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A NASC for Equicontinuous Maps for Integral Equations

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Abstract: We offer necessary and sufficient conditions for a mapping of the form

$$(P\phi)(t) = p(t) - \int_0^t C(t,s)g(s,\phi(s))ds$$

to send sets of bounded continuous functions on $[0,\infty)$ into equicontinuous sets. When that equicontinuity holds then one may study the problem of obtaining a bounded solution of the integral equation by means of a Schauder-type fixed point theorem. When the mapped sets are equicontinuous then we use Schaefer's fixed point theorem to show that we can obtain a bounded positive solution provided that we know that the resolvent kernel, $R(t,s)$, of C is non-negative and that

$$p(t) - \int_0^t R(t,s)p(s)ds$$

is bounded and positive, while $g(t,x)$ does not grow too fast near $x = 0$. The known literature shows that there are wide classes of important problems from applied mathematics and fractional equations for which these conditions hold. For those classes, the problem of obtaining a positive solution is largely solved when equicontinuity, characterized by our theorem, holds.

Keywords: *integral equations; compact maps; positive kernels; positive solutions; Schaefer's fixed point theorem.*

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1 Introduction

In the study of integral equations by fixed point methods of the Schauder type (see [11, pp. 25-34], for example) there commonly occurs an integral of the form

$$\int_0^t C(t,s)g(s,x(s))ds. \tag{1.1}$$

A main part of the investigation involves using that integral to map a set of bounded continuous functions into an equicontinuous set. Our objective here is to establish a necessary and sufficient condition on C and g to ensure that this will happen. The conditions needed are (1.2), (1.4), and sometimes (1.3).

The function $C : (0,\infty) \times (0,\infty) \to (0,\infty)$ is measurable and for any finite interval $J \subset [0,\infty)$ the integral $\int_J C(t,s)ds$ exists for each $t \in J$ with

$$\sup_{t\in J} \int_0^t C(t,s)ds < \infty. \tag{1.2}$$

The function $g : [0,\infty) \times \Re \to \Re$ is continuous and bounded when x is bounded, while

$$x > 0 \implies g(t,x) > 0. \tag{1.3}$$

The function C is of fading memory type by which we mean that

$$0 < s < t_2 < t_1 \implies C(t_2,s) \geq C(t_1,s). \tag{1.4}$$

There is a more elementary formulation in place of (1.2) which is worth noting. Instead of asking (1.2) and deriving a subsequent (2.2), in all the statements of Theorem 2.2 and Corollary 2.1 replace (1.2) with: Let $C : (0,\infty) \times (0,\infty) \to (0,\infty)$ be continuous for $0 < s < t$ and suppose that for each $\epsilon > 0$ there is a $\delta > 0$ such that

$$0 \leq t_2 \leq t_1, \quad t_1 - t_2 < \delta \implies \int_{t_2}^{t_1} C(t_1,s)ds < \epsilon.$$

Remark 1. These conditions are the only ones needed to prove the necessary and sufficient condition. In fact, stopping at the end of that proof makes a complete note which can stand alone. But it turns out that this result can be used in Schaefer's fixed point theorem to give a very simple solution to the classical problem of finding a positive solution of the integral equation

$$x(t) = p(t) - \int_0^t C(t,s)g(s,x(s))ds, \tag{1.5}$$

where $p : [0,\infty) \to (0,\infty)$ is continuous. By using our theorem, it turns out that all we need prove is a certain *a priori* bound on solutions of a related equation. This work offers a way of attacking Riemann-Liouville fractional equations after suitable transformations given in [4]. Using a transformation introduced in [5] that bound will be automatic if we use two results which have been well studied in the theory of integral equations. We need to know that the resolvent, $R(t,s)$, for $C(t,s)$ is non-negative and that

$$p(t) - \int_0^t R(t,s)p(s)ds > 0. \tag{1.6}$$

In the appendix we summarize the literature giving affirmative answers to both properties.

It is convenient to first deal with the requirement that when M is a set of bounded continuous functions then the mapping Q defined by $\phi \in M$ implies that

$$(Q\phi)(t) = \int_0^t C(t,s)\phi(s)ds \tag{1.7}$$

maps M into an equicontinuous subset of the Banach space $(\mathcal{B}, \|\cdot\|)$ of bounded continuous functions $\phi : [0,\infty) \to \Re$ with the supremum norm.

Problem. Let $\mathcal{M}$ be a collection of bounded sets M of continuous functions. Find necessary and sufficient conditions on C to ensure that QM is equicontinuous for each M in $\mathcal{M}$.

All scalar fractional differential equations of both Caputo and Riemann-Liouville type, as well as many problems from applied mathematics, typically including heat transfer problems, have kernels $C(t,s) = (t-s)^{q-1}$, $0 < q < 1$, which satisfy (1.2) and (1.4).

A short section is added to show that the basic idea also works for $g(t,s,x(s))$.

2 Equicontinuity

In our work on equicontinuity we always discuss properties on the entire interval $[0,\infty)$. But we see in Theorem 6.1 that we are working on an arbitrary interval $[0,E]$. The reason for this is that compactness of the map is then a consequence of the equicontinuity alone. If uniqueness holds then solutions on such finite intervals can be parlayed into solutions on $[0,\infty)$ without usual difficulties with compactness of such mappings. The results here can be restricted to finite intervals and the theorems will hold without further change. Uniqueness results are also given.

We begin with (1.7)

$$(Q\phi)(t) = \int_0^t C(t,s)\phi(s)ds$$

for which we have the following result. Theorems 2.1 and 2.2 will be put together in Corollary 2.1 to yield the promised necessary and sufficient condition for equicontinuity.

Theorem 2.1 *The mapping Q defined in (1.7) with condition (1.2) holding will map every bounded set M in $(B, \|\cdot\|)$ into an equicontinuous set only if*

$$\int_0^t C(t,s)ds \text{ is uniformly continuous} \tag{2.1}$$

on any finite interval $J \subset [0,\infty)$.

The result is obvious since we could choose M as a set of constant functions. The same result is true with $\phi(s)$ replaced by $g(\phi(s))$ if there is a constant c with $g(c) \neq 0$ since we could again take M to be a set of functions containing c. Notice that the idea fails when we replace $g(\phi(s))$ by $g(s,\phi(s))$ unless there is a constant c with $g(s,c)$ being a nonzero constant.

