\Pi_1 z^1)(t) - (\Pi_1 z^2)(t)\|^2 \quad & \le 2\mathbb{E}\|g(t, z_t^1 + y_t) - g(t, z_t^2 + y_t)\|^2 \\
& +2\mathbb{E}\| \textstyle\int_0^t AS(t-s)(g(s, z_s^1 + y_s) - g(s, z_s^2 + y_s))ds\|^2 \\
& \le 2M_g\|(-A)^{-\beta}\|^2\|z_s^1 - z_s^2\|_{\mathcal{B}_h}^2 \\
& +2T \textstyle\int_0^t \frac{M_{1-\beta}^2}{(t-s)^{2(1-\beta)}} M_g \|z_s^1 - z_s^2\|_{\mathcal{B}_h}^2 ds \\
& \le 2M_g \left\{\|(-A)^{-\beta}\|^2 + \tfrac{(M_{1-\beta}T^\beta)^2}{(2\beta-1)}\right\}(2l^2 \sup_{0\le s\le T} \\
& \mathbb{E}\|z^1(s) - z^2(s)\|^2 + 2(\|z_0^1\|_{\mathcal{B}_h}^2 + \|z_0^2\|_{\mathcal{B}_h}^2) \\
& \le \nu \sup_{0\le s\le T} \mathbb{E}\|z^1(s) - z^2(s)\|^2) \quad (\text{ since } \ z_0^1 = z_0^2 = 0)
\end{aligned}
$$

Taking supremum over t,

$$\|(\Pi_1 z^1)(t) - (\Pi_1 z^2)(t)\|_{\mathcal{B}_T^0} \le \nu \|z^1 - z^2\|_{\mathcal{B}_T^0},$$

where

$$\nu = 4M_g l^2(\|(-A)^{-\beta}\|^2 + \frac{(M_{1-\beta}T^\beta)^2}{2\beta - 1}) < 1.$$

Thus Π_1 is a contraction on $\mathcal{B}_T^0$.
Step 3. Π_2 is completely continuous $\mathcal{B}_T^0$.
Claim 1. Π_2 is continuous on $\mathcal{B}_T^0$.
Let z^n be a sequence such that $z^n \longrightarrow z$ in $\mathcal{B}_T^0$. Then, there exists a number $k > 0$ such that $\|z^n(t)\| \le k$, for all n and a.e. $t \in [0,T]$, so $z^n \in \mathcal{B}_k$ and $z \in \mathcal{B}_k$. By hypothesis $(\mathcal{H}.2)$, we have $f(t, z_t^n + y_t) \longrightarrow f(t, z_t + y_t)$ for each $t \in [0,T]$. Since

$\|f(t, z_t^n + y_t) - f(t, z_t + y_t)\|^2 \le 2p_{q'}(t)$. From $(\mathcal{H}.3)$, Hölder inequality and the dominated convergence theorem, we have

$$
\begin{aligned}
\mathbb{E}\|\Pi_2 z^n(t) - (\Pi_2 z)(t)\|^2 \quad & \le 2\mathbb{E}\| \textstyle\int_0^t S(t-s)B[u_{z^n+y} - u_{z+y}]ds\|^2 \\
& +2\mathbb{E}\| \textstyle\int_0^t S(t-s)[f(s, z_s^n + y_s) - f(s, z_s + y_s)]ds\|^2 \\
& \le 6M_w M_b MT \textstyle\int_0^T \{E\|g(T, z_T^n + y_T) - g(T, z_T + y_T)\|^2 \\
& +T \textstyle\int_0^T \mathbb{E}\|AS(T-s)g(s, z_s^n + y_s) - AS(T-s)g(s, z_s + y_s)\|^2 ds \\
& +MT \textstyle\int_0^T \mathbb{E}\|f(s, z_s^n + y_s) - f(s, z_s + y_s)\|^2 ds\}(\eta)d\eta \\
& +2MT \textstyle\int_0^T \mathbb{E}\|f(s, z_s^n + y_s) - f(s, z_s + y_s)\|^2 ds \\
& \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}
$$

Thus, Π_2 is continuous.

Claim 2. Π_2 maps $\mathcal{B}_k$ into equicontinuous family.
Let $z \in \mathcal{B}_k$ and τ_1, $\tau_2 \in [0, T]$, we have

$$\begin{aligned}\mathbb{E}\|(\Pi_2 z)(\tau_2) - (\Pi_2 z)(\tau_1)\|^2 \leq \quad & 4\mathbb{E}\| \textstyle\int_0^{\tau_1}(S(\tau_2 - s) - S(\tau_1 - s))f(s, z_s + y_s)ds\|^2 \\ & +4\mathbb{E}\| \textstyle\int_0^{\tau_1}(S(\tau_2 - s) - S(\tau_1 - s))Bu(s)ds\|^2 \\ & +4\mathbb{E}\| \textstyle\int_{\tau_1}^{\tau_2} S(\tau_2 - s)f(s, z_s + y_s)ds\|^2 \\ & +4\mathbb{E}\| \textstyle\int_{\tau_1}^{\tau_2} S(\tau_2 - s))Bu(s)ds\|^2.\end{aligned}$$

From (3.5), Hölder inequality, it follows that

$$\begin{aligned}\mathbb{E}\|(\Pi_2 z)(\tau_2) - (\Pi_2 z)(\tau_1)\|^2 \leq \quad & 4T\| \textstyle\int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|^2 p_{q'}(s)ds \\ & +4TM_b\mathcal{G} \textstyle\int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|^2 ds \\ & +4T \textstyle\int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|^2 p_{q'}(s)ds \\ & +4TM_b\mathcal{G} \textstyle\int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|^2 ds.\end{aligned}$$

The right-hand side is independent of $z \in \mathcal{B}_k$ and tends to zero as $\tau_2 - \tau_1 \longrightarrow 0$, since the compactness of $S(t)$ for $t > 0$ implies the continuity in the uniform operator topology. Thus, Π_2 maps $\mathcal{B}_k$ into an equicontinuous family of functions. The equicontinuities for the cases $\tau_1 < \tau_2 \leq 0$ and $\tau_1 < 0 < \tau_2$ are obvious.
Claim 3. $(\Pi_2 \mathcal{B}_k)(t)$ is precompact set in X.
Let $0 < t \leq T$ be fixed, $0 < \epsilon < t$, for $z \in \mathcal{B}_k$, we define

$$(\Pi_{2,\epsilon} z)(t) = S(\epsilon) \int_0^{t-\epsilon} S(t - s - \epsilon)f(s, z_s + y_s)ds + S(\epsilon) \int_0^{t-\epsilon} S(t - s - \epsilon)Bu(s)ds.$$

Using the estimation (3.5) as above and by the compactness of $S(t)$ $(t > 0)$, we obtain $V_\epsilon(t) = \{(\Pi_{2,\epsilon} z)(t) : \ z \in \mathcal{B}_k\}$ is relative compact in X for every ϵ, $0 < \epsilon < t$. Moreover, for every $z \in \mathcal{B}_k$, we have

$$\begin{aligned}\mathbb{E}\|\Pi_2 z)(t) - \Pi_{2,\epsilon} z)(t)\|^2 \leq \quad & 2T \textstyle\int_{t-\epsilon}^{t} \|S(t - s)\|^2 \mathbb{E}\|f(s, z_s + y_s)\|^2 ds \\ & +2TM_b\mathcal{G} \textstyle\int_{t-\epsilon}^{t} \|S(t - s)\|^2 ds \\ & \leq 2TM \textstyle\int_{t-\epsilon}^{t} p_{q'}(s)ds + 2TM_b\mathcal{G}M\epsilon.\end{aligned}$$

Therefore,

$$\mathbb{E}\|\Pi_2 z)(t) - \Pi_{2,\epsilon} z)(t)\|^2 \longrightarrow 0 \quad \text{as} \quad \epsilon \longrightarrow 0^+,$$

and there are precompact sets arbitrarily close to the set $V(t) = \{(\Pi_2 z)(t) : \ z \in B_k\}$, hence the set $V(t)$ is also precompact in X.

Thus, by Arzela-Ascoli theorem Π_2 is a compact operator. These arguments enable us to conclude that Π_2 is completely continuous, and by the fixed point theorem of Karasnoselskii there exists a fixed point $z(.)$ for $\widehat{\Pi}$ on $\mathcal{B}_k$. If we define $x(t) = z(t) + y(t)$, $-\infty < t \leq T$, it is easy to see that $x(.)$ is a mild solution of (1) satisfying $x_0 = \varphi$, $x(T) = x_1$. Then the proof is complete.

4 Conclusion

Our paper contains some controllability results for neutral stochastic functional differential equations with infinite delay driven by fractional Brownian motion in a real separable Hilbert space. The result proves that the fixed-point theorem can effectively be used in control problems to obtain sufficient conditions. We can extend the controllability result for neutral impulsive stochastic systems with different types of delays in our subsequent papers.

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Approximation of the Optimal Control Problem on an Interval with a Family of Optimization Problems on Time Scales

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Abstract: In this paper we consider a family of optimization problems defined on variable time scales $\mathbb{T}_\lambda$, which depend on the parameter λ We prove that the family of value functions $V_\lambda(t_0, x)$ of the optimal control problem on $[t_0, t_1]_{\mathbb{T}_\lambda}$ converges locally uniformly in $\mathbb{R}^d$ to the value function $V(t_0, x)$ of the optimal control problem on $[t_0, t_1]$, provided $\sup_{t\in[t_0,t_1]_{\mathbb{T}_\lambda}} \mu_\lambda(t) \to 0$ as $\lambda \to 0$, where $\mu_\lambda(t)$ is the graininess function of $\mathbb{T}_\lambda$.

Keywords: *time scale; value function; right-scattered point; right-dense point; graininess function.*

Mathematics Subject Classification (2010): 34K35, 34N99, 39A13, 49K15, 93C15, 93C56.

1 Introduction

This work is devoted to the study of the limiting behavior of the optimal control problem for dynamic equations, defined on a family of time scales $\mathbb{T}_\lambda$, in the regime when the graininess function μ_λ converges to zero as $\lambda \to 0$. At the same time the segment of the time scale $[t_0, t_1]_{\mathbb{T}_\lambda} = [t_0, t_1] \cap \mathbb{T}_\lambda$ approaches $[t_0, t_1]$ e.g. in the Hausdorff metric. The natural question that arises is how the optimal control problem on the time scale is related to the corresponding control problem on the interval $[t_0, t_1]$.

The answer to the above question is well understood for Eulerian time scales (according to classification [6]) that is, if $\mathbb{T}_\lambda = \lambda\mathbb{Z}_+$, $\lambda > 0$, and the equation on time scales becomes a difference equation.

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The results listed above are based on Euler polygonal method, which guarantees that the corresponding solutions of differential and difference equations on finite time intervals are close to each other, provided the steps are small. This method works really well if the right-hand sides of differential equations are continuous. In this case, both solutions are smooth, which makes it relatively easy to estimate the difference between them. However, the right-hand sides of the optimal control problem, considered in this paper, depend on the control parameter function $u(t)$. Generally speaking, $u(t)$ is only measurable. This makes the solution of the differential equation only absolutely continuous. In turn, this significantly complicates the estimates for the difference between corresponding solutions. Estimates of this type were obtained with convex analysis techniques in the works [9] – [11]. Using these estimates, the authors showed that the value function for the difference equation approximation converges to the corresponding value function for continuous differential equation as the approximation step goes to zero.

