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Function Projective Dual Synchronization of Chaotic Systems with Uncertain Parameters

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Abstract: This paper mainly concerns with the general methods for the function projective dual synchronization of a pair of chaotic systems with unknown parameters. The adaptive control law and the parameter update law are derived to make the states of a pair of chaotic systems asymptotically synchronized up to a desired scaling function by Lyapunov stability theory. The general approach for function projective dual synchronization of Lü system and Lorenz system is provided. Numerical simulation results show that the proposed method is effective and convenient.

Keywords: *function projective; dual synchronization; adaptive control; uncertain parameters; Lyapunov stability theory.*

Mathematics Subject Classification (2010): 34H10, 74H55, 74H65.

The essence of studying chaotic systems is to understand their structure and behavior. These systems are deemed important as they reflect and model natural phenomena. One of the main reasons for studying chaotic systems lies in the interest of controlling chaos. Many areas have branched from this study due to practical applications in many fields. The main property of chaotic dynamics is its critical sensitivity to initial conditions which is responsible for initially neighboring trajectories separating from each other exponentially in the course of time. For many years, this feature made chaos undesirable, insofar as the sensitivity to initial conditions of chaotic systems reduces their predictability over long time scales. On the other hand, the capability of chaotic dynamics to amplify small perturbations improves their utility for reaching specific desired states with very high flexibility and low energy cost. In contrast, the process of controlling chaos is directed to improving a desired behavior by making only small time-dependent perturbations in an

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accessible system parameter or dynamical variable. Therefore, understanding the behavior of chaos is crucial in the process of seeking beneficial applications to our lives [1,2,4].

The study of synchronization has been widely explored in a variety of systems including physical, chemical and ecological systems. In the broadest sense, synchronization is often understood as the tendency to undergo resembling evolution in time. Synchronization is an important mechanism for creating order in complex systems. Many nonlinear dynamical systems have been found to show a kind of behavior known as chaos, being characterized as chaotic systems by their extreme sensitivity to initial conditions and having noise-like behaviors. Several types of synchronization behaviors have been demonstrated and identified, such as, complete synchronization [3], phase synchronization [5, 6], anti-phase synchronization [7–10], lag synchronization [11, 12], generalized synchronization [13], projective synchronization [14–21] and so on. Function projective synchronization, which is the generalization of projective synchronization, is one of the important synchronization methods that have been widely investigated to obtain faster communication with its proportional feature. function projective synchronization means that the drive and response systems could be synchronized up to a scaling function. Recently, many authors have investigated the function projective synchronization. It is obvious that the unpredictability of the scaling function in function projective synchronization can additionally improve the security of communication [22–25].

However, the theory of dual synchronization has been intensively reviewed and studied recently. The first study on dual synchronization of chaotic systems has been reported by Tsimring and Sushchik in 1996 in [26]. Later, several dual synchronization methods have been reported, for example, dual synchronization of one dimensional discrete chaotic systems was undertaken in [27], where the authors achieved dual synchronization via specific classes of piecewise-linear maps with conditional linear coupling. In [28], the authors experimentally demonstrated dual synchronization of chaos in two pairs of microchip lasers in a one-way coupling configuration over one transmission channel. In [29], the authors demonstrated that dual synchronization of Lorenz and Rössler systems can be obtained by using the means of Lyapunov stabilization theory. In [30], the authors addressed dual synchronization via output feedback strategy in two different chaotic systems. In [31], the authors achieved dual synchronization of modulated time-delayed system by designing a delay feedback controller. In [32], the author investigated the existence of projective-dual-anticipating, projective-dual, and projective-dual-lag synchronization in a coupled time-delayed systems with modulated delay time using Krasovskii–Lyapunov stability theory. In [33], the authors studied the problem of dual synchronization of two different fractional-order chaotic systems by a linear controller. Finally, in [34–37], the authors investigated dual synchronization and dual anti-dual synchronization using nonlinear and adaptive control. To the best of our knowledge, the function projective dual synchronization of chaotic systems with unknown parameters has not yet been studied by any researcher. Inspired by the previous works, in this paper we propose a new analytic treatment of function projective dual synchronization of chaotic systems using adaptive control method in which a state variable of the drive system dual synchronizes with the state variable of the response system up to a scaling function. Numerical simulations are carried out for adaptive function projective dual synchronization behavior of two chaotic systems with uncertain parameters which are depicted through figures for different particular cases.