The convolution case is especially simple and it is the case most often encountered in applied mathematics. We then have

$$\int_0^t C(t-s)ds = \int_0^t C(s)ds$$

which we require to be uniformly continuous on any finite interval of $[0,\infty)$ when C is locally integrable on $(0,\infty)$, which is the case for the very large class of functions $A(t)$ discussed in Section 8 in which (A1) holds. That integral will always be uniformly continuous on any closed bounded interval. Moreover, one of the primary applications has $C(t) = t^{q-1}$ where $0 < q < 1$ so we always have uniform continuity on $[0,\infty)$. All of this brings in an important property of the integral of C; that is, the integral as a measure is absolutely continuous with respect to the Lebesgue measure. In fact, for a given $\epsilon > 0$ and $t_1, t_2 \geq 0$, we can certainly find a $\delta > 0$ so that

$$\left|\int_{t_2}^{t_1} C(t_1 - s)ds\right| = \left|\int_0^{t_1-t_2} C(u)du\right| < \epsilon,$$

if $|t_1 - t_2| < \delta$ because C is in $L^1(0,1)$. In general, the δ depends on t_1. If (1.2) holds with t_1 fixed, then $|t_1 - t_2| < \delta$ implies

$$\left|\int_{t_2}^{t_1} C(t_1, s)ds\right| < \epsilon. \tag{2.2}$$

The reader may wish to refer also to Section 8 in which we discuss (A1)-(A3) and, in particular, (A1) which requires the kernel $A \in L^1(0,1)$ giving us exactly that same property.

The following result shows that the uniform continuity is also sufficient for equicontinuity. The combined form for a necessary and sufficient condition will be given in Corollary 2.1.

Theorem 2.2 *Let M be a bounded subset of $(\mathcal{B}, \|\cdot\|)$. Suppose that $\int_0^t C(t,s)$ is uniformly continuous on any finite interval $J \subset [0,\infty)$. If Q is defined by (1.7) with condition (1.2) holding then QM is an equicontinuous set.*

Proof. Let f be a typical element of M, let $\epsilon > 0$ be given, and let $t_1 \geq 0$ be fixed. Let J be a finite interval of $[0,\infty)$ with $t_1 \in J$. Because of the assumed uniform continuity there is a $\delta > 0$ such that $|t_1 - t_2| < \delta$ with $t_2 \in J$ implies that

$$\left|\int_0^{t_2} C(t_2, s)ds - \int_0^{t_1} C(t_1, s)ds\right| < \epsilon \quad \text{so} \quad \left|\int_0^{t_2} [C(t_2,s) - C(t_1,s)]ds - \int_{t_2}^{t_1} C(t_1,s)ds\right| < \epsilon.$$

Since M is bounded, we find $K > 0$ so that $\|f\| \leq K$ for all $f \in M$. Because of (1.2), it follows from (2.2) that a δ can be chosen so that $0 \leq t_1 - t_2 < \delta$ implies

$$\left|\int_{t_2}^{t_1} C(t_1, s)f(s)ds\right| \leq K\left|\int_{t_2}^{t_1} C(t_1, s)ds\right| < \epsilon.$$

We now have

$$\int_0^{t_2} |C(t_2,s) - C(t_1,s)|ds = \int_0^{t_2} [C(t_2,s) - C(t_1,s)]ds$$

$$\leq \left| \int_0^{t_2} C(t_2,s)ds - \int_0^{t_1} C(t_1,s)ds \right| + \left| \int_{t_2}^{t_1} C(t_1,s)ds \right| < 2\epsilon. \tag{2.3}$$

Now from Q, checking equicontinuity we have

$$\left| \int_0^{t_2} C(t_2,s)f(s)ds - \int_0^{t_1} C(t_1,s)f(s)ds \right|$$

$$= \left| \int_0^{t_2} C(t_2,s)f(s)ds - \int_0^{t_2} C(t_1,s)f(s)ds - \int_{t_2}^{t_1} C(t_1,s)f(s)ds \right|$$

$$\leq \|f\| \left| \int_0^{t_2} |C(t_2,s) - C(t_1,s)|ds \right| + \|f\| \left| \int_{t_2}^{t_1} C(t_1,s)ds \right|$$

$$\leq \|f\| \left[\int_0^{t_2} |C(t_2,s) - C(t_1,s)|ds + \epsilon \right] \leq \|f\| 3\epsilon.$$

The same inequality holds if $0 \leq t_2 - t_1 < \delta$. □

We will be illustrating the results by invoking Schaefer's fixed point theorem which requires a compact mapping of a collection of sets. It is this which motivates our collection $\mathcal{M}$, below.

The last two results will now be combined into a necessary and sufficient condition. Let $\mathcal{M}$ be the class of all sets M for which there are positive constants L_M with the property that if $\phi \in M$ then $\phi : [0,\infty) \to \Re$ is continuous and $\|\phi\| \leq L_M$. For $M \in \mathcal{M}$ let W be the mapping defined by $\phi \in M$ which implies that

$$(W\phi)(t) = \int_0^t C(t,s)g(s,\phi(s))ds. \tag{2.4}$$

The next results need to be stated in the two parts because of the "only if" statement. We would need to ask that there is a constant c so that $g(t,c)$ is a nonzero constant function.

Corollary 2.1 *Let (1.2) and (1.4) hold.*

(i) Let g be continuous and independent of t and suppose there is a constant c with $g(c) \neq 0$. The mapping W defined in (2.4) will map every set $M \in \mathcal{M}$ into an equicontinuous subset of $(B, \|\cdot\|)$ if and only if (2.1) holds.

(ii) If (2.1) holds and if $g(t,x)$ is continuous and bounded for x bounded then the mapping W defined in (2.4) will map every set $M \in \mathcal{M}$ into an equicontinuous subset of $(B, \|\cdot\|)$.

Proof. For a given set $M \in \mathcal{M}$ with the constant function $c \in M$ construct a new set M^* defined by $\phi^* \in M^*$ which implies that $\phi^*(t) = g(\phi(t))$ for $\phi \in M$. The new set M^* is also in $\mathcal{M}$. If ϕ is the constant c then ϕ^* is a nonzero constant and $(W\phi)(t) = (Q\phi^*)(t)$

will reside in an equicontinuous set only if (2.1) holds, where Q is defined in (1.7). On the other hand, if (2.1) holds, by Theorem 2.2 we see that QM^* is an equicontinuous set. By the definition of M^*, we have $WM = QM^*$ so that WM is an equicontinuous set. This proves (i).

To prove (ii), proceed as in the proof of (i) with $g(t,x)$ and construct M^* again for a given set M so that W will map M into an equicontinuous set exactly as in part (i) above. □

3 Dependence on t

Frequently the mapping takes the form

$$(H\phi)(t) = \int_0^t C(t,s)v(t,s,\phi(s))ds, \tag{3.1}$$

where $v : [0,\infty) \times [0,\infty) \times \Re \to \Re$ is continuous and for each $L > 0$ there exists $D > 0$ so that for $0 \le s \le t$,

$$|x| \le L \implies |v(t,s,x)| \le D. \tag{3.2}$$

A treatment of this mapping may allow us to reduce (1.4) by asking for a $\beta : [0,\infty) \to (0,\infty)$ with the property that

$$C^*(t,s) =: C(t,s)/\beta(t) \tag{3.3}$$

satisfies (1.4). Then

$$(H\phi)(t) = \int_0^t C^*(t,s)\beta(t)\phi(s)ds \tag{3.4}$$

will have the form of (3.1). While we would expect $\beta(t)$ to tend to infinity, this will be no problem in our theorem below so long as we work on finite intervals $[0,E]$.