Our work extends the result [9] – [11] on the limiting behavior of the value function to the case of general time scales. However, we use different methods since the topological structure of the time scale we are considering may be complex. The main difficulty in our work is to establish the *uniform* convergence of solutions of the Cauchy problem on $[t_0, t_1] \cap_{\mathbb{T}_\lambda}$ to the solution of the corresponding Cauchy problem on $[t_0, t_1]$. This makes our analysis significantly different from [12], where only special pointwise convergence was obtained. More sophisticated approach is necessary because, in contrast with [12], the right-sides of our equations are not piecewise continuous, as well as we are dealing with much more general time scales (as opposed to the Eulerian time scale in [9] – [11]).

This paper is organized as follows. In Section 2 we provide some definitions and preliminary results on time scales calculus, and state the main result. The main result on the convergence of the family of the value functions to the value function of the limit problem is proved in Section 3.

2 Preliminaries and Main Result

2.1 Basic notions of time scales theory

The time scales theory was introduced by S. Hilger in his PhD thesis [13] (1988) as a unified theory for both discrete and continuous analysis. This theory was further developed by a number of authors, see [4] and references therein. For reader's convenience, we present several notions from this theory, which are used in this paper.

Time scale $\mathbb{T}$ is a non-empty closed subset of $\mathbb{R}$; $A_{\mathbb{T}} := A \cap \mathbb{T}$ for $A \subset \mathbb{R}$; $\sigma : \mathbb{T} \to \mathbb{T}$, $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ is the *forward jump operator*; $\rho : \mathbb{T} \to \mathbb{T}$, $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ is the *backward jump operator* (here $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$); $\mu : \mathbb{T} \to [0, \infty)$, $\mu(t) := \sigma(t) - t$ is called *the graininess function.* A point $t \in \mathbb{T}$ is called *left-dense* (LD) (*left-scattered* (LS), *right-dense* (RD) or *right-scattered* (RS)) if $\rho(t) = t$ ($\rho(t) < t$, $\sigma(t) = t$ or $\sigma(t) > t$); $\mathbb{T}^k := \mathbb{T} \setminus \{M\}$ if $\mathbb{T}$ has a left-scattered maximum M, $\mathbb{T}^k := \mathbb{T}$ otherwise.

A function $f : \mathbb{T} \to \mathbb{R}^d$ is called Δ*–differentiable* at $t \in \mathbb{T}^k$ if the limit

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}$$

exists in $\mathbb{R}^d$. The properties of the Lebesgue Δ – measure and the Lebesgue Δ – integrability are described, e.g. in [3].

2.2 Control theory on time scales

Let $\mathbb{T}$ be a time scale, such that $\sup \mathbb{T} = +\infty$, $t_0, t_1 \in \mathbb{T}$, and $U \subset \mathbb{R}^m$ is a compact set.

- ▷ An *optimal control problem on the time scale* $\mathbb{T}$ is the problem of the type

$$\begin{cases} x^{\Delta} = f(t, x, u), \\ x(t_0) = x_0, \\ J(u) = \int_{[t_0,t_1)_{\mathbb{T}}} L(s, x(s), u(s))\Delta s + \Psi(x(t_1)) \to \inf, u \in \mathcal{U}(t_0), \end{cases} \tag{2.1}$$

  where $f : [t_0, t_1]_{\mathbb{T}} \times \mathbb{R}^d \times U \to \mathbb{R}^d$, $L : [t_0, t_1]_{\mathbb{T}} \times \mathbb{R}^d \times U \to \mathbb{R}^1$ and $\Psi : \mathbb{R}^d \to \mathbb{R}^1$.

- ▷ $\mathcal{U}(t) := L^{\infty}([t, t_1]_{\mathbb{T}}, U)$, i.e. the set of bounded, Δ–measurable functions [5, Chapter 5.7] defined on $[t, t_1]_{\mathbb{T}}$ and taking values in U for each $t \in [t_0, t_1)_{\mathbb{T}}$, is called *the set of admissible controls.*

- ▷ The *Bellman function* (or *the value function*) is

$$V(t_0, x_0) := \inf_{u(\cdot)\in\mathcal{U}(t_0)} J(t_0, x_0, u). \tag{2.2}$$

2.3 Main result

Let $\Lambda \subset \mathbb{R}$, such that 0 is a limit point of Λ, be the set of indices. Consider the family of time scales $\mathbb{T}_\lambda, \lambda \in \Lambda$, such that $\sup \mathbb{T}_\lambda = \infty$. For any $t_0, t_1 \in \mathbb{T}_\lambda$, denote $[t_0, t_1]_{\mathbb{T}_\lambda} = [t_0, t_1] \cap \mathbb{T}_\lambda$ and $\mu_\lambda = \sup_{t\in[t_0,t_1]_{\mathbb{T}_\lambda}} \mu(t)$. Assume

$$\mu_\lambda(t) \to 0 \text{ as } \lambda \to 0. \tag{2.3}$$

In this case $\mathbb{T}_\lambda$ converges (e.g. in the Hausdorff metric) to a continuous time scale $\mathbb{T}_0$ (here we use the classification from [6]), and hence $[t_0, t_1]_{\mathbb{T}_\lambda}$ becomes $[t_0, t_1]$ in the limit $\lambda \to 0$. For every $\mathbb{T}_\lambda$ consider the optimal control problem on the time scale $[t_0, t_1]_{\mathbb{T}_\lambda}$:

$$\begin{cases} x^{\Delta} = f(t, x, u), \\ x(t_0) = x, \\ J_\lambda(u) = \int_{[t_0,t_1)_{\mathbb{T}_\lambda}} L(t, x(t), u(t))\Delta t \to \inf, u \in \mathcal{U}(t_0). \end{cases} \tag{2.4}$$

Along with (2.4), consider the corresponding continuous optimal control problem on the interval $[t_0, t_1]$:

$$\begin{cases} \frac{dx(t)}{dt} = f(t, x(t), u(t)), \\ x(t_0) = x, \\ J(u) = \int_{t_0}^{t_1} L(t, x(t), u(t))dt \to \inf, u \in \mathcal{U}(t_0). \end{cases} \tag{2.5}$$

Denote $V_\lambda(t_0, x)$ and $V(t_0, x)$ to be the corresponding Bellman functions for these problems, given by (2.2). Our main result is the following theorem.

Theorem 2.1 *Let $\mathbb{T}_\lambda$ be such that (2.3) holds. In addition, assume that*

1) *The functions f, f_x and L are continuous on $[t_0, t_1] \times \mathbb{R}^d \times U$;*

2) *f and L are globally Lipschitz in x, with Lipschitz constant $K > 0$.*

Then

$$V_\lambda(t_0, \cdot) \to V(t_0, \cdot) \textit{ in } C_{loc}(\mathbb{R}^d),\ \lambda \to 0. \tag{2.6}$$

3 Proof of Theorem 2.1.

Without loss of generality, we assume that $t_0 = 0$ and $t_1 = 1$. Consider an arbitrary time scale $\mathbb{T}_\lambda$ and an arbitrary admissible control $u_\lambda(t)$ on it. Let $x_\lambda(t)$ be a corresponding admissible trajectory. Denote $\tilde{u}_\lambda(t)$ to be the extension of $u_\lambda(t)$ to the entire interval $[0,1]$:

$$\tilde{u}_\lambda(t) := \begin{cases} u_\lambda(t), t \in [0,1]_{\mathbb{T}_\lambda}, \\ u_\lambda(r), t \in [r, \sigma(r)),\ r \in \mathrm{RS}. \end{cases} \tag{3.1}$$

This control is admissible for the problem (2.5). The proof of the main result will heavily rely on the following two lemmas.

Lemma 3.1 *Let $x(t)$ be a solution of*

$$\begin{cases} \frac{dx}{dt} = f(t, x, \tilde{u}_\lambda(t)), \\ x(0) = x_0. \end{cases}$$

Then

$$\left| \int_{[0,1)_{\mathbb{T}_\lambda}} L(t, x_\lambda(t), u_\lambda(t))\Delta t - \int_0^1 L(t, x(t), \tilde{u}_\lambda(t))dt \right| \to 0,\ \lambda \to 0. \tag{3.2}$$

Proof. Fix $\varepsilon > 0$. Our goal is to show that the expression in (3.2) can be made less than ε for all sufficiently small λ. Using Gronwall inequality and its analogue for time scales [4], one can show that for any $r > 0$ there is $C(r) > 0$ such that

$$|x_\lambda(t)| \leq C(r),\ t \in [0,1]_{\mathbb{T}_\lambda}, |x(t)| \leq C(r),\ t \in [0,1],\ |x_0| \leq r. \tag{3.3}$$

The estimates (3.3) are uniform for all admissible controls, since U is compact. Therefore, there is a constant $C_1(r) > 0$ such that

$$\begin{gathered} |L(t, x_\lambda(t), u_\lambda(t))| \leq C_1(r),\ |f(t, x_\lambda(t), u_\lambda(t))| \leq C_1(r), \\ |f_x(t, x_\lambda(t), u_\lambda(t))| \leq C_1(r),\ \forall t \in [0,1]_{\mathbb{T}_\lambda} \text{ and } \lambda \in \Lambda. \end{gathered} \tag{3.4}$$

Then we have

$$\begin{aligned} \int_{[0,1)_{\mathbb{T}_\lambda}} L(t, x_\lambda(t), u_\lambda(t))dt = \int_{[0,1)_{\mathbb{T}} \setminus \mathrm{RS}} L(t, x_\lambda(t), u_\lambda(t))dt \\ + \sum_{r \in \mathrm{RS}} L(r, x_\lambda(r), u_\lambda(r))\mu(r). \end{aligned} \tag{3.5}$$

In view of (3.4),

$$\sum_{r \in \mathrm{RS}} L(r, x_\lambda(r), u_\lambda(r))\mu(r) \leq C_1 \sum_{r \in \mathrm{RS}} \mu_\lambda(r),$$

which holds true regardless the sums are finite or infinite. Then

$$\begin{aligned} \sum_{r \in \mathrm{RS}} L(r, x_\lambda(r), u_\lambda(r))\mu(r) = \sum_{k=1}^{N} L(r_k, x_\lambda(r_k), u_\lambda(r_k))\mu(r_k) \\ + \sum_{k \geq N+1} L(r_k, x_\lambda(r_k), u_\lambda(r_k))\mu(r_k), \end{aligned} \tag{3.6}$$

where $N = N(\lambda) \geq 1$ is chosen so that

$$\sum_{k=N(\lambda)+1} \mu(r_k) \leq \frac{\mu_\lambda}{2}. \tag{3.7}$$

We now remove the right-scattered points, which appear in the sum (3.7), from the time scale. By construction, their total Δ–measure does not exceed $\frac{\mu_\lambda}{2}$. Denote $A = \bigcup_{k=N(\lambda)+1}[r_k, \sigma(r_k))$. Clearly, $|A| \leq \frac{\mu_\lambda}{2}$, where $|A|$ stands for Lebesgue measure of A. Denote $B := [0,1]\backslash A$.

