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Maximal Regularity of Non-autonomous Forms with Bounded Variation

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Abstract: We are concerned with the non-autonomous evolutionary problem

$$(P)\begin{cases} \dot{u}(t) + A(t)u(t) = f(t), & t \in [0, \eta], \\ u(0) = u_0. \end{cases}$$

Each operator $A(t)$ is associated with a symmetric sesquilinear form $\mathfrak{a}(t;.,.)$ on a Hilbert separable space $(H, \|\cdot\|)$. We show that the approximation method considered in [13] to redemonstrate the maximal regularity in H, is still valid to prove this property if the sesquilinear form is symmetric and time bounded variation. This result was already established in [5].

Keywords: *sesquilinear forms; non-autonomous evolution equations; maximal regularity.*

Mathematics Subject Classification (2010): 35K90, 35K50, 35K45, 47D06.

1 Introduction

Let $(H, \|\cdot\|)$ and $(V, \|\cdot\|_V)$ be Hilbert separable spaces such that V is continuously and densely embedded in H, $V \underset{d}{\hookrightarrow} H$. Let V' be the anti-dual of V and denote by $(.|.)$ the scalar product of H and by $\langle .;. \rangle$ the duality pairing $V' \times V$. By the standard identification of H with H' we obtain the continuous and dense embedding

$$V \underset{d}{\hookrightarrow} H \simeq H' \underset{d}{\hookrightarrow} V'.$$

Moreover, it is shown in [4] that there exists a constant c_H such that

$$\begin{aligned} &\|u\| \leqslant c_H \|u\|_V \quad \text{for all} \quad u \in V \\ \text{and} \quad &\|f\|_{V'} \leqslant c_H \|f\| \quad \text{for all} \quad f \in H. \end{aligned}$$

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Note that a result on existence, uniqueness and asymptotic behaviour was established in [11] for the problem (P) on a Banach space and with $t \in [0, \infty[$.

Let $\mathfrak{a}(.; u, v) : [0, \eta] \to \mathbb{C}$ be a measurable function for all $u, v \in V$. For each $t \in [0, \eta]$ the operator $A(t)$ is associated with a sesquilinear form $\mathfrak{a}(t; ., .) : V \times V \longrightarrow \mathbb{C}$ which satisfies:

$[H1]$ $D(\mathfrak{a}(t; ., .)) = V$.

$[H2]$ There exists $M > 0$ such that for all $t \in [0, \eta]$ and $u, v \in V$, we have $|\mathfrak{a}(t; u, v)| \leqslant M\|u\|_V \|v\|_V$, (V-boundedness).

$[H3]$ There exist $\alpha > 0, \delta \in \mathbb{R}$ such that for all $t \in [0, \eta]$ and all $u, v \in V$ we have $\alpha\|u\|_V^2 \leqslant Re\mathfrak{a}(t; u, u) + \delta\|u\|_H^2$, (quasi-coerciveness).

Let $t \in [0, \eta]$. For each fixed $u \in V$, the operator $\mathfrak{a}(t, u; .)$ defines a continuous anti-linear functional on V, then it induces a linear operator $\mathcal{A}(t) \in \mathcal{L}(V, V')$ such that $a(t; u, v) = \langle \mathcal{A}(t)u, v\rangle$ for all $u, v \in V$. In this case, $-\mathcal{A}(t)$ generates a strongly continuous holomorphic semigroup $(e^{-s\mathcal{A}(t)})_{s\geqslant 0}$ on V'. When the problem (P) is considered in the spaces V and H, the form $\mathfrak{a}(t, \cdot; \cdot)$ is associated with $A(t)$ which is a part of $\mathcal{A}(t)$ in H. Therefore the operator $A(t) : D(A(t)) \subset V \to H$ is defined as

$$D(A(t)) = \{u \in V, \ \mathcal{A}(t)u \in H\}, \qquad A(t)u = \mathcal{A}(t)u.$$

Moreover, $-A(t)$ generates a strongly continuous holomorphic semigroup $(e^{-sA(t)})_{s\geqslant 0}$ with $(e^{-.A(t)}) := (e^{-.\mathcal{A}(t)})_{|H}$. Note that all the above results can be found in [18, Chapter 2] or in [15].