Theorem 3.1 *Let (1.2), (1.4), (2.1), and (3.2) hold for (3.1). For M as in Theorem 2.2 and $M \in \mathcal{M}$ then (3.1) maps M into an equicontinuous set on $[0,E]$.*

Proof. Notice first that for each $E > 0$ there is a $K > 0$ such that $0 < t \le E$ implies that

$$\int_0^t C(t,s)ds \le K.$$

For a given $M \in \mathcal{M}$, $\epsilon_1 > 0$, and $t_1 \in [0,E]$, we seek $\delta > 0$ so that $\phi \in M$ and $0 \le t_2 < t_1$ and $t_1 - t_2 < \delta$ implies that

$$|(H\phi)(t_2) - (H\phi)(t_1)| < \epsilon_1. \tag{3.5}$$

Let L be the bound for this M and D be defined in (3.2). We have

$$\begin{aligned}
&(H\phi)(t_2) - (H\phi)(t_1) \\
&= \int_0^{t_2} C(t_2,s)v(t_2,s,\phi(s))ds - \int_0^{t_1} C(t_1,s)v(t_1,s,\phi(s))ds \\
&= \int_0^{t_2} [C(t_2,s)v(t_2,s,\phi(s)) - C(t_2,s)v(t_1,s,\phi(s))]ds \\
&+ \int_0^{t_2} [C(t_2,s)v(t_1,s,\phi(s)) - C(t_1,s)v(t_1,s,\phi(s))]ds \\
&- \int_{t_2}^{t_1} C(t_1,s)v(t_1,s,\phi(s))ds.
\end{aligned}$$

Now, v is uniformly continuous on M for $0 \le t \le E$ so for a given $\epsilon > 0$ there is a $\delta > 0$ such that $\phi \in M$, $|t_1 - t_2| < \delta$ implies that $|v(t_1, s, \phi(s)) - v(t_2, s, \phi(s))| < \epsilon$. At the same time, let δ be so small that for this ϵ then (2.3) holds. Thus

$$\begin{aligned}
&|(H\phi)(t_2) - (H\phi)(t_1)| \\
&\le \int_0^{t_2} C(t_2, s)\epsilon ds + \int_0^{t_2} [C(t_2, s) - C(t_1, s)]|v(t_1, s, \phi(s))|ds \\
&+ \int_{t_2}^{t_1} C(t_1, s)|v(t_1, s, \phi(s))|ds \\
&\le \epsilon \int_0^{t_2} C(t_2, s)ds + D\int_0^{t_2} [C(t_2, s) - C(t_1, s)]ds + D\int_{t_2}^{t_1} C(t_1, s)ds \\
&< \epsilon \int_0^{t_2} C(t_2, s)ds + 2\epsilon D + D\epsilon \\
&< \epsilon K + 2D\epsilon + D\epsilon < \epsilon_1 \ \text{ if } \ \epsilon < \epsilon_1/(K + 3D).
\end{aligned}$$

Similarly, we can show that (3.5) holds if $0 \le t_1 < t_2$ and $t_2 - t_1 < \delta$. □

4 Schauder's Theorem and Measures of Noncompactness

Study of the literature shows that investigators using fixed point theory frequently pursue either a contraction, a Schauder type fixed point theorem based on compactness of the mapping, or Darbo's fixed point theorem based on measures of noncompactness. If a contraction is present, it is usually the most elementary, but if it is not available then a compactness type result is usually far more elementary than Darbo's theorem . Theorem 2.2 can be a definite asset in determining if the compactness path is feasible. Darbo's path can require far less structure in the kernel. See, for example the lengthy expository paper of Appell [3, p. 195] for discussions of measures of non-compactness related to the present discussion.

The point of the second half of this paper is to show that in the choice of theorems in that compactness path, Schaefer's theorem can be so very natural, simple, and direct. To see this we start with Schauder's theorem [11, p. 25] and then in the next section compare it with Schaefer's for this class of problems.

Theorem 4.1 (Schauder) *Let M be a non-empty convex subset of a normed space $(\mathcal{B}, \|\cdot\|)$. Let P be a continuous mapping of M into a compact set $K \subset M$. Then P has a fixed point.*

To apply the theorem we see that:
1. We must find a self-mapping set as described.
2. The natural mapping defined by (1.5) needs to be continuous and into a compact set.

The next section will show that Schaefer's theorem can get us past both of these in a very smooth way.

5 Schaefer's Fixed Point Theorem

In this and the following sections we work our way up to application of Schaefer's theorem.

The object of this section is to point out two requirements for a positive solution. We need $R(t,s) \geq 0$ and $p(t) - \int_0^t R(t,s)p(s)ds > 0$. There is large literature detailed in the appendix giving sufficient conditions for them to hold.

We now show how Schaefer's fixed point theorem [11, p. 29] gets us past the Items 1 and 2 discussed in the previous section. We place this discussion in the context of the search for a positive solution of (6.1). Much has been written about such existence, as may be seen, for example, in the books ([1], [2]). In many problems, such as population studies, only a positive solution has any meaning.

Theorem 5.1 (Schaefer) *Let $(\mathcal{B}, \|\cdot\|)$ be a normed space, P be a continuous mapping of $\mathcal{B}$ into $\mathcal{B}$ which is compact on each bounded subset X of $\mathcal{B}$. Then either*

(i) *the equation $x = \lambda Px$ has a solution for $\lambda = 1$, or*

(ii) *the set of all such solutions x, for $0 < \lambda < 1$, is unbounded.*

Notice the difference between this and Schauder's theorem in the search for a positive solution of (1.5).

1. The most challenging part of application of Schauder's theorem is locating a self mapping set. Schaefer's theorem does not require it.

2. We will restrict our mapping to an arbitrary interval $[0, E]$ which will later be extended to $[0, \infty)$. The mapping P will be the natural mapping defined by (1.5). Our Theorem 2.2 will take care of the requirements that $P : \mathcal{B} \to \mathcal{B}$ and that the equicontinuous mapping is compact on bounded sets.

3. Because we are working on a bounded interval, pointwise continuity of $g(t,x)$ will take care of continuity of the map.