1 Problem Statement

Consider the following two chaotic systems with uncertain parameters as the drive system:

$$\left\{ \begin{array}{l} \dot{x}_1 = f_1(x_1) + F_1(x_1)\alpha, \\ \dot{y}_1 = g_1(y_1) + G_1(y_1)\beta, \end{array} \right. \tag{1}$$

where $x_1 = (x_{11}, x_{12}, ..., x_{1n})^T \in R^n$ and $y = (y_{11}, y_{12}, ..., y_{1n})^T \in R^n$ are the state vectors of the systems, $f_1 : R^n \to R^n$ and $g_1 : R^n \to R^n$ are two continuous vector functions, $F_1 : R^n \to R^{n\times m}$, $G_1 : R^n \to R^{n\times m}$ are two matrix functions and $\alpha, \beta \in R^m$ are the unknown parameter vectors of the two drive systems. The systems studied in this paper depend linearly on the parameters and many resemble well-known chaotic systems. By a linear combination of the drive systems states, a scalar signal is generated in the form of

$$\varepsilon_d = \sum_{i=1}^{n} (a_i x_1 + b_i y_1) = A^T x_1 + B^T y_1 = C^T x, \tag{2}$$

where $A = (a_1, a_2, ..., a_n)^T$ and $B = (b_1, b_2, ..., b_n)^T$ are known matrices and $C = (A^T \;\; B^T)^T$ and $x = (x_1^T \;\; y_1^T)^T$. This generated scalar signal is fed to the response systems which are corresponding to the drive systems. The response systems are

$$\left\{ \begin{array}{l} \dot{x}_2 = f_2(x_2) + F_2(x_2)\hat{\alpha} + u_1, \\ \dot{y}_2 = g_2(y_2) + G_2(y_2)\hat{\beta} + u_2, \end{array} \right. \tag{3}$$

where $x_2 = (x_{21}, x_{22}, ..., x_{2n})^T \in R^n$ and $y_2 = (y_{21}, y_{22}, ..., y_{2n})^T \in R^n$ are the state vectors, $f_2 : R^n \to R^n$ and $g_2 : R^n \to R^n$ are two continuous vector functions, $F_2 : R^n \to R^{n\times m}$, $G_2 : R^n \to R^{n\times m}$ are two matrix functions and $\hat{\alpha}, \hat{\beta} \in R^m$ represent the estimated vectors of unknown parameter vectors α, β and $u = (u_1 \;\; u_2)^T \in R^{2n}$ is a controller. By the linear combination of the response systems states a scalar signal is generated in the form of

$$\varepsilon_r = \sum_{i=1}^{n} (a_i x_2 + b_i y_2) = A^T x_2 + B^T y_2 = C^T y. \tag{4}$$

Our goal is to obtain the function projective dual synchronization between the drive and the response systems. Now define the error function between the drive and the response systems as $e_s = \varepsilon_r - h(t)\varepsilon_d = C^T(y - h(t)x)$, where $h(t) = \mathrm{diag}(h_1(t), h_2(t), ..., h_{2n}(t))$ is a scaling matrix. Therefore, for function projective dual synchronization we use adaptive control method to design the control in such a way that the origin becomes asymptotically stable equilibrium point of the error dynamics i.e., $\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|x_2 - h(t)x_1\| = 0$, $\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|y_2 - h(t)y_1\| = 0$, where the scaling function $h(t) \in C^1(0, +\infty)$ and $0 < h(t) < N_h$ for all $t > 0$, (N_h is a positive constant for the function $h(t)$).