Next, in the same way as it was done in (3.1), we may define a piecewise-constant extension of $x_\lambda(t)$ to the entire interval $[0,1]$. This extension is denoted with $\tilde{x}_\lambda(t)$. Similarly, the function $L(t,x,u)$, which is defined only for $t \in \mathbb{T}_\lambda$, may be extended to $\widetilde{L}(t,x,u)$, defined for $t \in [0,1]$. Clearly, this extension satisfies the same bound $|\widetilde{L}(t,x,u)| \leq C$. Therefore, using the results from [7, Theorem 2.9.],

$$\int_{[0,1]_{\mathbb{T}_\lambda}} L(t, x_\lambda(t), u_\lambda(t))\Delta t = \int_0^1 \widetilde{L}(t, \tilde{x}_\lambda(t), \tilde{u}_\lambda(t))dt.$$

Consequently,

$$\left| \int_{[0,1)_{\mathbb{T}_\lambda}} L(t, x_\lambda(t), u_\lambda(t))\Delta t - \int_0^1 L(t, x(t), \tilde{u}_\lambda(t))dt \right| \\ \leq C\mu_\lambda + \int_B \left| (\widetilde{L}(t, \tilde{x}_\lambda(t), \tilde{u}_\lambda(t)) - L(t, x(t), \tilde{u}_\lambda(t))) \right| dt. \tag{3.8}$$

Let us estimate the last integral in (3.8). The set B consists of a finite number of right-scattered points $(r_1, \ldots, r_N)$ and possibly intervals between them, consisting of limit points. In view of (3.3) and the compactness of U, the functions $f(t,x,u)$ and $L(t,x,u)$, without loss of generality, are defined on a compact set, hence they are uniformly continuous. Therefore, there exists $\varepsilon_1 = \varepsilon_1(\varepsilon) > 0$ such that

$$|L(t,x,u) - L(s,x,u)| < \varepsilon, \; |f(t,x,u) - f(s,x,u)| < \varepsilon, \text{ if } |t-s| < \varepsilon_1. \tag{3.9}$$

In view of (2.3), we can choose λ small enough so that $\mu_\lambda < \varepsilon_1$. Denoting $B_1 = B \setminus \bigcup_{i=1}^N [r_i, \sigma(r_i))$, we have

$$\int_B \widetilde{L}(t, \tilde{x}_\lambda(t), \tilde{u}_\lambda(t))dt = \int_{B_1} L(t, x_\lambda(t), u_\lambda(t))dt + \sum_{i=1}^N \int_{r_i}^{\sigma(r_i)} L(r_i, \tilde{x}_\lambda(t), \tilde{u}_\lambda(t))dt.$$

Hence,

$$\int_B |\widetilde{L}(t, \tilde{x}_\lambda(t), \tilde{u}_\lambda(t)) - L(t, x(t), \tilde{u}_\lambda(t))|dt \leq K \int_B |x_\lambda(t) - x(t)|dt + \varepsilon, \tag{3.10}$$

where we used (3.9) and the Lipschitz property of L. Now, we estimate the difference $|\tilde{x}_\lambda(t) - x(t)|$. Without loss of generality, assume that the time scale $\mathbb{T}_\lambda$ has the following structure (Figure 1).

Here

1) the solid line indicates the line segments, which consist of limit points;

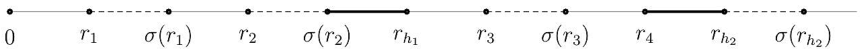

Figure 1: The structure of time scale.

2) the dashed line indicates the line segments $[r_i, \sigma(r_i))$, i.e. r_i are the remaining right-scattered points;

3) the boldface solid line indicates the set of points which were removed, i.e. the set A.

The argument is similar for other structures of time scales.

1) For $t \in [0, r_1]$ we have $u_\lambda(t) = \tilde{u}_\lambda(t)$, therefore $\tilde{x}_\lambda(t) = x(t)$.

2) For $t \in [r_1, \sigma(r_1))$, clearly $\tilde{x}_\lambda(t) = x_\lambda(r_1) = x(r_1)$ and $\tilde{u}_\lambda(t) = u_\lambda(r_1)$. It follows from the integral representation of the solution

$$x(t) = x(r_1) + \int_{r_1}^{t} f(s, x(s), u_\lambda(r_1))ds$$

that $x \in C^2[r_1, \sigma(r_1))$. Hence, using Taylor's expansion with the remainder in the Lagrange form, we obtain

$$\begin{aligned} x(t) &= x(r_1) + f(r_1, x(r_1), u_\lambda(r_1))(t - r_1) \\ &+ f'_x(s_1, x(s_1), u_\lambda(r_1)) \cdot f(s_1, x(s_1), u_\lambda(r_1)) \frac{(t - r_1)^2}{2}, \end{aligned} \tag{3.11}$$

for some $s_1 \in [r_1, \sigma(r_1)]$. Here f'_x is the Jacobian matrix. It follows from (3.4) that

$$\max_{t \in [t_0, t_1]} |f'_x(t, x(t), u_\lambda(t)) f(t, x(t), u_\lambda(t))| \leq C_1^2. \tag{3.12}$$

Thus, when $t \in [r_1, \sigma(r_1))$, we obtain

$$|x(t) - \tilde{x}_\lambda(t)| \leq \int_{r_1}^{\sigma(r_1)} |f(t, x(t), u_\lambda(r_1))|\, dt \leq C_1 \mu(r_1). \tag{3.13}$$

But when $t = \sigma(r_1)$ we have

$$\tilde{x}_\lambda(\sigma(r_1)) = x_\lambda(r_1) + f(r_1, x_\lambda(r_1), u_\lambda(r_1))\mu(r_1) = x(r_1) + f(r_1, x(r_1), u_\lambda(r_1))\mu(r_1).$$

Thus, from (3.11) and (3.12) we get

$$|x(\sigma(r_1)) - \tilde{x}_\lambda(\sigma(r_1))| \leq C_1^2 \frac{\mu_\lambda^2(r_1)}{2}. \tag{3.14}$$

3) For $t \in [\sigma(r_1), r_2]$, it follows from (3.14) and Gronwall inequality

$$|\tilde{x}_\lambda(t) - x(t)| \leq \frac{\mu_\lambda^2(r_1)}{2} C_1^2 e^{K(r_2 - \sigma(r_1))}. \tag{3.15}$$

4) For $t \in [r_2, \sigma(r_2))$ we may argue the same way as for $t \in [r_1, \sigma(r_1))$ to get

$$|x(t) - \tilde{x}_\lambda(t)| \le \frac{\mu_\lambda^2(r_1)}{2} C_1^2 e^{K(r_2 - \sigma(r_1))} + \mu_\lambda(r_2) C_1. \tag{3.16}$$

$$|x(\sigma(r_2)) - \tilde{x}_\lambda(\sigma(r_2))| \le \frac{\mu_\lambda^2(r_1)}{2} C_1^2 [(1 + K\mu_\lambda(r_2)) e^{K(r_2 - \sigma(r_1))} + 1].$$

5) On the line segment $t \in [\sigma(r_2), r_{h_1}]$ we have

$$|\tilde{x}(r_{h_1}) - \tilde{x}(\sigma(r_2))| \le C_1(r_{h_1} - r_2) = C_1\mu_1,\ |x(r_{h_1}) - x(\sigma(r_2))| \le C_1\mu_1,$$

$$|\tilde{x}_\lambda(r_{h_1}) - x(r_{h_1})| \le 2C_1\mu_1 + (1 + K\mu_\lambda(r_2)) \frac{\mu_\lambda^2(r_1)}{2} C_1^2 e^{K(r_2 - \sigma(r_1))} + \frac{\mu_\lambda^2(r_2)}{2} C_1^2. \tag{3.17}$$

Continuing this procedure to the remaining intervals of Figure 1 for $t \in [r_{h_1}, r_5]$, we have the following estimate

$$\begin{aligned}
&|\tilde{x}_\lambda(t) - x(t)| \le ((1 + K\mu_\lambda(r_{h_2}))(1 + K\mu_\lambda(r_3)) 2C_1\mu_1 \\
&* e^{K((r_4 - \sigma(r_3)) + (r_3 - r_{h_1}))} + \frac{1}{2}(1 + K\mu_\lambda(r_{h_2})) e^{K((r_4 - \sigma(r_3))} \mu_\lambda^2(r_1) C_1^2 \\
&* e^{K((r_2 - \sigma(r_1)) + (r_3 - r_{h_1}))} (1 + K\mu_\lambda(r_2))(1 + K\mu_\lambda(r_3)) + \frac{1}{2}(1 + K\mu_\lambda(r_{h_2})) \\
&* e^{K((r_4 - \sigma(r_3))} \mu_\lambda^2(r_2) C_1^2 e^{K(r_3 - \sigma(r_{h_1}))} (1 + K\mu_\lambda(r_3)) + \frac{1}{2}(1 + K\mu_\lambda(r_{h_2})) \\
&* e^{K((r_4 - \sigma(r_3))} \mu_\lambda^2(r_3) C_1^2 + 2C_1\mu_2(1 + K\mu_\lambda(r_{h_2})) + \frac{1}{2}\mu_\lambda^2(r_{h_2}) C_1^2) e^{K(r_5 - \sigma(r_{h_2}))}.
\end{aligned} \tag{3.18}$$

Denote

$$\Pi := (1 + K\mu_\lambda(r_1))(1 + K\mu_\lambda(r_2))(1 + K\mu_\lambda(r_3))(1 + K\mu_\lambda(r_{h_2})) \dots (1 + K\mu_\lambda(r_N)),$$

where the product is taken over all right-scattered points of Figure 1. Then

$$\ln \Pi \le K(\mu_\lambda(r_1) + \mu_\lambda(r_2) + \mu_\lambda(r_3) \dots) \le K.$$

Note also that the sum of all powers of e that appear in (3.18) does not exceed K, since those arguments involve the lengths of disjoint subintervals of $[0, 1]$. Consequently, for $t \notin [r_k, \sigma(r_k))$ we have

$$|\tilde{x}_\lambda(t) - x(t)| \le \mu_\lambda \left(\Pi e^K C_1 + \frac{1}{4} \Pi C_1^2 e^K \right) \to 0,\ \lambda \to 0. \tag{3.19}$$

For $t \in [r_k, \sigma(r_k))$, arguing as in (3.19), we get

$$|\tilde{x}_\lambda(t) - x(t)| \le \mu_\lambda \left(\Pi e^K \left(C_1 + \frac{C_1^2}{4} \right) + 3C_1 \right). \tag{3.20}$$

Therefore, $|\tilde{x}_\lambda(t) - x(t)| \to 0, \lambda \to 0$, uniformly for $t \in [0, 1]$. Combining (3.8) and (3.10) for any time scale $\mathbb{T}_\lambda$ and admissible control $u_\lambda(t)$ for (2.4), there is an admissible control $\tilde{u}_\lambda(t)$ for (2.5), such that

$$|J_\lambda(u_\lambda) - J(\tilde{u}_\lambda)| \to 0,\ \lambda \to 0. \tag{3.21}$$

This completes the proof of Lemma 3.1. □

Lemma 3.2 *For any admissible control $u(\cdot)$ for the problem (2.5) and for every time scale $\mathbb{T}_\lambda$ there is an admissible control $u^\lambda_{ts}(\cdot)$ for the problem (2.4) such that*

$$|J(u) - J_\lambda(u^\lambda_{ts})| \to 0,\ \lambda \to 0. \tag{3.22}$$

Proof. Let $u^\lambda_{ts}(\cdot)$ be an arbitrary admissible control for the problem (2.4) and $x^\lambda_{ts}(\cdot)$ be the corresponding trajectory. Similarly, let $x(\cdot)$ be an admissible trajectory of the problem (2.5) which corresponds to the admissible control $u(\cdot)$. Then

$$\begin{aligned}\int_{[0,1)_{\mathbb{T}_\lambda}} L(t, x^\lambda_{ts}(t), u^\lambda_{ts}(t))\Delta t = \int_{[0,1)_{\mathbb{T}}\setminus \mathrm{RS}} L(t, x^\lambda_{ts}(t), u^\lambda_{ts}(t))\Delta t \\ + \sum_{r\in \mathrm{RS}} L(r, x^\lambda_{ts}(r), u^\lambda_{ts}(r))\mu(r).\end{aligned} \tag{3.23}$$