4. We only have to show that there is an *a priori* bound on all possible solutions of our (1.5) when we insert a parameter. This is a two step process.

a. With $p(t) > 0$ and $g(t,x) > 0$ for $x > 0$ it is clear from (1.5) that a solution begins positive and is bounded above by $p(t)$ so long as it remains positive. Thus, we need to provide a non-negative lower bound for the solution.

b. To obtain a lower bound, in Section 6 we transform (1.5) (with a parameter λ) into an equation, later designated as (6.10)

$$x(t) = \lambda\left[p(t) - \int_0^t R(t,s)p(s)ds\right] + \int_0^t R(t,s)\left[x(s) - \frac{g(s,x(s))}{J}\right]ds.$$

Here, we repeat some classical theory of integral equations [10, pp. 189-193, 207-213] with detail in Section 6. The function R is a resolvent satisfying

$$R(t,s) = \lambda JC(t,s) - \int_s^t \lambda JC(t,u)R(u,s)du \tag{5.1}$$

for $0 < s < t < \infty$ with

$$\int_0^t R(t,s)\phi(s)ds \tag{5.2}$$

continuous for any continuous function $\phi : [0,\infty) \to \Re$.

We observe that the resolvent R is also a function of λ. For brevity in notation, we will suppress the λ in the expression of R here. But this will cause us no trouble in Theorem 6.1 below since we ask that (5.2)-(5.4) hold for each λ, $0 < \lambda \leq 1$.

Now, in order to ensure that $x(t)$ remains positive we require **three** things. The first two are

$$R(t,s) \geq 0 \tag{5.3}$$

and

$$p(t) - \int_0^t R(t,s)p(s)ds > 0, \quad t \geq 0 \tag{5.4}$$

for each $\lambda,\ 0 < \lambda \leq 1$. These two properties have been studied for decades in the standard integral equation theory and several sufficient conditions are offered in the appendix which cover major areas in applied mathematics and fractional differential equations. A prominent example satisfying (5.3), (5.4), and Theorem 2.2 is $C(t,s) = (t-s)^{q-1}, 0 < q < 1$ and $p(t)$ non-decreasing. For such problems, all conditions of Schaefer's theorem will immediately hold except for the *a priori* bound. And the third requirement is treated next and can be absolutely elementary.

c) Main note. We used the negative integral in (1.5) to get an upper bound on all possible solutions. When (5.3) and (5.4) hold and when we can find a $J > 0$ with $0 < g(t,x)/(Jx) < 1$ when $0 < x \leq p(t)$ and $0 \leq t \leq E$, then a transformation will change (6.1) into an equivalent equation with positive right-hand side, thereby making $x = 0$ a lower bound. This will satisfy Schaefer's theorem and we will have a positive solution on an arbitrary interval $[0,E]$. This is the content of Theorem 6.1 and we notice that the inequality $0 < g(t,x)/(Jx) < 1$ must fail for such functions as $g(t,x) = x^{1/3}$, but Theorem 6.2 will pick up just such cases.

6 The Resolvent and a Transformation

In the previous section we gave an outline indicating that the conditions $\int_0^t C(t,s)ds$ is continuous, $R(t,s) \geq 0$, $p(t) - \int_0^t R(t,s)p(s)ds > 0$, and $0 < g(t,x)/(Jx) < 1$ "locally" imply the existence of a positive solution on any interval $[0,E]$. These offer a major contrast to conditions required in Schauder's theorem. Here are the details of the promised transformation.

Schaefer's fixed point theorem requires the introduction of a parameter $\lambda \in (0,1]$ into the integral equation. We return to (1.2), (1.3), and (1.6) which we restate here with the parameter for reference as

$$x(t) = \lambda\left[p(t) - \int_0^t C(t,s)g(s,x(s))ds\right], \tag{6.1}$$

where $0 < \lambda \leq 1$,

$$C : (0,\infty) \times (0,\infty) \to (0,\infty) \tag{6.2}$$

satisfies (1.2), g and p are continuous where

$$g : [0,\infty) \times \Re \to \Re,\ x > 0 \implies g(t,x) > 0 \tag{6.3}$$

and

$$p : [0,\infty) \to (0,\infty). \tag{6.4}$$

Let J be an arbitrary positive constant and write (6.1) as

$$\begin{aligned} x(t) &= \lambda p(t) + \int_0^t C(t,s)[-\lambda J x(s) + \lambda J x(s) - \lambda g(s,x(s))]ds \\ &= \lambda p(t) - \lambda J \int_0^t C(t,s)x(s)ds + \lambda J \int_0^t C(t,s)\left[x(s) - \frac{g(s,x(s))}{J}\right]ds. \end{aligned} \tag{6.5}$$

Define

$$D(t,s) = \lambda J C(t,s) \tag{6.6}$$

and write the linear part as

$$z(t) = \lambda p(t) - \int_0^t D(t,s)z(s)ds \tag{6.7}$$

with resolvent equation

$$R(t,s) = D(t,s) - \int_s^t D(t,u)R(u,s)du \tag{6.8}$$

so that by the linear variation of parameters formula

$$z(t) = \lambda p(t) - \int_0^t R(t,s)\lambda p(s)ds. \tag{6.9}$$

We have again suppressed the λ in the expressions of R and D for brevity here.

The nonlinear variation of parameters formula [10, pp. 190-193] yields

$$x(t) = z(t) + \int_0^t R(t,s)\left[x(s) - \frac{g(s,x(s))}{J}\right]ds. \tag{6.10}$$

Main note. We emphasize that this equation is used in Theorems 6.1 and 6.2 only for establishing a lower bound on the solution. It is never the mapping equation. It is used in Theorem 7.1 to show uniqueness.

The reader will note that (iii) in Theorem 6.1 below requires a near Lipschitz condition centered on the $t-$axis. This eliminates such functions as $g(t,x) = x^{1/3}$. But those functions can be included with a simple translation device which we show in Theorem 6.2.

There is so much to be gained by working on an arbitrary finite interval $[0,E]$. In that case an equicontinuous map becomes a compact map. Moreover, if $p(t)$ is unbounded, then we obtain a solution which may be unbounded, but the unboundedness of p then causes us no trouble with the compactness arguments. When we examine proofs of continuity of the mapping, if we were working directly on the whole interval $[0,\infty)$ we would be needing some severe uniform continuity conditions on $g(t,x)$. The introduction of $\beta(t)$ in (3.3) can completely save a problem for this program. Since $\beta(t)\phi(s)$ in (3.4) could be unbounded if $\beta(t) \to \infty$ as $t \to \infty$ the finite interval avoids any problem with that property.

When we argue that by uniqueness we can continue a solution to all of $(0,\infty)$, notice that we are only using uniform convergence on compact sets to obtain that solution. By contrast, if we examine the end of Miller's proof [10, pp. 210–212] of Theorem 6.1 we will see that he does not have uniqueness and obtains a solution on a finite interval, stating then in the last line of the proof that the solution can be extended by continuation methods. However, that requires another significant argument. As noted in [9, p. 42], this type of argument is not elementary and relies on Zorn's lemma.

Finally, there is another significant advantage of work on $[0, E]$ and it is something of a surprise. In Theorem 2.2 if we had asked for uniform continuity of $\int_0^t C(t,s)ds$ on $[0,\infty)$, we would have restricted the growth rate of $g(t,x)$ in x to essentially linear growth. But when we work on $[0, E]$ we need only ask for continuity, leaving growth rate completely unrestricted.