Recall that, if the problem (P) is considered in V' we have the following powerful result.

Theorem 1.1 (Lions' theorem) *For each* $(f, u_0) \in L^2(0, \eta; V') \times H$ *there is a unique solution* $u \in MR(V, V') := L^2(0, \eta; V) \cap H^1(0, \eta; V')$ *of the Cauchy problem*

$$\dot{u}(t) + \mathcal{A}(t)u(t) = f(t), \quad t \in (0, \eta), \quad u(0) = u_0. \tag{1}$$

We refer to [17, p. 112], [6, XVIII Chapter 3, p. 513] for the proof of this result. It is noteworthy to state that although Lions' theorem proves well-posedness of the Cauchy problem (P) with maximal regularity in V', the result remains unsatisfying in concrete applications to elliptic boundary value problems for which one needs solutions taking values in H. For this type of problems, only the part $A(t)$ of $\mathcal{A}(t)$ in H does really satisfy the boundary conditions. Hence, the central problem is whether maximal regularity in H is valid in the following sense:

Problem 1.1 For $(f, u_0) \in L^2(0, \eta; H) \times V$, does the solution u of (P) belong to $MR(V, H) := L^2(0, \eta; V) \cap H^1(0, \eta; H)$?

We will treat this question in three steps.

Step 1: $t \longmapsto A(t) := A$ for all $t \in [0, \eta]$. For this autonomous case, Problem 1.1 has been treated intensively, and has a positive answer.

Step 2: $t \longmapsto A(t)$ is a step function. This case was studied in [13] in a more general context and the authors have obtained a positive answer.

Step 3: The general case. The measurability condition assumed in Lions' theorem is not sufficient to establish the H-maximal regularity [5]. Extra conditions should be imposed on the regularity of $(\mathfrak{a}(t; ., .))_{0\leqslant t\leqslant\eta}$ with respect to t, or (and) on the space containing u_0. It is proved in [12] that u_0 has to be in a specified interpolation space. In the literature there are various conditions that ensure the H-maximal regularity. In the works of Lions we distinguish two cases. For $u_0 = 0$, he assumed that $\mathfrak{a}$ is symmetric and $\mathfrak{a}(., u, v) \in C^1[0, \eta]$ for all $u, v \in V$ [14, page 65]. For $u_0 \in D(A(0))$ he obtained a positive answer if $\mathfrak{a}(., u, v) \in C^2[0, \eta]$ [14, page 95], or if the forms are symmetric and $\mathfrak{a}(., u, v) \in C^1[0, \eta]$ (a combination of [14, Theorem 1.1, p. 129] and [14, Theorem 5.1, p. 138]). However, Bardos assumed that the domains of both $A(t)^{1/2}$ and $A(t)^{*1/2}$ coincide with V as spaces and topologically with constants independent of t, and that $\mathcal{A}(.)^{1/2}$ is continuously differentiable with values in $\mathcal{L}(V, V')$ [3]. The results of Bardos were extended in Arendt et al. [2] by assuming the piecewise continuity of $\mathfrak{a}$ instead of continuous differentiability. As Bardos in [3], Arendt et al. [2] assumed the same square property of the domains of $A(.)^{\frac{1}{2}}$ and $A(.)^{*\frac{1}{2}}$. Dier [5] improved the result of Arendt et al. by considering symmetric and bounded variation form: for all $u, v \in V$ and $t, s \in [0, \eta]$ the form satisfies $\mathfrak{a}(t; u, v) = \overline{\mathfrak{a}(t; v, u)}$ and