For $|x_0| \le R$ and $t \in [0,1]_{\mathbb{T}_\lambda}$ the estimates (3.4) hold for any fixed $R > 0$. Hence

$$\sum_{r\in \mathrm{RS}} L(r, x^\lambda_{ts}(r), u^\lambda_{ts}(r))\mu(r) \le C_1 \sum_{r\in \mathrm{RS}} \mu_\lambda(r), \tag{3.24}$$

uniformly for all $u^\lambda_{ts}(\cdot)$. In particular, the sum in (3.23) is convergent, similarly to (3.6). Once again, for every $\lambda > 0$ we choose $N(\lambda) \ge 1$ such that $\sum_{k=N+1} \mu(r_k) \le \frac{\mu_\lambda}{2}$. As before, denote $A = \bigcup_{k=N+1} [r_k, \sigma(r_k))$. Its Lebesgue measure is small: $|A| \le \frac{\mu_\lambda}{2}$. Introduce $B := [0,1]_{\mathbb{T}_\lambda}\setminus A$. In other word, B contains only finitely many right-scattered points $r_1, \ldots, r_N$. Now, for any admissible $u(\cdot)$ and $u^\lambda_{ts}(\cdot)$ we write

$$\begin{aligned}&\left|\int_0^1 L(t, x(t), u(t))dt - \int_{[0,1)_{\mathbb{T}_\lambda}} L(t, x^\lambda_{ts}(t), u^\lambda_{ts}(t))\Delta t\right| \\ &\le \mu_\lambda + \left|\int_{[0,1]\setminus A} L(t, x(t), u(t))dt - \int_B L(t, x^\lambda_{ts}(t), u^\lambda_{ts}(t))dt\right|.\end{aligned} \tag{3.25}$$

Fix $\varepsilon > 0$. By Luzin's theorem, there is function $u_\varepsilon(t)$, which is continuous on $[0,1]$ and such that $|A_\varepsilon| < \varepsilon$, where $A_\varepsilon := \{t \in [0,1] : u(t) \ne u_\varepsilon(t)\}$, $\lambda(A_\varepsilon) < \varepsilon$. Denote $B_\varepsilon = [0,1]\setminus A_\varepsilon$. Since f and L are uniformly continuous on the compact set $[0,1]\times \overline{B(0,C_1)}\times U$, for any $0 < \varepsilon_1 < \varepsilon$ there is $0 < \varepsilon_2 = \varepsilon_2(\varepsilon_1)$ such that if $|u - u_1| < \varepsilon_2$, then

$$|f(t,x,u) - f(t,x,u_1)| + |L(t,x,u) - L(t,x,u_1)| < \varepsilon_1 \tag{3.26}$$

for any $t \in [0,1]$ and $|x| \le C_1$. Without loss of generality, assume that $\varepsilon_2 < \varepsilon_1$. Note that $u_\varepsilon(t)$ is uniformly continuous on $[0,1]$. Therefore, one can find $0 < \varepsilon_3 < \varepsilon_2$ such that if $|t - s| < \varepsilon_3$, then $|u_\varepsilon(t) - u_\varepsilon(s)| < \varepsilon_2$. Note that, for sufficiently small λ, $\mu_\lambda < \varepsilon_3$.

We are now in position to construct a new admissible control u^λ_c, which would take into account the structure of $\mathbb{T}_\lambda$. The construction is done separately on each of the intervals as follows:

- ▷ for $t \in A$, i.e. for $t \in [r_i, \sigma(r_i)), i \ge N(\lambda) + 1$, set $u^\lambda_c(t) := u(r_i)$;
- ▷ for $t \in [\sigma(r_i), r_{i+1}), i = 1, ..., N(\lambda) - 1$, set $u^\lambda_c(t) = u(t)$;
- ▷ if $t \in [r_i, \sigma(r_i))$ and $B_\varepsilon \cap [r_i, \sigma(r_i)) = \emptyset$, define $u^\lambda_c(t) := u(r_i)$, $1 \le i \le N$;

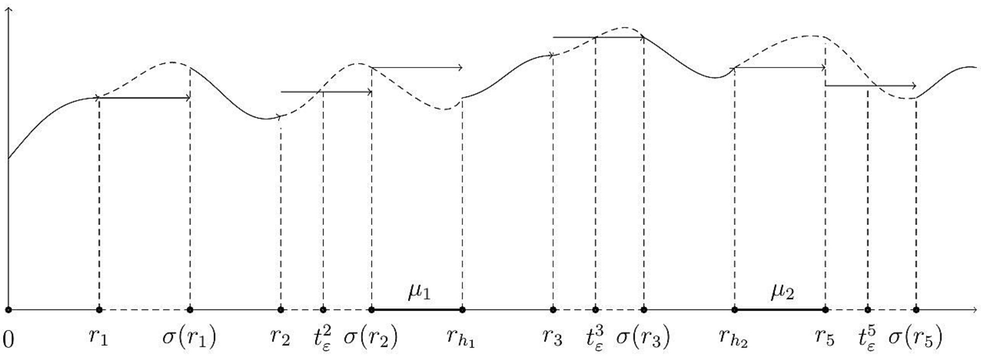

Figure 2: The construction of control.

▹ finally, if $t \in [r_i, \sigma(r_i))$ and $B_\varepsilon \cap [r_i, \sigma(r_i)) \neq \emptyset$, for such $1 \leq i \leq N$ set $u_c^\lambda(t) := u_\varepsilon(t_\varepsilon^i)$, where t_ε^i is an arbitrary point of the set $B_\varepsilon \cap [r_i, \sigma(r_i))$. Since $t_\varepsilon^i \in B_\varepsilon$, then $u_\varepsilon(t_\varepsilon^i) = u(t_\varepsilon^i) \in U$, therefore the control $u_c^\lambda(t)$ on $[r_i, \sigma(r_i))$ is admissible.

Figure 2 visualizes this construction. Here, on the x-axis, the intervals $[r_i, \sigma(r_i))$, $0 \leq i \leq N$, are denoted with dashed lines, the set A is comprised of solid boldface intervals, and the intervals, on which $u(t)$ is continuous, are denoted with solid thin lines. In addition, the graph of $u(t)$ is dashed, and the graph of the new control $u_c^\lambda(t)$ is solid.

In what follows, we are going to analyze the time scale, depicted in Figure 2. The analysis is similar in other cases. Let $x_c^\lambda(t)$ be an admissible trajectory for (2.5). By construction, $u_c^\lambda(t)$ is an extension (in the sense of (3.1)) of some admissible control $u_{ts}^\lambda(t)$ on the time scale $\mathbb{T}_\lambda$. Then, it follows from (3.21) that

$$|J_\lambda(u_{ts}^\lambda) - J(u_c^\lambda)| \to 0, \ \lambda \to 0. \tag{3.27}$$

Let us show that

$$|J(u) - J(u_c^\lambda)| \to 0, \ \lambda \to 0. \tag{3.28}$$

We have

$$\left|\int_0^1 (L(t, x(t), u(t)) - L(t, x_c^\lambda(t), u_c^\lambda(t)))dt\right| \leq \left|\int_{A_\varepsilon} (L(t, x(t), u(t)) - L(t, x_c^\lambda(t), u_c^\lambda(t)))dt\right| \\ + \left|\int_{B_\varepsilon} (L(t, x(t), u(t)) - L(t, x_c^\lambda(t), u_c^\lambda(t)))dt\right|. \tag{3.29}$$

The first term in the right-hand side of (3.29) can be bounded by $C_1(R)|A_\varepsilon| \leq C_1(R)\varepsilon$. We now estimate the second term in the right-hand side of (3.29):

$$\left|\int_{B_\varepsilon} (L(t, x(t), u(t)) - L(t, x_c^\lambda(t), u_c^\lambda(t)))dt\right| \leq K \int_{B_\varepsilon} |x(t) - x_c^\lambda(t)|dt \\ + \int_{B_\varepsilon} \left|L(t, x_c^\lambda(t), u(t)) - L(t, x_c^\lambda(t), u_c^\lambda(t))dt\right|. \tag{3.30}$$

Next,

$$\int_{B_\varepsilon} \left|L(t,x_c^\lambda(t),u(t)) - L(t,x_c^\lambda(t),u_c^\lambda(t))\right| dt \le C_1(R)\mu_\lambda + \int_{B_\varepsilon\cap\bar{A}} \left|L(t,x_c^\lambda(t),u(t)) - L(t,x_c^\lambda(t),u_c^\lambda(t))\right| dt \tag{3.31}$$

and

$$\begin{aligned}\int_{B_\varepsilon\cap\bar{A}} &|L(t,x_c^\lambda(t),u(t)) - L(t,x_c^\lambda(t),u_c^\lambda(t))|dt \\ &= \sum_{i=1}^{N-1}\int_{[\sigma(r_i),r_{i+1})\cap B_\varepsilon} |L(t,x_c^\lambda(t),u(t)) - L(t,x_c^\lambda(t),u_c^\lambda(t))|dt \\ &\quad + \sum_{i=1}^{N}\int_{[r_i,\sigma(r_i))\cap B_\varepsilon} |L(t,x_c^\lambda(t),u(t)) - L(t,x_c^\lambda(t),u_c^\lambda(t))|dt. \end{aligned}\tag{3.32}$$

By construction of u_c^λ, the first term in the right-hand side of (3.32) is zero, and some of the terms in the second sum may vanish if there are no points from the set B_ε in the interval $[r_i,\sigma(r_i))$. Since $\mu_\lambda < \varepsilon_3$, by uniform continuity of $u_\varepsilon(t)$ and (3.26), we have

$$\sum_{i=1}^{N}\int_{[r_i,\sigma(r_i))\cap B_\varepsilon} |L(t,x_c^\lambda(t),u(t)) - L(t,x_c^\lambda(t),u_c^\lambda(t))|dt \le \varepsilon_1 \sum_{i=1}^{N}\mu(r_i) \le \varepsilon_1. \tag{3.33}$$

Then from (3.30), (3.31) and (3.33) we have

$$\int_{B_\varepsilon} |L(t,x(t),u(t)) - L(t,x_c^\lambda(t),u_c^\lambda(t))|dt \le K\int_{B_\varepsilon}|x(t)-x_c^\lambda(t)|dt + C_1(R)\mu_\lambda + \varepsilon_1. \tag{3.34}$$

It remains to estimate the difference $|x(t) - x_c^\lambda(t)|$ in (3.34). We are going to do this in the setting of Figure 2, the analysis in the general case is analogous.

1) For $t\in[0,r_1]$, $u(t) = u_c^\lambda(t)$, therefore $x(t) = x_c^\lambda(t)$.