Theorem 6.1 *Suppose that:*

(i) Conditions (1.2), (1.3), and (1.4) hold.

(ii) $\int_0^t C(t,s)ds$ *is continuous on any interval* $[0, E]$.

(iii) For each $E > 0$ *there are* $K < 1$, $J > 0$, *and* $L = \sup_{0\le t\le E} p(t)$ *with* $0 < \frac{g(t,x)}{Jx} < K$ *if* $0 < x \le L$ *and* $0 \le t \le E$.

(iv) The unique solution $R(t,s)$ *of (6.8) is non-negative for* $0 < s < t < \infty$ *and (5.4) holds.*

Then for $\lambda = 1$ *(6.1) has a positive solution on* $[0, E]$ *for any* $E > 0$. *If solutions are uniquely determined by the initial condition then the solution exists on* $[0,\infty)$.

Proof. While $\mathcal{B}$ is defined for functions on $[0,\infty)$, in this theorem all of the functions are restricted to the interval $[0, E]$ until we come to (iv) when we then develop the solution on $[0,\infty)$. Define a mapping P from (6.1) so that for $\phi \in \mathcal{B}$

$$(P\phi)(t) = p(t) - \int_0^t C(t,s)g(s,\phi(s))ds \tag{6.11}$$

and item (i) of Schaefer's theorem will be

$$x(t) = \lambda\left[p(t) - \int_0^t C(t,s)g(s,x(s))ds\right]. \tag{6.12}$$

I. First notice that in (6.12) if there is a solution, then $x(0) = \lambda p(0) > 0$ and, so long as $x(t) > 0$, the integrand in (6.12) is positive. Therefore

$$0 < x(t) \le \lambda p(t) \le L. \tag{6.13}$$

Notice also that the upper bound on $x(t)$ is independent of λ. If we can show that $x(t)$ remains positive then (6.13) will represent an *a priori* bound on all possible solutions of (6.12) and (ii) in Schaefer's theorem will be excluded.

II. We now show that $P : \mathcal{B} \to \mathcal{B}$. First, p is uniformly continuous on $[0, E]$. Next, if ϕ is in $\mathcal{B}$ then $g(t,\phi(t))$ is continuous and bounded on $[0, E]$ so by Theorem 2.2 $g(t,\phi(t))$ will be mapped by (6.11) into an equicontinuous set, from which we conclude that $P\phi \in \mathcal{B}$.

III. Continuity of the mapping P follows from that of $g(t,x)$ and the fact that $\int_0^t C(t,s)ds$ is uniformly continuous. Details are very similar to those in [6].

IV. Now we must show that *zero* is a lower bound on all possible solutions of (6.12). For this we go to the equivalent transformed equation (6.10). That transformation is reversible so a bound on solutions of (6.10) means a bound on the solutions of (6.12). It is easy to see that if 0 is a lower bound of solutions of (6.10), then it is independent of the λ, just as in the case of the upper bound. Notice that (iii) implies that if a solution is positive on an interval $[0, t_1)$, then the integrand in (6.10) is positive. This means that the solution can not vanish at t_1 because $z(t) > 0$. Hence, if a solution exists on $[0, E]$ then it is positive. We have established that any solution satisfies (6.13) on $[0, E]$.

V. Concerning the compact map, note that while we have two forms for our equation, it is only (6.11) which is the mapping equation. Equation (6.10) is only used to establish the lower bound on solutions, although in the next section it is also used for uniqueness. In order to show that P is a compact map, we only need to show that P maps bounded sets into equicontinuous sets since we are working on $[0, E]$. We noted in II that p is uniformly continuous. Let M be any bounded set in $\mathcal{B}$ on $[0, E]$ and determine $H > 0$ so that $\phi \in M$ implies $\|\phi\| \leq H$. This means that $g(t, x)$ is bounded for $|x| \leq H, 0 \leq t \leq E$. By Corollary 2.1 the set M will be mapped by P in (6.11) into an equicontinuous set. Adding in the uniformly continuous function p shows that P maps the given bounded set into an equicontinuous set.

The conditions of Schaefer's theorem are satisfied and P has a fixed point satisfying (6.12) for $\lambda = 1$.

In the event that solutions are unique then a solution, $x_n(t)$, is uniquely determined on any interval $[0, n]$ for n an arbitrary positive integer. Notice that the x_{n+k} agrees with x_n on $[0, n]$. Extend each $x_n(t)$ to a function $y_n(t)$ which is continuous on $[0, \infty)$ and agreeing with x_n on $[0, n]$. The sequence $y_n(t)$ converges uniformly on compact sets to a single function $x(t)$ on $[0, \infty)$ which does satisfy (6.1) with $\lambda = 1$ at every point on $[0, \infty)$. □

We now outline the changes needed in order to accommodate functions like $g(t, x) = x^{1/3}$. All of the conditions of Theorem 6.1 will be retained except (iii). The fact is that, unlike the classical result of Miller [10, p. 210] in which he obtains a non-negative solution which is not necessarily unique, our result will be a strictly positive solution on any interval $[0, E]$. Because the solution is positive, for each E and $\lambda \in (0, 1]$ if we can find $D_\lambda > 0$ and construct a line $x = \lambda D_\lambda > 0$ above which the solution lies, then condition (iii) holds, again above the line, so long as we are working on the fixed interval $[0, E]$. In that region functions like $x^{1/3}$ will allow the dominance displayed in (iii).

The big change here from Theorem 6.1 is that the region in (6.16) depends on each fixed λ, but we always keep the solution in the region $0 < \lambda D_\lambda \leq x(t) \leq L$, $0 \leq t \leq E$, yielding the *a priori* bound of L for every λ. When we invoke Schaefer's theorem then we obtain a solution in that region for $\lambda = 1$. Non-uniqueness problems for $g(t, x) = x^{1/3}$ will vanish since any pair of solutions must reside in that region, and that is the topic of the next section.

Now consider the region $0 < \lambda D_\lambda \leq x(t) \leq L$. Notice that it gets closer to the $x-$axis as $\lambda \to 0$ if D_λ is bounded. For $g(t, x) = x^{1/3}$ the region in (iii) of Theorem 6.1 of $0 < x \leq L$ can not possibly yield the inequality $0 < g(t, x)/(Jx) < K < 1$. For each λ we must go back to our transformation discussed in (6.5)-(6.10) and pick a larger value of J. This is no cause for concern because the transformation is reversible and the solution of (6.10) with each such J will correspond to the solution of (6.12).