$$|\mathfrak{a}(t; u, v) - \mathfrak{a}(s; u, v)| \leqslant |g(t) - g(s)| \|u\|_V \|v\|_V, \tag{2}$$

where $g : [0, \eta] \to \mathbb{R}^+$ is a nondecreasing function. Ouhabaz and Spina followed another way in [16] when $u(0) = 0$ and $\mathfrak{a}$ is α-Holder continuous for some $\alpha > \frac{1}{2}$. This result was improved in [10] where the authors imposed that $\mathfrak{a}$ satisfies some Dini-type condition, which is a generalisation of the Holder continuity. Laasri and Sani in [13] gave another approach by approximating the problem (P) and using the frozen coefficient method developed in [7,8]. The authors gave explicitly an approximate solution $u_\Lambda \in MR(V, H)$ which converges to the solution u of the problem (P) in $MR(V, H)$ if the form $\mathfrak{a}$ is symmetric and time Lipschitz continuous. In this work we develop the last approach to re-demonstrate the result of [5]. In fact, Theorem 2.2 shows that the approximate solution converges weakly in $MR(V, H)$ to the solution u of (P).

2 Main Results

Let us recall some known results for the autonomous case that we use in the proof. In the following the constant $c > 0$ varies but does not depend on the variable to be estimated. Let $[a, b]$ be an arbitrary subinterval of $[0, \eta]$ and let $(f, u_0) \in L^2(a, b; V') \times H$. Lions theorem ensures the existence of a unique solution $u \in MR(a, b; V, V') := L^2(a, b; V) \cap H^1(a, b; V')$ of the autonomous problem

$$\dot{u}(t) + Au(t) = f(t), \quad \text{t. a.e. on } (a, b) \subset [0, \eta], \qquad u(a) = u_0. \tag{P_0}$$

It is shown in [17, Chapter III, Proposition 1.2] and in [18, Lemma 5.5.1] that if $u \in MR(a, b; V, V')$, then $\|u(.)\|^2$ is absolutely continuous on $[a, b]$ and

$$\frac{d}{dt}\|u(.)\|^2 = 2Re\langle \dot{u}; u\rangle. \tag{3}$$

For $(f, u_0) \in L^2(a, b; H) \times V$ the solution u of (P_0) belongs to the maximal regularity space $MR(a, b; D(A), H) := L^2(a, b; D(A)) \cap H^1(a, b; H)$ which is continuously embedded into $C([a, b], V)$, see [6, Example 1, page 577]. In addition, if the form $\mathfrak{a}$ is symmetric, W. Arendt and R. Chill proved in [1] the following results.

Proposition 2.1 *Let $\mathfrak{a}$ be a continuous symmetric sesquilinear form satisfying hypotheses $[H1] - [H3]$. Let $(f, u_0) \in L^2(a, b; H) \times V$ and $u \in MR(a, b; D(A), H)$. Then*

the following results hold:
i) The function $\mathfrak{a}(u(.)) \in W^{1,1}(a,b)$. *Moreover, the following product formula holds*

$$\frac{d}{dt}\mathfrak{a}(u(t)) = 2(Au(t)|\dot{u}) \quad \text{for a.e.} \quad t \in [a,b]. \tag{4}$$

In this case we infer the following estimate

$$\frac{d}{dt}\mathfrak{a}(u(t)) \leqslant \|f(t)\|^2 \quad \text{for a.e.} \quad t \in [a,b]. \tag{5}$$

ii) If the function u *satisfies* (P_0), *then there exists a constant* $c(M,\alpha,\delta,\eta) > 0$ *independent of* f, u_0 *and* $[a,b] \subset [0,\eta]$ *for which*

$$\sup_{s\in[a,b]} \|u(s)\|_V^2 \leqslant c\left[\|u(a)\|_V^2 + \|f\|_{L^2(a,b;H)}^2\right]. \tag{6}$$