2) For $t\in(r_1,\sigma(r_1)]$, we have

$$\begin{aligned}|x(t) - x_c^\lambda(t)| &\le \int_{[r_1,\sigma(r_1))\cap A_\varepsilon} \left|f(t,x_c^\lambda(t),u(t)) - f(t,x_c^\lambda(t),u(r_1))\right| dt \\ &+ \int_{r_1}^{t} K|x(s)-x_c^\lambda(s)|ds + \int_{[r_1,\sigma(r_1))\cap B_\varepsilon} \left|f(t,x_c^\lambda,u_\varepsilon(t)) - f(t,x_c^\lambda(t),u_\varepsilon(t_\varepsilon^1))\right| dt \\ &\le \int_{r_1}^{t} K|x(s)-x_c^\lambda(s)|ds + 2C_1(R)\,|[r_1,\sigma(r_1))\cap A_\varepsilon| + \varepsilon_1\mu(r_1),\end{aligned}$$

where we used the uniform continuity of f on $[0,1]\times\overline{B(0,C_1)}\times U$. Then by Gronwall inequality, we obtain

$$|x(t)-x_c^\lambda(t)| \le (\varepsilon_1\mu(r_1) + 2C_1(R)|[r_1,\sigma(r_1))\cap A_\varepsilon|)e^{K\mu(r_1)} = \delta_1 e^{K\mu(r_1)}, \tag{3.35}$$

where $\delta_1 = \varepsilon_1\mu(r_1) + 2C_1(R)|[r_1,\sigma(r_1))\cap A_\varepsilon|$.

3) For $t \in [\sigma(r_1), r_2)$ we have $|x(t) - x_c^\lambda(t)| \le \delta_1 e^{K\mu(r_1)} e^{K(r_2-\sigma(r_1))}$.

Continuing this procedure to the remaining intervals of Figure 2 for $t \in [r_2, \sigma(r_5))$ we have the following estimate

$$\begin{aligned}
&|x(t) - x_c^\lambda(t)| \le (\varepsilon_1 \mu(r_1) + 2C_1(R)\,|[r_1, \sigma(r_1)) \cap A_\varepsilon| \\
&\times e^{\{K(\mu(r_1)+(r_2-\sigma(r_1))+\mu(r_2)+\mu(r_3)+(r_{h_2}-\sigma(r_3))+\mu(r_5)+(r_3-r_{h_1}))\}} \\
&+ (2C_1(R)\,|[r_2, \sigma(r_2)) \cap A_\varepsilon| + \varepsilon_1 \mu(r_2))e^{\{K(\mu(r_2)+\mu(r_3)+(r_{h_2}-\sigma(r_3))+\mu(r_5)+(r_3-r_{h_1}))\}} \\
&+ 2C_1(R)\mu_1 e^{\{K((r_3-r_{h_1})+\mu(r_3)+(r_{h_2}-\sigma(r_3))+\mu(r_5))\}} + (2C_1(R)\,|[r_3, \sigma(r_3)) \cap A_\varepsilon| \\
&+ \varepsilon_1 \mu(r_3))e^{\{K(\mu(r_3)+(r_{h_2}-\sigma(r_3))+\mu(r_5))\}} + 2C_1(R)\,|[r_5, \sigma(r_5)) \cap A_\varepsilon|\, e^{K\mu(r_5)} \\
&+ 2C_1(R)\mu_2 e^{K\mu(r_5)} + \varepsilon_1 \mu(r_5) e^{K\mu(r_5)}.
\end{aligned} \tag{3.36}$$

Once again, the sum of all powers of e in (3.36) does not exceed K, since it is the sum of lengths of disjoint subintervals of $[0,1]$. Altogether, for $t \in [0,1]$ we have

$$|x(t) - x_c^\lambda(t)| \le (\varepsilon_1 + 2C_1(R)\varepsilon + C_1(R)\mu_\lambda)e^K. \tag{3.37}$$

From (3.29)–(3.34) we get

$$\begin{aligned}
&\left| \int_0^1 (L(t, x(t), u(t)) - L(t, x_c^\lambda(t), u_c^\lambda(t))) \right| dt \\
&\qquad \le K(\varepsilon_1 + 2C_1(R)\varepsilon + C_1(R)\mu_\lambda)e^K + C_1(R)\mu_\lambda + \varepsilon_1 + C_1(R)\varepsilon.
\end{aligned} \tag{3.38}$$

Since ε and ε_1 can be chosen arbitrarily small, we have $|J(u) - J(u_c^\lambda)| \to 0, \lambda \to 0$, hence the proof of Lemma 3.2 follows from (3.27). □

We now return to the proof of Theorem 2.1. In Lemma 3.1 we have shown that for an arbitrary time scale $\mathbb{T}_\lambda$ and an arbitrary admissible control for the problem (2.4) $u_\lambda(t)$, there is an admissible control $\tilde{u}_\lambda$ for the problem (2.5), such that $|J_\lambda(u_\lambda) - J(\tilde{u}_\lambda)| = \varphi(\lambda) \to 0$, $\lambda \to 0$. Consequently, $J(\tilde{u}_\lambda) \le J_\lambda(u_\lambda) + \varphi(\lambda)$. Using the definition of the value function, we have $V(0,x) \le J_\lambda(u_\lambda) + \varphi(\lambda)$. We may take the infimum over all admissible controls to get $V(0,x) \le V_\lambda(0,x) + \varphi(\lambda)$. There exists a uniformly converging subsequence $V_{\lambda_n}(0,x)$: $V_{\lambda_n}(0,x) \rightrightarrows V_0(0,x)$, $|x| \le r$, with $\lambda_n \to 0$ as $n \to \infty$. Passing to the limit as $\lambda_n \to 0$, we have $V(0,x) \le V_0(0,x)$.

Let us show that inequality $V(0,x) < V_0(0,x)$ is impossible. By contradiction, assume $V(0,x) < V_0(0,x)$. Then there are $\delta > 0$ and $n_0 \ge 1$ such that for $\lambda_n \le \lambda_{n_0}$ we have $V_{\lambda_n}(0,x) > V(0,x) + \delta$. However, for such $\delta > 0$ we may construct an admissible control $u(t)$ for the system (2.5), such that $J(u) + \frac{\delta}{2} < V_{\lambda_n}(0,x)$. For such $u(t)$ we now apply Lemma 3.2 to construct an admissible control $u_{ts}^{\lambda_n}$, such that (3.22) holds. Then for sufficiently small λ_n we have $J_{\lambda_n}(u_{ts}^{\lambda_n}) < V_{\lambda_n}(0,x)$, which leads to a contradiction. Therefore, $V(0,x) = V_0(0,x)$, i.e. any convergent sequence $V_{\lambda_n}(0,x)$ has $V(0,x)$ as its limit. Since the family $V_\lambda(0,x)$ is compact, we have $V_\lambda(0,x) \to V_0(0,x)$, $\lambda \to 0$, which proves Theorem 2.1. □

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Stability Analysis of Impulsive Hopfield-Type Neuron System on Time Scale

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Abstract: Impulsive Hopfield-type neural systems on time scale are investigated. Sufficient conditions for the existence and uniqueness of the equilibrium state are obtained. Based on the generalized Lyapunov function method the sufficient conditions of global exponential stability are established for the neuron system under investigation. Efficiency of the obtained sufficient conditions is illustrated by a numerical example.

Keywords: *stability; time scale; Hopfield neural networks; impulsive system; exponential stability; Lyapunov function.*

Mathematics Subject Classification (2010): 92B20, 93D05, 93D30, 34K45, 34N05.

1 Introduction

Hopfield neural networks and their generalizations are important models of biological processes that are widely used now for solution of the applied problems in different areas of the modern technologies such as the optoelectronics, image reconstruction, speech synthesis, computer vision [1]– [6], and in the solution of different optimization problems, see also [7, 8].

Neural networks with impulses, both continuous and discrete ones, are widely used in the modeling of artificial intelligence, in robotics and electronics, and are intensively studied lately [9]– [13], with the most results obtained for neural networks with continuous time. Therefore, it makes sense to consider impulsive neural systems on time scale, which allows a simultaneous description of the system dynamics both in the discrete and the

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continuous case. In addition, this approach allows us to obtain new results for discrete neural systems, similar to those already known for the continuous case.

An extensive literature is devoted to the differential systems with impulsive action on general time scale [14]– [16] while the neural networks with pulses on the time scale are not well studied [17].

The purpose of this paper is to obtain the sufficient conditions of the global exponential stability of the equilibrium state for the impulsive neural Hopfield network on time scale. The study was carried out within the framework of the generalized second Lyapunov method on the basis of the scalar non-autonomous function on time scale.

2 Main Definitions and Necessary Theorems

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$. Fundamental notions and theorems of mathematical analysis on time scale, as well as the definitions of the derivative and the integral, the rules of differentiation and integration, the definitions and properties of rd-contiguous function, regressive function, the jump operator $\sigma(t)$, the graininess of the time scale $\mu(t)$ and the exponential function are explicity given in [18]– [20].

We need the following properties of the Δ-derivative.

Theorem 2.1 *Assume that f, g are Δ-differentiable at $t \in \mathbb{T}^k$. Then the following assertions are true:*

(1) *the product fg is Δ-differentiable at $t \in \mathbb{T}^k$ and*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t));$$

(2) $f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t)$;

(3) *if $f^{\Delta}(t) \geq 0$ then f is non-decreasing on $\mathbb{T}^k$.*

We denote by $e_p(t, t_0)$ an exponential function on time scale. Further the following properties of an exponential function will be used.

Theorem 2.2 *If $p \in \mathcal{R}^+$, $\lambda > 0$, then for all $t, t_0 \in \mathbb{T}$ and $t \geq t_0$*

(1) $e_p(t_0, t_0) = 1$, $e_p(t, t_0) > 0$;

(2) $e_p(t, t_0) = 1/e_p(t_0, t)$;

(3) $e_p^{\Delta}(t, t_0) = p(t)e_p(t, t_0)$;

(4) $e_p(\sigma(t), t_0) = (1 + \mu(t)p(t))e_p(t, t_0)$;

(5) $\frac{1}{e_p(t,t_0)} = e_{\ominus p}(t, t_0)$, *where* $\ominus p \in \mathcal{R}^+$, $(\ominus p)(t) = -\frac{p(t)}{1+\mu(t)p(t)}$;

(6) $e_{\ominus\lambda}(t, t_0) \leq 1$, $\lim_{t\to+\infty} e_{\ominus\lambda}(t, t_0) = 0$ (*see* [21]);

(8) *if* $\mathbb{T} = \mathbb{R}$, *that* $e_{\ominus\lambda}(t, t_0) = e^{-\lambda(t-t_0)}$;

(9) *if* $\mathbb{T} = \mathbb{Z}$, *that* $e_{\ominus\lambda}(t, t_0) = (1 + \lambda)^{-(t-t_0)}$.

Here $\mathcal{R}^+$ is the set of all rd-continuous and positively regressive functions $f : \mathbb{T} \to \mathbb{R}$.

We denote by $\|x\| = (\sum_{i=1}^n x_i^2)^{1/2}$ an Euclidean vector norm of the vector $x \in \mathbb{R}^n$, $\|A\| = (\lambda_M(A^{\mathrm{T}}A))^{1/2}$ denotes a matrix norm of the matrix $A = \{a_{ij}\} \in \mathbb{R}^{n\times n}$, $\lambda_M(A)$ is a maximal eigenvalue of the matrix A, $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \le t \le b\}$ for $a, b \in \mathbb{T}$, the intervals $[a, b)_{\mathbb{T}}$, $(a, b]_{\mathbb{T}}$, $[a, +\infty)_{\mathbb{T}}$ are defined similarly.

Further we shall need the following result.