Up to this point we have been suppressing the λ in the expression of $R(t, s)$ for brevity. We now reinsert the λ in the notation, say $R_\lambda(t, s)$, to emphasize that $R_\lambda(t, s)$ is the unique solution of (5.1) for the fixed λ. Unlike the case in Theorem 6.1 where J is an arbitrary positive constant, in Theorem 6.2 we choose J as a function of λ so we write

$J = J_\lambda$. We now restate (5.1) here with the new parameters as

$$R_\lambda(t,s) = \lambda J_\lambda C(t,s) - \int_s^t \lambda J_\lambda C(t,u) R_\lambda(u,s)du \tag{6.14}$$

for $0 < s < t$ and $\lambda \in (0,1]$. Here J_λ is to be determined.

In Theorem 6.2 below we ask that (5.4) hold and for a given $\gamma > 1$, define

$$D_\lambda = \min_{0 \le t \le E} \left[p(t) - \int_0^t R_\lambda(t,s)p(s)ds \right] / \gamma \tag{6.15}$$

for each λ, $0 < \lambda \le 1$.

In particular, the solution obtained by Schaefer's theorem satisfies $x(t) \ge D$ with $D = D_1$. And this D will be critical in proving the uniqueness result in the next section, enabling us to continue our solution on an arbitrarily large interval $[0,E]$ to a positive solution on $[0,\infty)$.

Theorem 6.2 *Let the conditions of Theorem 6.1 hold except for (iii). Let* $0 < K < 1$, $E > 0$ *be given and let* $L = \sup_{0 \le t \le E} p(t) > 0$. *Suppose that for each* $\lambda \in (0,1]$, *there exists* $J_\lambda > 0$ *so that*

$$\lambda D_\lambda \le x \le L, \;\; t \in [0,E] \implies 0 < \frac{g(t,x)}{J_\lambda x} \le K. \tag{6.16}$$

Then for $\lambda = 1$ *(6.12) has a positive solution on* $[0,E]$ *and any solution of (6.12) for* $\lambda \in (0,1]$ *satisfies* $0 < \lambda D_\lambda \le x(t) \le L$.

Proof. In the second to last sentence of the theorem recall that (6.10) and (6.12) share solutions. We explain the second sentence of Theorem 6.2 as follows. Recall from (6.9) that

$$z(t) = \lambda \left[p(t) - \int_0^t R_\lambda(t,s)p(s)ds \right]. \tag{6.17}$$

From this and (6.15) it follows that $z(t) > \lambda D_\lambda$ on $[0,E]$. All of the work on continuity and compactness needed for Schaefer's theorem was given in the proof of Theorem 6.1.

The only thing left to be proved here is the *a priori* bound. The upper bound of L still holds. We need only to show a lower bound. Suppose a solution $x(t)$ exists on $[0,E]$ and, since $\lambda p(0) > \lambda D_\lambda > 0$, we can assume $x(t) > \lambda D_\lambda$ on an interval $[0,t_1)$. It follows from (6.16) that the integrand in (6.10) is positive on $[0,t_1]$. We now investigate the possibility that $x(t_1) = \lambda D_\lambda$. Note that on $[0,t_1)$ we have

$$x(t) = z(t) + \int_0^t R_\lambda(t,s)x(s) \left[1 - \frac{g(s,x(s))}{J_\lambda x(s)} \right] ds$$

and this integral is positive on $[0,t_1]$ and $x(t) > \lambda D_\lambda$ on $[0,t_1)$. Because the integral is positive we have $x(t) \ge z(t)$ at t_1. However, at t_1 we have $x(t_1) \ge z(t_1) > \lambda D_\lambda$, a contradiction to our assumption that $x(t_1) = \lambda D_\lambda$. We conclude that the solution remains above λD_λ on $[0,E]$. □

In Example 6.1 below, we will outline some basic steps that can be taken for determining the number J_λ. To this end, we examine (5.4) and (6.15). First we see that, in

(6.14), if $C(t,s) = A(t-s)$, and if A satisfies conditions (A1)-(A3) in Section 8, then for each $\lambda \in (0,1]$, $R_\lambda(t,s) = R_\lambda(t-s)$, $R_\lambda(t) > 0$, and

$$\int_0^\infty R_\lambda(t)dt = \lambda J_\lambda A^*(1+\lambda J_\lambda A^*)^{-1}, \tag{6.18}$$

if $A^* = \int_0^\infty A(s)ds < \infty$ (see Section 8 and Miller [10, pp. 212–213]). If the function A is defined on $[0,E]$, we can easily extend its domain to $[0,\infty)$ with A satisfying (A1)-(A3) on $[0,\infty)$ and $A \in L^1[0,\infty)$.

Next, we observe that if $p(t)$ is non-decreasing, $R_\lambda(t,s) \geq 0$, and $\int_0^t R_\lambda(t,s)ds < 1$ for $t \in [0,E]$, then

$$p(t) - \int_0^t R_\lambda(t,s)p(s)ds \geq p(t)\left[1 - \int_0^t R_\lambda(t,s)ds\right] > 0. \tag{6.19}$$

This implies that (5.4) holds. The lower bound of $\left(1 - \int_0^t R_\lambda(t,s)ds\right)$ for $t \in [0,E]$ is essential for determining J_λ. For the convolution case, we have from (6.18) that

$$1 - \int_0^t R_\lambda(t-s)ds = (1+\lambda J_\lambda A^*)^{-1} + \int_t^\infty R_\lambda(u)du. \tag{6.20}$$

For the non-covolution case, in Example 6.1 we ask that the resolvent $R_\lambda(t,s)$ in (6.14) satisfy

$$1 - \int_0^t R_\lambda(t,s)ds \geq (1+\lambda J_\lambda N)^{-1} \tag{6.21}$$

for all $t \in [0,E]$ and a fixed positive number N. Condition (6.21) holds for a general class of convex kernels. We will not go into the details here and refer the readers to Section 8 for reference.

We now consider the equation

$$x(t) = p(t) - \int_0^t C(t,s)x^{1/3}(s)ds \quad \text{for} \quad t \in [0,E]. \tag{6.22}$$

Example 6.1 *Let $E > 0$ be given, let $p(t)$ be continuous, positive, and non-decreasing for $t \in [0,E]$, and let $L = \sup_{0\leq t\leq E} p(t)$. Suppose that*

(i) Conditions (1.2), (1.4), (5.2), and (5.3) hold.

(ii) $\int_0^t C(t,s)ds$ is continuous on any interval $[0,E]$.

If (6.21) holds, then (6.22) has a positive solution on $[0,E]$.