The method considered in [13] consists in the approximation of $\mathfrak{a}$ and $\mathcal{A}$ by step function. Let $\Lambda = (0 = \lambda_0 < \lambda_1 < ... < \lambda_{n+1} = \eta)$ be a subdivision of $[0,\eta]$. Let

$$\mathfrak{a}_k : V \times V \to \mathbb{C} \quad \text{for } k = 0,1,...,n$$

be a finite family of continuous and H-elliptic forms. The associated operators are denoted by $A_k \in \mathcal{L}(V,V')$. The function $\mathfrak{a}$ is approximated by $\mathfrak{a}_\Lambda : [0,\eta] \times V \times V \to \mathbb{C}$ for each $k = 0,1,...,n$ and $\lambda_k \leq t < \lambda_{k+1}$

$$\begin{cases} \mathfrak{a}_\Lambda(t;u,v) := \mathfrak{a}_k(u,v) = \dfrac{1}{\lambda_{k+1}-\lambda_k}\displaystyle\int_{\lambda_k}^{\lambda_{k+1}} \mathfrak{a}(r;u,v)dr, \\ \mathfrak{a}_\Lambda(\eta;u,v) := \mathfrak{a}_n(u,v). \end{cases}$$

Thus, the approximate $\mathcal{A}_\Lambda : [0,\eta] \to \mathcal{L}(V;V')$ of $\mathcal{A}$ is given by

$$\begin{cases} \mathcal{A}_\Lambda(t) := \mathcal{A}_k = \dfrac{1}{\lambda_{k+1}-\lambda_k}\displaystyle\int_{\lambda_k}^{\lambda_{k+1}} \mathcal{A}(r)u\mathrm{d}r \quad \text{for} \;\; \lambda_k \leqslant t < \lambda_{k+1},\; k = 0,1,...,n, \\ \mathcal{A}_\Lambda(\eta) := \mathcal{A}_n. \end{cases}$$

For $u_0 \in H$ and $f \in L^2(0,T;V')$ there exists a unique $u_\Lambda \in MR(V,V')$ such that

$$(P_\Lambda)\begin{cases} \dot{u}_\Lambda(t) + \mathcal{A}_\Lambda(t)u_\Lambda(t) = f(t), \quad \text{for a.e } t \in [0,\eta], \\ u_\Lambda(0) = u_0. \end{cases}$$

Note that on each interval $[\lambda_k, \lambda_{k+1}[$ the solution u_Λ coincides with the solution of the autonomous Cauchy problem

$$(P_k)\begin{cases} \dot{u}_k(t) + A_k u_k(t) = f(t) \;\; t\text{−a.e.} \quad \text{on } (\lambda_k, \lambda_{k+1}), \\ u_k(\lambda_k) = u_{k-1}(\lambda_k) \in V, \end{cases} \tag{7}$$

which belongs to $MR(\lambda_k, \lambda_{k+1}; D(A_k), H)$.

The problem (P) is invariant under shifting the operator by a scalar multiplication. Then, for the sake of simplicity, we may assume without loss of generality that $\delta = 0$.

Proposition 2.2 *[13, Theorem 3.2]. Let $(f, u_0) \in L^2(a, b; V^{'}) \times H$. Let u and u_Λ be the solutions of (P) and (P_Λ) respectively. Then*
i) There exists a constant $c > 0$ witch is independent of $\{f, u_0, \Lambda\}$ such that

$$\int_0^t \|u_\Lambda(s)\|_V^2 ds \leqslant c \left[\int_0^t \|f(s)\|_{V'}^2 ds + \|u_0\|^2 \right] \quad \text{for a.e. } t \in [0, \eta], \tag{8}$$

ii) The solution u_Λ converges weakly to u in $MR(V, V^{'})$ as $|\Lambda| \to 0$.

If the conditions $(f, u_0) \in L^2(0, \eta; H) \times V$ are fulfilled, then the solution u_Λ of (P_Λ) belongs to the maximal regularity space $MR(V, H)$ which is continuously embedded into $C([0, \eta], V)$. In this case, the same estimate as in 8 is provided with the following theorem.