Lemma 2.1 *Let* $\widetilde{C} = \widetilde{C}[a, b]_{\mathbb{T}}$ *be a set of all continuous on* $[a, b]_{\mathbb{T}}$ *functions* $f\colon [a, b]_{\mathbb{T}} \to \mathbb{R}^n$, $p > 0$ *and* $\|\cdot\|^{\sim}$ *be a norm defined on* $\widetilde{C}$ *by the formula*

$$\|f\|^{\sim} = \sup_{t\in[a,b]_{\mathbb{T}}} \{p^{-1}\, e_{\ominus p}(t, a)\|f(t)\|\}.$$

Then $\left(\widetilde{C}, \|\cdot\|^{\sim}\right)$ *is a Banach space.*

Proof. Let $\|\cdot\|_1$ be a norm given on the set of $\widetilde{C}$ by the formula $\|f\|_1 = \sup_{t\in[a,b]_{\mathbb{T}}} \|f(t)\|$. We show that the norms $\|\cdot\|^{\sim}$ and $\|\cdot\|_1$ are equivalent. For all $f \in \widetilde{C}$ and $t \in [a, b]_{\mathbb{T}}$ we have $p^{-1}e_{\ominus p}(t, a)\|f(t)\| \le p^{-1}\|f(t)\| \le p^{-1}\|f\|_1$ or $\|f\|^{\sim} \le p^{-1}\|f\|_1$. Since the function $e_p(t, a)$ is continuous on $[a, b]_{\mathbb{T}}$, there exists a constant $\mathcal{E} > 0$ such that $e_p(t, a) \le \mathcal{E}$ for all $t \in [a, b]_{\mathbb{T}}$, whence for any $f \in \widetilde{C}$ we have $\|f(t)\| = p\, e_p(t, a)p^{-1}e_{\ominus p}(t, a)\|f\| \le p\,\mathcal{E}p^{-1}e_{\ominus p}(t, a)\|f(t)\| \le p\,\mathcal{E}\|f\|^{\sim}$ or $\|f\|_1 \le p\,\mathcal{E}\|f\|^{\sim}$. Thus, the norms $\|\cdot\|^{\sim}$ and $\|\cdot\|_1$ are equivalent.

As is known from the mathematical analysis, since $[a, b]_{\mathbb{T}}$ is a compact set, the space $\left(\widetilde{C}, \|\cdot\|_1\right)$ is a Banach space. Consequently, $\left(\widetilde{C}, \|\cdot\|^{\sim}\right)$ is also a Banach space. Lemma 2.1 is proved.

3 Impulsive Neural Network on Time Scale

Let $\mathbb{T}$ be an arbitrary time scale, $\sup \mathbb{T} = +\infty$, the sequence $\{t_k\}_{k=1}^{+\infty} \subset \mathbb{T}$ so that $t_1 < t_2 < \ldots$, $t_k \to +\infty$ and $k \to +\infty$ and points t_k are dense.

We consider the impulsive neural system

$$x^{\Delta}(t) = -Bx(t) + Ts(x(t)) + u, \quad t \in \mathbb{T}_\tau, \quad t \ne t_k, \tag{1}$$

$$x(t_k^+) = x(t_k) + I_k(x(t_k)), \quad k \in \mathbb{N}, \tag{2}$$

with the initial condition

$$x(t_0) = x_0, \quad t_0 \in \mathbb{T}_\tau, \quad x_0 \in \mathbb{R}^n. \tag{3}$$

Here $\mathbb{T}_\tau = [\tau, +\infty)_{\mathbb{T}}$, $\tau \in \mathbb{T}$, $\tau < t_1$, $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, x_i is the activation of the i-th neuron, $T = \{t_{ij}\} \in \mathbb{R}^{n\times n}$, the components t_{ij} describe the interaction between the ith and jth neurons, $s : \mathbb{R}^n \to \mathbb{R}^n$, $s(x) = (s_1(x_1), s_2(x_2), \ldots, s_n(x_n))^{\mathrm{T}}$, the function s_i describes the response of the ith neuron, $B \in \mathbb{R}^{n\times n}$, $B = \mathrm{diag}\,\{b_1, b_2, \ldots, b_n\}$, $b_i > 0$, $i = 1, 2, \ldots, n$, $u \in \mathbb{R}^n$ is a constant external input vector, the function $I_k\colon \mathbb{R}^n \to \mathbb{R}^n$ describes impulsive perturbations of the neural system.

By the solution of the impulsive system (1), (2) we mean the function $x(t)$, which for $t \ne t_k$ satisfies the equation (1) and for $t = t_k$ satisfies the equation (2), where $x(t_k^+) = \lim_{t\to t_k+0,\ t\in\mathbb{T}} x(t)$, $x(t_k) = x(t_k^-) = \lim_{t\to t_k-0,\ t\in\mathbb{T}} x(t)$ are one-sided right-hand and left-hand limits of the function $x(t)$, respectively.

Concerning the system (1) and the time scale $\mathbb{T}$, we introduce the following assumptions.

H_1. There are positive constants $l_i > 0$, $i = 1, 2, \ldots, n$ such that $|s_i(u) - s_i(v)| \le l_i|u - v|$ for all $u, v \in \mathbb{R}$.

H_2. There is a constant $\mu^* > 0$ such that $\mu(t) \le \mu^*$ for all $t \in \mathbb{T}_\tau$.

Existence conditions for a unique equilibrium state of the system (1) without impulses be given by the following theorem, the proof of which is similar to the proof of Theorem 1 from [22].

Theorem 3.1 *Let the assumption* H_1 *be valid and there exist a constant* $d_i > 0$, $i = 1, 2, \ldots, n$, *such that the inequalities*

$$b_i - \frac{1}{2}\sum_{j=1}^{n}\big(l_j|t_{ij}| + \frac{d_j}{d_i}\, l_i|t_{ji}|\big) > 0, \quad i = 1, 2, \ldots, n, \tag{4}$$

are true. Then there is a unique equilibrium state of the system (1).

The equilibrium state of the system (1), (2) will be referred to as the constant function $x(t) \equiv x^*$, which is the solution of the system (1), (2). Using Theorem 3.1 it is easy to get the following result.

Theorem 3.2 *Let the assumption* H_1 *and inequalities* (4) *be valid and let* $x(t) \equiv x^*$ *be an equilibrium state of the system* (1). *If* $I_k(x^*) = 0$ *for all* $k \in \mathbb{N}$, *then* $x(t) \equiv x^*$ *is a unique equilibrium state of the system* (1), (2).

We prove the following theorem on the existence and uniqueness of the solution of impulsive neural system.

Theorem 3.3 *Let the assumption* H_1 *be valid, then there exists a unique solution of problem* (1) – (3) *on* $[t_0, +\infty)_{\mathbb{T}}$ *for all initial data* $(t_0, x_0) \in \mathbb{T}_\tau \times \mathbb{R}^n$.

Proof. For an arbitrary $t_0 \in \mathbb{T}_\tau$ two cases are possible: $t_0 \in [\tau, t_1)_{\mathbb{T}}$ or $t_0 \in [t_{k-1}, t_k)_{\mathbb{T}}$ for some $k = 2, 3, \ldots$. We first choose $t_0 \in [\tau, t_1)_{\mathbb{T}}$ and we denote $L = \max\{l_1, l_2, \ldots, l_n\}$, $\gamma = \|B\| + L\|T\|$, $p = \gamma + 1$.

Let $\widetilde{C}_1 = \widetilde{C}_1[t_0, t_1]$ be the space of continuous functions $f\colon [t_0, t_1]_{\mathbb{T}} \to \mathbb{R}^n$ with the norm

$$\|f\|_1^{\sim} = \sup_{t \in [t_0,\, t_1]_{\mathbb{T}}} \{p^{-1}\, e_{\ominus p}(t, t_0)\|f(t)\|\}.$$

Consider the operator $F_1\colon \widetilde{C}_1 \to \widetilde{C}_1$ acting according to the formula

$$F_1(x)(t) = x_{01} + \int_{t_0}^{t} [-Bx(\lambda) + Ts(x(\lambda)) + u]\Delta\lambda,$$

where $x_{01} = x_0$. The function $-Bx(t) + Ts(x(t)) + u$ is continuous on the segment $[t_0, t_1]_{\mathbb{T}}$, hence it is rd-continuous on $[t_0, t_1]_{\mathbb{T}}$. In accordance with Theorem 1.74 from [18] the function $F_1(x)(t)$ is differentiable on $[t_0, t_1]_{\mathbb{T}}$ (and, as a consequence, it is continuous on $[t_0, t_1]_{\mathbb{T}}$) and

$$[F_1(x)(t)]^{\Delta} = -Bx(t) + Ts(x(t)) + u, \quad t \in [t_0, t_1]_{\mathbb{T}}.$$

We verify the fulfillment of the conditions of the contraction map principle. For any $x, y \in \widetilde{C}_1$ for all $t \in [t_0, t_1]_{\mathbb{T}}$ we obtain the inequalities

$$\begin{aligned}
\|F_1(x)(t) - F_1(y)(t)\| &= \| \int_{t_0}^{t} [-B(x(\lambda) - y(\lambda)) + Ts(x(\lambda) - y(\lambda))]\Delta\lambda \| \le \\
&\le \|B\| \int_{t_0}^{t} \|x(\lambda) - y(\lambda)\|\Delta\lambda + L\|T\| \int_{t_0}^{t} \|x(\lambda) - y(\lambda)\|\Delta\lambda = \\
&= \gamma \int_{t_0}^{t} \|x(\lambda) - y(\lambda)\|\Delta\lambda = \gamma \int_{t_0}^{t} p\, e_p(\lambda, t_0) p^{-1} e_{\ominus p}(\lambda, t_0) \|x(\lambda) - y(\lambda)\|\Delta\lambda \le \\
&\le \gamma \sup_{\lambda \in [t_0,\, t_1]_{\mathbb{T}}} \{p^{-1} e_{\ominus p}(\lambda, t_0)\|x(\lambda) - y(\lambda)\|\} \int_{t_0}^{t} p\, e_p(\lambda, t_0)\Delta\lambda = \\
&= \gamma \|x - y\|_1^{\sim} (e_p(t, t_0) - 1) \le \gamma\, e_p(t, t_0)\|x - y\|_1^{\sim},
\end{aligned}$$

whence we have

$$\begin{aligned}
\frac{1}{p}\, e_{\ominus p}(t, t_0)\|F_1(x)(t) - F_1(y)(t)\| &\le \frac{\gamma}{p}\, \|x - y\|_1^{\sim}, \\
\sup_{t \in [t_0,\, t_1]_{\mathbb{T}}} \{p^{-1} e_{\ominus p}(t, t_0)\|F_1(x)(t) - F_1(y)(t)\|\} &\le \frac{\gamma}{p}\, \|x - y\|_1^{\sim}, \\
\|F_1(x) - F_1(y)\|_1^{\sim} &\le \frac{\gamma}{\gamma + 1}\, \|x - y\|_1^{\sim}.
\end{aligned}$$

Thus, the map F_1 is a contraction and consequently, there exists a unique fixed point $\widetilde{x}_1 \in \widetilde{C}_1$ of the operator F_1 for which we have

$$\begin{aligned}
&[\widetilde{x}_1(t)]^{\Delta} = -B\widetilde{x}_1(t) + Ts(\widetilde{x}_1(t)) + u, \quad t \in [t_0, t_1]_{\mathbb{T}}, \\
&\widetilde{x}_1(t_0) = x_{01}.
\end{aligned}$$