Proof. Let $g(t,x) = x^{1/3}$ and $0 < K < 1$ be given. We first observe that (6.21) implies (5.4). Thus, to apply Theorem 6.2 we only need to verify that (6.16) holds. For each $\lambda \in (0,1]$, to determine J_λ in (6.16) we need to find a lower bound of D_λ in terms of J_λ. To this end, we apply (6.21) and we proceed as follows.

$$\begin{aligned} D_\lambda &= \min_{0\leq t\leq E}\left[p(t) - \int_0^t R_\lambda(t,s)p(s)ds\right]/\gamma \\ &\geq \min_{0\leq t\leq E} p(t)\left[1 - \int_0^t R_\lambda(t,s)ds\right]/\gamma \geq p_0\,(1+\lambda J_\lambda N)^{-1}/\gamma, \end{aligned} \tag{6.23}$$

where $p_0 = p(0)$. We now define

$$G_\lambda = \max\left\{\frac{g(t,x)}{x},\ 0 \le t \le E,\ \lambda D_\lambda \le x \le L\right\}$$
$$= \max\left\{\frac{x^{1/3}}{x},\ 0 \le t \le E,\ \lambda D_\lambda \le x \le L\right\} = \frac{1}{(\lambda D_\lambda)^{2/3}}. \tag{6.24}$$

Thus, to show that (6.16) holds, it suffices to solve the inequality

$$0 < \frac{g(t,x)}{J_\lambda x} \le \frac{G_\lambda}{J_\lambda} \le K \tag{6.25}$$

on $\lambda D_\lambda \le x \le H,\ t \in [0,E]$ for a positive J_λ. It follows from (6.23) and (6.24) that

$$\frac{G_\lambda}{J_\lambda} = \frac{1}{J_\lambda(\lambda D_\lambda)^{2/3}} \le \frac{1}{J_\lambda \lambda^{2/3}\left[p_0(1+\lambda J_\lambda N)^{-1}\gamma^{-1}\right]^{2/3}} = \frac{\left(1+\lambda J_\lambda N\right)^{2/3}\gamma^{2/3}}{J_\lambda\left(\lambda p_0\right)^{2/3}} \le K. \tag{6.26}$$

It is easy to see that (6.26) has a positive solution J_λ (infinitely many). Thus, (6.16) is satified and (6.22) has a positive solution on $[0,E]$ by Theorem 6.2. □

7 Uniqueness

We will begin with an example in which Theorem 6.2 already shows that there is a positive solution and for positive x we will write $g(t,x) = x^{1/2}$ which is very instructive for several reasons. First, early in our study of differential equations we find that $x' = -x^{1/2}$ for $x \ge 0$ can generate non-uniqueness so we are immediately on guard. It is especially effective here in that there are no inequalities; everything is absolutely exact and we can see at each step exactly what is promoting uniqueness and where uniqueness is likely to fail.

Example 7.1 *Let $g(t,x) = x^{1/2}$ for $x \ge 0$. If the other conditions of Theorem 6.2 hold, then there is a unique positive solution of (6.1) on $[0,\infty)$ for $\lambda = 1$.*

Proof. According to the proof of Theorem 6.2 for any $E > 0$ any solution of (6.1) for $\lambda = 1$ on $[0,E]$ satisfies $0 < D \le x(t) \le L$ for a pair of fixed numbers D and L with $D = D_1$. Then $x(t)$ satisfies (6.10) for any positive number J. By way of contradiction to uniqueness, if x_1 and x_2 are two solutions on some interval $[0,E]$ then we will rationalize the subsequent numerator and have

$$\begin{aligned}
x_1(t) - x_2(t) &= \int_0^t R(t,s)\left[x_1(s) - \frac{x_1^{1/2}(s)}{J} - x_2(s) + \frac{x_2^{1/2}(s)}{J}\right]ds \\
&= \int_0^t R(t,s)\left[x_1(s) - x_2(s) - \frac{x_1^{1/2}(s) - x_2^{1/2}(s)}{J}\right]ds \\
&= \int_0^t R(t,s)\left[x_1(s) - x_2(s) - \frac{(x_1(s) - x_2(s))}{J(x_1^{1/2}(s) + x_2^{1/2}(s))}\right]ds \\
&= \int_0^t R(t,s)(x_1(s) - x_2(s))\left[1 - \frac{1}{J(x_1^{1/2}(s) + x_2^{1/2}(s))}\right]ds.
\end{aligned} \tag{7.1}$$

We proceed to estimate the right-hand side of (7.1) by choosing a sufficiently large J in (6.10) so that $\beta := \dfrac{1}{2JL^{1/2}} \leq \dfrac{1}{2JD^{1/2}} < 1$. It follows from (7.1) that

$$\begin{aligned}|x_1(t) - x_2(t)| &\leq \int_0^t R(t,s)|x_1(s) - x_2(s)|\left[1 - \frac{1}{J(L^{1/2} + L^{1/2})}\right] ds \\ &= \int_0^t R(t,s)|x_1(s) - x_2(s)| ds\, (1 - \beta).\end{aligned}$$

Thus, taking the supremum of both sides

$$\|x_1 - x_2\| \leq (1 - \beta) \sup_{t \in [0,E]} \int_0^t R(t,s) ds\, \|x_1 - x_2\|, \tag{7.2}$$

we now show that

$$\int_0^t R(t,s) ds \leq 1 \quad \text{for} \quad t \in [0, E]. \tag{7.3}$$

In fact, since $R(t,s) \geq 0$ and $C(t,s)$ is non-increasing in t, we have for $t > s$

$$\begin{aligned}\int_s^t R(t,u) du &\leq \int_s^t R(t,u) C(u,s) du / C(t,s) \\ &= [C(t,s) - R(t,s)]/C(t,s) = 1 - R(t,s)/C(t,s) \leq 1.\end{aligned}$$

Thus, (7.3) holds and (7.2) would yield a contradiction. □

Notice that the denominator in the last term of (7.1) is a type of average value of the derivative of $g(x) = x^{1/2}$. We will see this in the general case with a very explicit application of the mean value theorem for derivatives.

Theorem 7.1 *Let the conditions of Theorem 6.2 hold, let $E > 0$ be given so that D and L are known with $D = D_1$, and let $g(t,x) = g(x)$ with $(d/dx)g(x) > 0$ and continuous for $D \leq x \leq L$. If $C(t,s)$ is non-increasing in t for $t > s$, then the positive solution of Theorem 6.2 is unique.*

Proof. By way of contradiction, assume that x_1 and x_2 are two positive solutions on $[0, E]$ so that by Theorem 6.2 they must both satisfy $D \leq x \leq L$. Pick $J > (d/dx)g(x)$ for $D \leq x \leq L$.

By the mean value theorem for derivatives, for each $s \in [0, E]$ either $x_1(s) = x_2(s)$ or there is an ξ between x_1 and x_2 with

$$g(x_1) - g(x_2) = \frac{dg(\xi)}{dx}(x_1 - x_2).$$

Thus, we have

$$\begin{aligned}x_1(t) - x_2(t) &= \int_0^t R(t,s)\left[x_1(s) - x_2(s) - \frac{g(x_1(s)) - g(x_2(s))}{J}\right] ds \\ &= \int_0^t R(t,s)\left[(x_1(s) - x_2(s)) - \left[\frac{\frac{dg(\xi(s))}{dx}[x_1(s) - x_2(s)]}{J}\right]\right] ds \\ &= \int_0^t R(t,s)[x_1(s) - x_2(s)]\left[1 - \frac{\frac{dg(\xi(s))}{dx}}{J}\right] ds.\end{aligned}$$

Now let $\alpha = \inf \left\{ \frac{dg(x)}{dx} : D \le x \le L \right\}$. Then $\alpha > 0$. Note that $\alpha < J$. Therefore,

$$\|x_1 - x_2\| \le \|x_1 - x_2\|(1 - \alpha/J) \sup_{t \in [0,E]} \int_0^t R(t,s)ds < \|x_1 - x_2\|,$$

is a contradiction.