Theorem 2.1 *Let $g : [0, \eta] \to \mathbb{R}^+$ be a nondecreasing function. Let $(f, u_0) \in L^2(0, \eta; H) \times V$. Let $\mathfrak{a}$ be a symmetric sesquilinear form satisfying $[H1] - [H3]$ and*

$$|\mathfrak{a}(t; u, v) - \mathfrak{a}(s; u, v)| \leqslant (g(t) - g(s))\|u\|_V \|v\|_V \qquad (t, s \in [0; \eta], s \leqslant t).$$

If u_Λ is the solution of (P_Λ), then there exists a constant $c(\alpha, c_H, M, \eta, g)$ such that

$$\|u_\Lambda(t)\|_V^2 \leqslant c \left[\|u_0\|_V^2 + \|f\|_{L^2(0,\eta;H)}^2 \right], \qquad \forall t \in [0, \eta]. \tag{9}$$

Proof. Let $t \in [0, \eta]$, then there exists $k \in \{0, 1, 2, ..., n\}$ such that $t \in [\lambda_k, \lambda_{k+1}[\subset [0, \eta]$. Since the solution u_Λ coincides with the solution u_k of the autonomous problem (P_k) on each interval $[\lambda_k, \lambda_{k+1}[$, then the coercivity property and (5) yield

$$\begin{aligned}
\alpha \|u_\Lambda(t)\|_V^2 &\leqslant \mathfrak{a}_k(u_\Lambda(t)) \\
&= [\mathfrak{a}_k(u_k(t)) - \mathfrak{a}_k(u_k(\lambda_k))] + \sum_{i=0}^{i=k-1} \mathfrak{a}_i(u_i(\lambda_{i+1})) - \mathfrak{a}_i(u_i(\lambda_i)) \\
&+ \sum_{i=1}^{i=k} \mathfrak{a}_i(u_i(\lambda_i)) - \mathfrak{a}_{i-1}(u_i(\lambda_i)) + \mathfrak{a}_0(u_0(\lambda_0)) \\
&= \int_{\lambda_k}^t \frac{d}{ds}\mathfrak{a}_k(u_k(s)) ds + \sum_{i=0}^{i=k-1} \int_{\lambda_i}^{\lambda_{i+1}} \frac{d}{ds}\mathfrak{a}_i(u_i(s)) ds \\
&+ \sum_{i=1}^{i=k} \mathfrak{a}_i(u_i(\lambda_i)) - \mathfrak{a}_{i-1}(u_i(\lambda_i)) + \mathfrak{a}_0(u_0(\lambda_0)) \\
&\leqslant \int_0^t \|f(s)\|^2 ds + M\|u_0\|_V^2 + \sum_{i=1}^{i=k} \mathfrak{a}_i(u_i(\lambda_i)) - \mathfrak{a}_{i-1}(u_i(\lambda_i)).
\end{aligned}$$

First, we give for each $k = 0, 1, 2, .., n-2$ an estimate of

$$|\mathfrak{a}_k(u_\Lambda(\lambda_{k+1}) - \mathfrak{a}_{k+1}(u_\Lambda(\lambda_{k+1}))| \,.$$

Obviously, the function g is of bounded variation. And since $\|u_\Lambda(.)\|^2$ is continuous on $[0, \eta]$, it is Riemann-Stieltjes integrable with respect to g.