Now let $\widetilde{C}_2 = \widetilde{C}_2[t_1, t_2]$ be a space of continuous functions $f\colon [t_1,\, t_2]_{\mathbb{T}} \to \mathbb{R}^n$ with the norm

$$\|f\|_2^{\sim} = \sup_{t \in [t_1,\, t_2]_{\mathbb{T}}} \{p^{-1} e_{\ominus p}(t, t_1)\|f(t)\|\}.$$

Consider the operator $F_2\colon \widetilde{C}_2 \to \widetilde{C}_2$ acting according to the formula

$$F_2(x)(t) = x_{02} + \int_{t_1}^{t} [-Bx(\lambda) + Ts(x(\lambda)) + u]\Delta\lambda,$$

where $x_{02} = \widetilde{x}_1(t_1) + I_1(\widetilde{x}_1(t_1))$. As above, there exists a unique fixed point $\widetilde{x}_2 \in \widetilde{C}_2$ of the operator F_2 for which we obtain that

$$\begin{aligned}
&[\widetilde{x}_2(t)]^{\Delta} = -B\widetilde{x}_2(t) + Ts(\widetilde{x}_2(t)) + u, \quad t \in [t_1, t_2]_{\mathbb{T}}, \\
&\widetilde{x}_2(t_1) = x_{02}.
\end{aligned}$$

Similarly, at the kth step, let $\widetilde{C}_k = \widetilde{C}_k[t_{k-1}, t_k]$ be a space of continuous functions $f\colon [t_{k-1},\, t_k]_{\mathbb{T}} \to \mathbb{R}^n$ with the norm

$$\|f\|_k^{\sim} = \sup_{t \in [t_{k-1},\, t_k]_{\mathbb{T}}} \{p^{-1} e_{\ominus p}(t, t_{k-1})\|f(t)\|\},$$

and $F_k\colon \widetilde{C}_k \to \widetilde{C}_k$ is the operator acting according to the formula

$$F_k(x)(t) = x_{0k} + \int_{t_{k-1}}^{t} [-Bx(\lambda) + Ts(x(\lambda)) + u]\Delta\lambda,$$

where $x_{0k} = \widetilde{x}_{k-1}(t_{k-1}) + I_{k-1}(\widetilde{x}_{k-1}(t_{k-1}))$. As above, there exists a unique fixed point $\widetilde{x}_k \in \widetilde{C}_k$ of the operator F_k and

$$\begin{aligned}&[\widetilde{x}_k(t)]^{\Delta} = -B\widetilde{x}_k(t) + Ts(\widetilde{x}_k(t)) + u, \quad t \in [t_{k-1}, t_k]_{\mathbb{T}},\\ &\widetilde{x}_k(t_1) = x_{0k}.\end{aligned}$$

We now consider the function

$$x(t) = \begin{cases} \widetilde{x}_1(t), & t \in [\tau, t_1]_{\mathbb{T}},\\ \widetilde{x}_k(t), & t \in (t_{k-1}, t_k]_{\mathbb{T}}, \quad k = 2, 3, \dots. \end{cases} \tag{5}$$

It is clear that the function (5) is a solution of the Cauchy problem (1)-(3) on $[t_0, +\infty)_{\mathbb{T}}$ and moreover, it is unique.

The case $t_0 \in [t_{k-1}, t_k)_{\mathbb{T}}$ for some $k = 2, 3, \dots$ is investigated similarly. Theorem 3.3 is proved.

4 Stability of the Neural Network

Let $x(t) \equiv x^*$ be an isolated equilibrium state of the system (1), (2).

Definition 4.1 The equilibrium state $x(t) \equiv x^*$ of the system (1), (2) is called globally uniformly exponentially stable, if there exist constants $p > 0$, $\alpha > 0$ and $\mathcal{N} = \mathcal{N}(x_0) > 0$ such that $\|x(t; t_0, x_0) - x^*\| < \mathcal{N}(e_{\ominus p}(t, t_0))^{\alpha}$ for all $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{T}_{\tau}$ and $t \in [t_0, +\infty)_{\mathbb{T}}$.

We make the change of variables $y(t) = x(t) - x^*$ and rewrite the initial problem (1)–(3) in the form

$$y^{\Delta}(t) = -By(t) + Tg(y(t)), \quad t \in \mathbb{T}_{\tau}, \quad t \neq t_k, \tag{6}$$

$$y(t_k^+) = y(t_k) + J_k(y(t_k)), \quad k \in \mathbb{N}, \tag{7}$$

$$y(t_0; t_0, y_0) = y_0, \quad t_0 \in \mathbb{T}_{\tau}, \quad y_0 \in \mathbb{R}^n,$$

where

$$y \in \mathbb{R}^n, \quad g(y) = (g_1(y_1), g_2(y_2), \dots, g_n(y_n))^{\mathrm{T}}, \;\; J_k(y) = (J_{k1}(y), J_{k2}(y), \dots, J_{kn}(y))^{\mathrm{T}},$$

$$g(y) = s(y + x^*) - s(x^*), \quad J_k(y) = I_k(y + x^*) - I_k(x^*).$$

It is clear that the behavior of the solution $x(t)$ of the system (1), (2) in the neighborhood of the equilibrium state x^* is equivalent to the behavior of solution $y(t)$ of the system (6), (7) in the neighborhood of zero.

If for the system (1) the assumption H_1 is valid, then for the system (6), (7) the following statements are true.

G_1. There are positive constants $l_i > 0$ such that $|g_i(r) - g_i(v)| \le l_i|\, r - v|$, for all $r, v \in \mathbb{R}$, $i = 1, 2, \dots, n$.

G_2. $g(0) = 0,\ J_k(0) = 0,\ k \in \mathbb{N}$.

Theorem 4.1 *We assume that assumptions* G_1, G_2 *and* H_2 *are valid. Let for all* $y \in \mathbb{R}^n$ *the inequalities*

$$J_{ki}^2(y) + 2y_i J_{ki}(y) \le 0, \quad k \in \mathbb{N}, \quad i = 1, 2, \dots, n,$$

are true and there exist constants $d_i > 0,\ i = 1, 2, \dots, n$ *such that the inequalities*

$$b_i - \frac{1}{2} \sum_{j=1}^n (l_j |t_{ij}| + \frac{d_j}{d_i} l_i |t_{ji}|) - \frac{1}{2} \mu^*(n+1)(b_i^2 + l_i^2 \sum_{j=1}^n t_{ji}^2) > 0, \quad i = 1, 2, \dots, n, \tag{8}$$

are valid. Then the equilibrium state $y(t) \equiv 0$ *of the system* (6), (7) *is globally uniformly exponentially stable.*

Proof. We denote

$$\xi_i = \sum_{j=1}^n (d_i l_j |t_{ij}| + d_j l_i |t_{ji}|),$$
$$\nu_i = (n+1)(d_i b_i^2 + \eta_i),$$
$$\eta_i = d_i l_i^2 \sum_{j=1}^n t_{ji}^2, \quad i = 1, 2, \dots, n,$$

and write the inequalities (8) in the form

$$2d_i b_i - \xi_i - \mu^* \nu_i > 0, \quad i = 1, 2, \dots, n.$$

Now choose a constant

$$0 < p < \min_{i=1,2,\dots,n} (2d_i b_i - \xi_i - \mu^* \nu_i) d_i^{-1}$$

and apply the Lyapunov function to the proof of the theorem

$$v(t, y) = \sum_{i=1}^n d_i y_i^2(t) e_p(t, t_0), \quad d_i > 0, \quad i = 1, 2, \dots, n.$$

Let $t \neq t_k$. For convenience, in what follows we shall write $y_i,\ \sigma,\ \mu_i$ and $g(y_i)$ instead of $y_i(t),\ \sigma(t),\ \mu_i(t)$ and $g(y_i(t))$ respectively. Since

$$(y_i^2)^\Delta = y_i^\Delta y_i + y_i^\Delta y_i(\sigma) = 2y_i^\Delta y_i + \mu (y_i^\Delta)^2,$$

for the derivative of the function y_i^2 along the solutions of the system (6) at the point t we have the estimate

$$(y_i^2)^\Delta|_{(6)} = 2y_i \big(-b_i y_i + \sum_{j=1}^n t_{ij} g_j(y_j) \big) + \mu \big(-b_i y_i + \sum_{j=1}^n t_{ij} g_j(y_j) \big)^2 \le$$
$$\le -2b_i y_i^2 + 2 \sum_{j=1}^n |t_{ij}||y_i||g_j(y_j)| + \mu(n+1) \big(b_i^2 y_i^2 + \sum_{j=1}^n t_{ij}^2 g_j^2(y_j) \big) \le$$
$$\le -2b_i y_i^2 + 2 \sum_{j=1}^n l_j |t_{ij}||y_i||y_j| + \mu(n+1) \big(b_i^2 y_i^2 + \sum_{j=1}^n l_j^2 t_{ij}^2 y_j^2 \big) =$$
$$\le (-2b_i y_i^2 + \mu(n+1) b_i^2) y_i^2 + 2 \sum_{j=1}^n l_j |t_{ij}||y_i||y_j| + \mu(n+1) \sum_{j=1}^n l_j^2 t_{ij}^2 y_j^2.$$

Further, using the formula

$$[y_i^2 e_p(t,t_0)]^\Delta = \big[py_i^2 + (y^\Delta)^2(1+\mu p)\big]e_p(t,t_0),$$

for the derivative of the function $y_i^2 e_p(t,t_0)$ along the solutions of system (6) we obtain the estimates

$$\begin{aligned}[y_i^2 e_p(t,t_0)]^\Delta|_{(6)} \le \Big[&py_i^2 + (1+\mu p)\big\{(-2b_i y_i^2 + \mu(n+1)b_i^2)y_i^2 + \\ &+ 2\sum_{j=1}^n l_j|t_{ij}||y_i||y_j| + \mu(n+1)\sum_{j=1}^n l_j^2 t_{ij}^2 y_j^2\big\}\Big]e_p(t,t_0) = \\ &= \Big[\big\{p + (1+\mu p)(-2b_i + \mu(n+1)b_i^2)\big\}y_i^2 + \\ &+ 2(1+\mu p)\sum_{j=1}^n l_j|t_{ij}||y_i||y_j| + \mu(n+1)(1+\mu p)\sum_{j=1}^n l_j^2 t_{ij}^2 y_j^2\Big]e_p(t,t_0).\end{aligned}$$

Now we can estimate the derivative of the function $v(t,y(t))$ along solutions (6)

$$\begin{aligned}v^\Delta(t,y(t))|_{(6)} &= \sum_{i=1}^n d_i[y_i^2 e_p(t,t_0)]^\Delta|_{(6)} \le \\ \le \sum_{i=1}^n d_i e_p(t,t_0)\Big[&\big\{p + (1+\mu p)(-2b_i + \mu(n+1)b_i^2)\big\}y_i^2 + \\ +2(1+\mu p)\sum_{j=1}^n l_j|t_{ij}||y_i||y_j| &+ \mu(n+1)(1+\mu p)\sum_{j=1}^n l_j^2 t_{ij}^2 y_j^2\Big] = \\ = e_p(t,t_0)\Big[\sum_{i=1}^n d_i&\big\{p + (1+\mu p)(-2b_i + \mu(n+1)b_i^2)\big\}y_i^2 + \\ +(1+\mu p)\sum_{i,j=1}^n 2d_i l_j|t_{ij}||y_i||y_j| &+ \mu(n+1)(1+\mu p)\sum_{i,j=1}^n d_i l_j^2 t_{ij}^2 y_j^2\Big].\end{aligned} \tag{9}$$