8 Appendix: Survey of Non-negative Resolvents

Note carefully that we always ask that $C(t,s) > 0$ and then notice in (5.1) that if there is a (t,s) at which the integral in (5.1) is negative, then $R(t,s)$ is positive at that (t,s). This is extremely important; the resolvent can never be a negative function for all (t,s). Everything we do here will depend on the resolvent being always non-negative. Thus we survey some of the main conditions known to ensure that property.

The convolution case

The first, and certainly the main, result is given by Miller [10, p. 209] and it concerns the case of

$$C(t,s) = A(t-s) \tag{8.1}$$

with the resolvent equation now reducing to

$$R(t) = A(t) - \int_0^t A(t-u)R(u)du. \tag{8.2}$$

The conditions on A are:
(A1) A is continuous on $(0,\infty)$ and is in $L^1(0,1)$.
(A2) $A(t)$ is positive and non-increasing for $t > 0$.
(A3) For each $T > 0$ the function $A(t)/A(t+T)$ is non-increasing in t for $0 < t < \infty$.

The classical example is $A(t) = t^{q-1}, 0 < q < 1$ and that is the kernel in all fractional differential equations of both Caputo and Riemann-Liouville type, many problems in heat transfer, and in a virtually endless list of other prominent problems from applied mathematics.

When A satisfies those conditions then Miller [10, pp. 212–213] establishes that
a) $R(t)$ is continuous on $(0,\infty)$.

b) $0 \le R(t) \le A(t)$ for all $t > 0$.

c) If $\int_0^\infty A(s)ds = \infty$ then $\int_0^\infty R(s)ds = 1$.

d) If $\int_0^\infty A(s)ds = A^* < \infty$ then $\int_0^\infty R(s)ds = A^*\left(1 + A^*\right)^{-1}$.

e) It is also true that if $A(t)$ is completely monotone on $(0,\infty)$ with $A(t) \not\equiv 0$, so is $R(t)$ with $R(t) > 0$ for all $t > 0$ [10, p. 224], and $A(t)$ satisfies (A1) - (A3) [10, p. 221].

Gripenberg [7, p.381] improves b) obtaining
f) $0 < R(t) \le A(t)/(1 + \int_0^t A(s)ds)$.

This is a result giving us the non-negativity of $R(t-s)$. We will now give two extreme examples for the companion result that

$$z(t) = p(t) - \int_0^t A(t-s)z(s)ds = p(t) - \int_0^t R(t-s)p(s)ds > 0.$$

The point of this theorem is that $\int_0^t R(s)ds < 1$. Thus, $1 - \int_0^t R(s)ds > 0$, a property which is critical.

Proposition 8.1 *Let $A(t)$ satisfy $(A1) - (A3)$, and let $p(t)$ be continuous, positive, and non-decreasing for $t \geq 0$. Then $R(t) > 0$ for all $t > 0$, $\int_0^T R(s)ds < 1$ for each finite $T > 0$, and $z(t) > 0$ for all $t \geq 0$.*

Proof. The assertion that $R(t) > 0$ is from Item f), above. It is clear from c) and d) that $\int_0^T R(s)ds < 1$ for each finite $T > 0$ since $R(t) > 0$ for all $t > 0$. The last assertion now follows from

$$z(t) = p(t) - \int_0^t R(t-s)p(s)ds \geq p(t)\left[1 - \int_0^t R(s)ds\right] > 0 \tag{8.3}$$

since $p(t)$ is positive and non-decreasing. This completes the proof.

Until we get to Item e), above, we know little about the behavior of $R(t)$. But with e) things change radically. $A(t)$ is monotone decreasing and so is $R(t)$. Moreover, it is true that $R(t) > 0$ if $A(t) \not\equiv 0$ so that the assertions of Proposition 8.1 will follow from Item e).

Proposition 8.2 *Let $A(t)$ be completely monotone on $(0, \infty)$, and let $p(t)$ be continuous, positive, and non-decreasing for $t \geq 0$. If $A(t) \not\equiv 0$, then $R(t) > 0$ for all $t > 0$, $\int_0^T R(s)ds < 1$ for each finite $T > 0$, and $z(t) > 0$ for all $t \geq 0$.*

The non-convolution case

The first step toward the non-convolution case is found in Miller [10, p. 217] where it is shown that if $A(t)$ satisfies (A1) - (A3) and if $B(s)$ is bounded, non-negative, and continuous with $\beta = \sup\{B(s) \colon 0 \leq s < \infty\}$, then the resolvent function $R(t,s)$ associated with the equation

$$R(t,s) = A(t-s)B(s) - \int_s^t A(t-u)B(u)R(u,s)du \tag{8.4}$$

exists, is measurable in (t, s) and satisfies

$$0 \leq R(t,s) \leq \beta A(t-s). \tag{8.5}$$

We want to establish a result that is parallel to that of Proposition 8.1. This requires a repetition of Proposition 8.1 and further analysis on $R(t, s)$.

Proposition 8.3 *Let $A(t)$ satisfy $(A1) - (A3)$, let $B(t)$ be bounded, continuous, and non-negative for $t \geq 0$, and let $p(t)$ be continuous, positive, and non-decreasing for $t \geq 0$. Then*

(i) $R(t,s) \geq 0$ *for* $t > s \geq 0$,

(ii) $\int_0^t R(t,s)ds < 1$ *for* $t \geq 0$,

(iii) $p(t) - \int_0^t R(t,s)p(s)ds > 0 \quad$ *for* $t \geq 0$.

Proof. Note that (i) follows from (8.5) and the proof of (iii) is exactly the same as that for (8.3) with $R(t,s)$ in the place of $R(t-s)$ if (ii) holds. To prove (ii), we set $C(t,s) = A(t-s)B(s)$ and observe that

$$C(t,u) \leq C(v,u) \text{ if } u \leq v \leq t \tag{8.6}$$

since A is positive and non-increasing for $t > 0$ by (A2). It then follows from Theorem 8.7 of Gripenberg et al. [8, p. 263, lines 11-13 from the bottom] with $f(t) \equiv 1$ that the solution $Z(t)$ of

$$Z(t) = 1 - \int_0^t C(t,s)Z(s)ds$$

is positive yielding

$$Z(t) = 1 - \int_0^t R(t,s)ds > 0.$$

Thus, (ii) holds. The proof is complete. □

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