For each $k = 0,1,2,..,n$ and for each arbitrary $t_k \in [\lambda_k, \lambda_{k+1}[$ there exists, by the inequality (6), a constant $c \geqslant 0$ depending only on M, δ, α, c_H, and η such that $u_{\Lambda|_{[t_k,\lambda_{k+1}[}} \in MR(t_k, \lambda_{k+1}; D(A_k), H)$ and

$$\|u_\Lambda(\lambda_{k+1})\|_V^2 \leqslant c\left[\|u(t_k)\|_V^2 + \|f\|^2_{L^2(\lambda_k,\lambda_{k+1};H)}\right]. \tag{10}$$

By the mean value theorem, the t_k is chosen such that

$$(g(\lambda_{k+1}) - g(\lambda_k)\|u_\Lambda(t_k)\|^2) = \int_{\lambda_k}^{\lambda_{k+1}} \|u_\Lambda(t)\|^2 d(g(t)). \tag{11}$$

Thus, the estimates (2), (10) and (11) yield

$$\begin{aligned}
&|\mathfrak{a}_k(u_\Lambda(\lambda_{k+1}) - \mathfrak{a}_{k+1}(u_\Lambda(\lambda_{k+1}))| \\
&\qquad \leqslant (g(\lambda_{k+1}) - g(\lambda_k))\|u_\Lambda(\lambda_{k+1})\|^2 \\
&\qquad \leqslant c(g(\lambda_{k+1}) - g(\lambda_k))\left[\|u_\Lambda(t_k)\|^2 + \|f\|^2_{L^2(0,\eta;H)}\right] \\
&\qquad \leqslant c\int_{\lambda_k}^{\lambda_{k+1}} \|u_\Lambda(t)\|_V^2 d(g(s)) + c\left((g(\lambda_{k+1}) - g(\lambda_k))\right)\|f\|^2_{L^2(0,\eta;H)}.
\end{aligned} \tag{12}$$

Thus,

$$\begin{aligned}
&\sum_{i=1}^{i=k} |\mathfrak{a}_i(u_i(\lambda_i)) - \mathfrak{a}_{i-1}(u_i(\lambda_i))| \\
&\qquad \leqslant \sum_{i=1}^{i=k} c\int_{\lambda_{i-1}}^{\lambda_i} \|u_\Lambda(t)\|_V^2 d(g(s)) + \sum_{i=1}^{i=k} c\left((g(\lambda_i) - g(\lambda_{i-1}))\right)\|f\|^2_{L^2(0,\eta;H)} \\
&\qquad \leqslant c\int_0^t \|u_\Lambda(t)\|_V^2 d(g(s)) + c\left((g(\eta) - g(0))\right)\|f\|^2_{L^2(0,\eta;H)}.
\end{aligned} \tag{13}$$

Consequently,

$$\alpha\|u_\Lambda(t)\|_V^2 \leqslant c\left[\|f\|^2_{L^2(0,\eta;H)}) + \|u_0\|_V^2\right] + c\int_0^t \|u_\Lambda(s)\|_V^2 d(g(s)).$$

By Gronwall's inequality, see [9, Theorem 5.1, page 498], we obtain that

$$\|u_\Lambda(t)\|_V^2 \leqslant c\left[\|f\|^2_{L^2(0,\eta;H)}) + \|u_0\|_V^2\right]. \tag{14}$$

□

The following theorem shows that the solution u_Λ converges weakly in $MR(V,H)$ to the solution u of (P) which belongs to the maximal regularity space $MR(V,H)$.

Theorem 2.2 *Let* $(f, u_0) \in L^2(a,b;V') \times H$. *We suppose that the forms* $(a(t;.,.))_{0\leqslant t\leqslant \eta}$ *satisfy the standing hypotheses* $[H1]$-$[H3]$ *and the regularity condition*

$$|\mathfrak{a}(t;u,v) - \mathfrak{a}(s;u,v)| \leqslant (g(t) - g(s))\|u\|_V\|v\|_V \quad (0 \leqslant s \leqslant t \leqslant \eta), \tag{15}$$

where $g : [0,\eta] \to [0,\infty)$ *is a non-decreasing function. Then the solution* u_Λ *of* (P_Λ) *converges weakly in* $MR(V,H)$ *as* $|\Lambda| \to 0$ *to the solution* u *of* (P). *Moreover*

$$\|u\|_{MR(V,H)} \leqslant c\left[\|u_0\|_V^2 + \|f\|^2_{L^2(0,\eta;H)}\right],$$

the constant c *depends only on* α, c_H, M, η *and* g.