Let us consider separately the last two double sums

$$\begin{aligned}\sum_{i,j=1}^n 2d_i l_j|t_{ij}||y_i||y_j| &\le \sum_{i,j=1}^n 2d_i l_j|t_{ij}|\frac{y_i^2 + y_j^2}{2} = \sum_{i,j=1}^n \big(d_i l_j|t_{ij}|y_i^2 + d_i l_j|t_{ij}|y_j^2\big) = \\ &= \sum_{i,j=1}^n d_i l_j|t_{ij}|y_i^2 + \sum_{i,j=1}^n d_i l_j|t_{ij}|y_j^2 = \sum_{i,j=1}^n d_i l_j|t_{ij}|y_i^2 + \sum_{i,j=1}^n d_j l_i|t_{ji}|y_i^2 = \\ &= \sum_{i=1}^n\Big[\sum_{j=1}^n(d_i l_j|t_{ij}| + d_j l_i|t_{ji}|)\Big]y_i^2 = \sum_{i=1}^n \xi_i y_i^2, \\ \sum_{i,j=1}^n d_i l_j^2 t_{ij}^2 y_j^2 &= \sum_{i,j=1}^n d_j l_i^2 t_{ji}^2 y_i^2 = \sum_{i=1}^n l_i^2\big(\sum_{j=1}^n d_j t_{ji}^2\big)y_i^2 = \sum_{i=1}^n \eta_i y_i^2\end{aligned}$$

and continue the estimate (9)

$$
\begin{aligned}
v^{\Delta}(t,y(t))|_{(6)} \le e_p(t,t_0)\Big[\sum_{i=1}^{n} d_i\big\{p+(1+\mu p)(-2b_i+\mu(n+1)b_i^2)\big\}y_i^2+\\
+(1+\mu p)\sum_{i=1}^{n}\xi_i y_i^2+\mu(n+1)(1+\mu p)\sum_{i=1}^{n}\eta_i y_i^2\Big] =\\
= e_p(t,t_0)\sum_{i=1}^{n}\Big[d_i\big\{p+(1+\mu p)(-2b_i+\mu(n+1)b_i^2)\big\}+(1+\mu p)(\xi_i+\mu(n+1)\eta_i)\Big]y_i^2 =\\
= e_p(t,t_0)\sum_{i=1}^{n}\Big[d_i p+(1+\mu p)\{d_i(-2b_i+\mu(n+1)b_i^2)+\xi_i+\mu(n+1)\eta_i\}\Big]y_i^2 =\\
= e_p(t,t_0)\sum_{i=1}^{n}\Big[d_i p+(1+\mu p)\{-2d_ib_i+\mu(n+1)d_ib_i^2+\xi_i+\mu(n+1)\eta_i\}\Big]y_i^2 =\\
= e_p(t,t_0)\sum_{i=1}^{n}\Big[d_i p+(1+\mu p)\{-2d_ib_i+\xi_i+\mu(n+1)(d_ib_i^2+\eta_i)\}\Big]y_i^2 =\\
= e_p(t,t_0)\sum_{i=1}^{n}\Big[d_i p+(1+\mu p)\{-2d_ib_i+\xi_i+\mu\nu_i\}\Big]y_i^2.
\end{aligned} \tag{10}
$$

For the quadratic trinomial $\psi_i(z)=\nu_i z^2-(2d_ib_i-\xi_i)z+d_i$, taking into account the fact that $\nu_i>0$ and the discriminant

$$
\begin{aligned}
D = (2d_ib_i-\xi_i)^2-4\nu_i d_i =\\
= 4d_i^2b_i^2-4d_ib_i\xi_i+\xi_i^2-4d_i(n+1)(d_ib_i^2+\eta_i) =\\
= 4d_i^2b_i^2-4d_ib_i\xi_i+\xi_i^2-4d_i^2(n+1)b_i^2-4d_i(n+1)\eta_i =\\
= -4nd_i^2b_i^2-\xi_i^2+\xi_i^2-4d_ib_i\xi_i+\xi_i^2-4d_i(n+1)\eta_i =\\
= -4nd_i^2b_i^2-\xi_i^2-2\xi_i(2d_ib_i-\xi_i)-4d_i(n+1)\eta_i<0,
\end{aligned}
$$

we have that $\psi_i(z)>0$ for all $z\in\mathbb{R}$. Thus, for all $i=1,2,\dots,n$, $t\ne t_k$

$$d_i-\mu(2d_ib_i-\xi_i-\mu\nu_i)=\nu_i\mu^2-(2d_ib_i-\xi_i)\mu+d_i>0$$

and, beside,

$$2d_ib_i-\xi_i-\mu\nu_i>2d_ib_i-\xi_i-\mu^*\nu_i>0.$$

Therefore, by the choice of the constant p the inequalities

$$0<p<\frac{2d_ib_i-\xi_i-\mu^*\nu_i}{d_i}\le\frac{2d_ib_i-\xi_i-\mu\nu_i}{d_i}\le\frac{2d_ib_i-\xi_i-\mu\nu_i}{d_i-\mu(2d_ib_i-\xi_i-\mu\nu_i)},$$

are true, whence we obtain

$$
\begin{aligned}
p(d_i-\mu(2d_ib_i-\xi_i-\mu\nu_i))\le 2d_ib_i-\xi_i-\mu\nu_i,\\
d_ip+(1+\mu p)(-2d_ib_i+\xi_i+\mu\nu_i)<0.
\end{aligned}
$$

Continuing the estimate (10), finally, for all $t \neq t_k$, we will have

$$v^{\Delta}(t, y(t))|_{(6)} \leq 0.$$

If $t \in [t_0, t_1]_{\mathbb{T}}$, then by Theorem 2.1 and definition of $y(t_k^+)$ we have

$$v(t, y(t)) \leq v(t_0, y_0). \tag{11}$$

Similarly, for all $t \in (t_k, t_{k+1}]_{\mathbb{T}}$, $k \in \mathbb{N}$, the inequality

$$v(t, y(t)) \leq v(t_k^+, y(t_k^+))$$

is true. Since

$$v(t_k^+, y(t_k^+)) - v(t_k, y(t_k)) = \sum_{i=1}^{n} d_i (y_i^2(t_k^+) - y_i^2(t_k)) e_p(t_k, t_0) =$$

$$= \sum_{i=1}^{n} d_i \Big[2y(t_k) J_{ki}(y(t_k)) + J_{ki}^2(y(t_k)) \Big] e_p(t_k, t_0) \leq 0,$$

we have

$$v(t, y(t)) \leq v(t_k, y(t_k)), \quad t \in (t_k, t_{k+1}]_{\mathbb{T}}, \;\; k \in \mathbb{N}.$$

In view of (11), the last estimate leads to the inequality

$$v(t, y(t)) \leq v(t_0, y_0) \;\; \text{for all} \;\; t \in [t_0, +\infty]_{\mathbb{T}},$$

from which it is easy to obtain the following estimate

$$\|y(t)\| \leq m \|y_0\| (e_{\ominus p}(t, t_0))^{\frac{1}{2}},$$

where $m = (\max_{i=1,2,\dots,n}\{d_i\} / \min_{i=1,2,\dots,n}\{d_i\})^{1/2}$, for all $y_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{T}_\tau$ and $t \in [t_0, +\infty)_{\mathbb{T}}$. Theorem 4.1 is proved.

We now consider the system (1), (2) in the particular case, when the impulsive action is given by a linear function.

Corollary 4.1 *Suppose that the assumptions* $\mathrm{H}_1, \mathrm{H}_2$ *are satisfied and there exist constants* $d_i > 0$, $i = 1, 2, \dots, n$ *such that the following inequalities hold*

$$b_i - \frac{1}{2} \sum_{j=1}^{n} (l_j |t_{ij}| + \frac{d_j}{d_i} l_i |t_{ji}|) - \frac{1}{2} \mu^*(n+1)(b_i^2 + l_i^2 \sum_{j=1}^{n} t_{ji}^2) > 0, \quad i = 1, 2, \dots, n.$$

Let $x(t) \equiv x^*$ *be the only equilibrium state of the systems* (1), (2) *and*

$$I_{ki}(x_i(t_k) - x^*) = -\gamma_{ik}(x_i(t_k) - x^*), \qquad k \in \mathbb{N}, \;\; i = 1, 2, \dots, n,$$

where $0 < \gamma_{ik} \leq 2$. *Then the equilibrium state* $x(t) \equiv x^*$ *of the system* (1), (2) *is globally uniformly exponentially stable.*

5 Example

On the time scale

$$\mathbb{P}_{1,\beta} = \cup_{j=0}^{\infty} \big[j(1+\beta), 1+j(1+\beta)\big]$$

we consider a three-component neural network with impulses

$$\begin{aligned} x_1(t)^{\Delta} &= -x_1(t) + 0,1s_1(x_1(t)) + 0,09s_2(x_2(t)) - 0,1s_3(x_3(t)) - 2,09, \\ x_2(t)^{\Delta} &= -x_2(t) + 0,05s_1(x_1(t)) - 0,1s_2(x_2(t)) + 0,1s_3(x_3(t)) + 1,25, \\ x_3(t)^{\Delta} &= -x_3(t) - 0,1s_1(x_1(t)) + 0,05s_2(x_2(t)) + 0,06s_3(x_3(t)) + 0,96, \\ &\qquad t \neq t_k, \end{aligned} \tag{12}$$

$$\begin{aligned} x_1(t_k^+) &= x_1(t_k) + \gamma(x_1(t_k) - 2), \\ x_2(t_k^+) &= x_2(t_k) + \gamma(x_2(t_k) + 1), \\ x_3(t_k^+) &= x_3(t_k) + \gamma(x_1(t_k) + 1,5), \quad k \in \mathbb{N}, \end{aligned} \tag{13}$$

where $x_1, x_2, x_3 \in \mathbb{R}$, $s_1(r) = s_2(r) = s_3(r) = \frac{1}{2}\big(|r+1| - |r-1|\big)$, $t_k = (k-1)(1+\beta) + 0,5$. Since the inequalities (4) are satisfied with the constants $l_i = d_i = 1$, the state $x^* = (2; -1; -1,5)^{\mathrm{T}}$ is the only equilibrium state of the systems (12). In view of the fact that $\mu^* = \beta$, the inequalities (8) take the form

$$\begin{aligned} &0,32 - 2,045\beta > 0, \\ &0,805 - 2,0412\beta > 0, \\ &0,765 - 2,0472\beta > 0, \end{aligned}$$

from which we find $\beta < 0,1564$. According to Corollary 4.1 we conclude that for $\beta < 0,1564$ the equilibrium $x^* = (2; -1; -1,5)^{\mathrm{T}}$ of the system (12), (13) is globally uniformly exponentially stable.

6 Conclusion

In the framework of the approach proposed in the paper [23] sufficient conditions of global uniform exponential stability are obtained for the equilibrium state of a neural network with impulses on an arbitrary time scale. The case is considered when the impulse action is given by a linear function. We note that in [10] similar results are obtained for $\mathbb{T} = \mathbb{R}$ under the assumption that the functions s_i are bounded. Corollary 4.1 of the present paper for $\mathbb{T} = \mathbb{R}$ gives sufficient conditions under which such a restriction is absent. In addition, sufficient conditions for the existence of a unique equilibrium state of a neural impulsive system on time scale are obtained.

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