Proof. Let $(f,u_0)\in L^2(0,\eta;H)\times V$. Let $u_\Lambda\in MR(V,H)$ be the solution of (P_Λ). Taking into account the weak convergence of the function u_Λ to u in the space $MR(V,V')$, it is enough to show that u_Λ is bounded in $MR(V,H)$. Theorem 2.1 assures the boundedness of u_Λ in $L^2(0,\eta;V)$, so it remains to prove this property for the derivative in $L^2(0,\eta;H)$.

$$\begin{aligned}\int_0^\eta \|\dot u_\Lambda(t)\|^2dt &= \int_0^\eta Re(-\mathcal{A}_\Lambda u_\Lambda(t)|\dot u_\Lambda(t))dt+\int_0^\eta Re(f(t);\dot u_\Lambda(t))_H dt\\ &= -\sum_{k=0}^{n-1}\int_{\lambda_k}^{\lambda_{k+1}}\frac{d}{dt}\frac{1}{2}\mathfrak{a}_k(u_\Lambda(t))dt+\int_0^\eta Re(f(t)|\dot u_\Lambda(t))dt\\ &= -\sum_{k=0}^{n-1}(\mathfrak{a}_k(u_\Lambda(\lambda_{k+1})-\mathfrak{a}_k(u_\Lambda(\lambda_k))+\int_0^\eta Re(f(t)|\dot u_\Lambda(t))dt\\ &= -\sum_{k=0}^{n-2}\mathfrak{a}_k(u_\Lambda(\lambda_{k+1}))-\mathfrak{a}_{k+1}(u_\Lambda(\lambda_{k+1}))+\int_0^\eta Re(f(t)|\dot u_\Lambda(t))dt\\ &\quad [-\mathfrak{a}_{n-1}(u_\Lambda(\lambda_n))+\mathfrak{a}_0(u_\Lambda(0))]. \end{aligned} \tag{16}$$

For the first term on the right-hand side of the equality (16) the inequality (13) yields

$$\begin{aligned}|\sum_{k=0}^{n-2}(\mathfrak{a}_k(u_\Lambda(\lambda_{k+1})-\mathfrak{a}_{k+1}(u_\Lambda(\lambda_{k+1}))| &\leqslant *c\int_0^\eta \|u_\Lambda(t)\|_V^2 d(g(t))+c[g(\eta)-g(0)]\|f\|^2_{L^2(0,\eta;H)}\\ &\leqslant c\left[\|f\|^2_{L^2(0,\eta;H)})+\|u_0\|_V^2\right].\end{aligned}$$

By the Cauchy-Schwarz and the Young inequalities

$$\begin{aligned}\int_0^\eta \|\dot u_\Lambda(t)\|^2dt &\leqslant c\left[\|f\|^2_{L^2(0,\eta;H)})+\|u_0\|_V^2\right]+\int_0^\eta (f(t)|\dot u_\Lambda(t))dt\\ &\leqslant c\left[\|f\|^2_{L^2(0,\eta;H)})+\|u_0\|_V^2\right]+\frac{1}{2}\int_0^\eta \|f(t)\|^2dt+\frac{1}{2}\int_0^\eta\|\dot u_\Lambda(t)\|^2dt.\end{aligned}$$

Thus, by the inequality (9), there exists a constant $c>0$ depending on $(c_H,M,\alpha,g(\eta),g(0))$ such that

$$\int_0^\eta \|\dot u_\Lambda(t)\|^2dt+\int_0^\eta\|u_\Lambda(t)\|_V^2dt\leqslant c\left[\|f\|^2_{L^2(0,\eta;H)}+\|u_0\|_V^2\right].$$

□

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