1.1 Adaptive function projective dual synchronization controller design

System (1) can be rewritten in the following form:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \end{pmatrix} = \begin{pmatrix} f_1(x_1) \\ g_1(y_1) \end{pmatrix} + \begin{pmatrix} F_1(x_1) & 0 \\ 0 & G_1(y_1) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \dot{x} = f(x) + F(x)\Phi, \tag{5}$$

where $\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \end{pmatrix} \in R^{2n}$, $f(x) = \begin{pmatrix} f_1(x_1) \\ g_1(y_1) \end{pmatrix} \in R^{2n}$, $F(x) = \begin{pmatrix} F_1(x_1) & 0 \\ 0 & G_1(y_1) \end{pmatrix} :$ $R^{2n} \to R^{2n\times 2m}$ and $\Phi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in R^{2m}$. Similarly, system (3) can be rewritten in the following form:

$$\begin{pmatrix} \dot{x}_2 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f_2(x_2) \\ g_2(y_2) \end{pmatrix} + \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \dot{y} = g\left(y\right) + G\left(y\right)\hat{\Phi} + u, \tag{6}$$

where $\dot{y} = \begin{pmatrix} \dot{x}_2 \\ \dot{y}_2 \end{pmatrix} \in R^{2n}$, $g(y) = \begin{pmatrix} f_2(x_2) \\ g_2(y_2) \end{pmatrix} \in R^{2n}$, $G(x) = \begin{pmatrix} F_2(x_2) & 0 \\ 0 & G_2(y_2) \end{pmatrix} :$ $R^{2n} \to R^{2n\times 2m}$ and $\hat{\Phi} = \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} \in R^{2m}$ and $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in R^{2n}$. Now, define the error vector as

$$e = y - h(t)x. \tag{7}$$

The time derivative of equation (7) is

$$\begin{aligned} \dot{e}(t) &= \dot{y} - h(t)\dot{x} - \dot{h}(t)x \\ &= g(y) + G(y)\hat{\Phi} - h(t)f(x) - h(t)F(x)\Phi - \dot{h}(t)x + u \\ &= h(t)F(x)\tilde{\Phi} + \tilde{F}\hat{\Phi} + \tilde{f} - \dot{h}(t)x + u, \end{aligned} \tag{8}$$

where $\tilde{f} = g(y) - h(t)f(x)$, $\tilde{F} = G(y) - h(t)F(x)$ and $\tilde{\Phi} = \hat{\Phi} - \Phi$. In practical situation, the parameters belonging to the drive and the response systems are always unknown. Therefore, by using adaptive control and the parameters identification techniques, the controller can be designed as:

$$u = -\tilde{f} - \tilde{F}\hat{\Phi} + \dot{h}(t)x - ke - e_s, \tag{9}$$

where

$$e_s = C^T e, \tag{10}$$

denotes the linear coupling of the drive and response systems and the adaptive parameter update laws are chosen as

$$\dot{\hat{\Phi}} = -F^T(x)h(t)e. \tag{11}$$

Definition 1.1 For the drive system (5) and the response system (6), it is said that the systems (5) and (6) are function projective dual synchronization if there exists a scaling function $h(t)$, such that $\lim_{t\to\infty} \|e(t)\| = 0$.

Theorem 1.1 *For given synchronization scaling function $h(t)$ and any initial conditions $x(0), y(0)$, the function projective dual synchronization between drive system (5) and response system (6) will occur by the control law (9) and the parameter update law (11).*

Proof. Construct dynamical Lyapunov function candidate in the form of:

$$V = \frac{1}{2}[e^T e + \tilde{\Phi}^T \tilde{\Phi}], \tag{12}$$

with the choice of the controller (9) and the parameter update law (11), the time derivative of V along the trajectories of equation (8) is

$$\dot{V} = e^T \dot{e} + \dot{\tilde{\Phi}}^T \tilde{\Phi} = e^T[h(t)F(x)\tilde{\Phi} - ke - e_s] + [-F(x)^T h(t)e]^T \tilde{\Phi} = -e^T Pe < 0. \tag{13}$$

Suppose we select an appropriate positive definite matrix P such that $\dot{V} < 0$, that is, $\dot{V}$ is negative definite. Then, according to the Lyapunov stability theorem [38], the function projective dual synchronization of the systems (5) and (6) is achieved under the certain chosen controller u and parameters update law. This completes the proof.

2 Adaptive Function Projective Dual Synchronization of Chaotic Systems

In this section, we realized the adaptive projective dual synchronization behavior in a pair of chaotic Lorenz and Lü systems, using proposed the technique. Now, define the pair of the drive system equations and the pair of the response system equations as

Drive 1: Lü system [40] is given by

$$\begin{aligned} \dot{x_1} &= \alpha(y_1 - x_1), \\ \dot{y_1} &= -x_1 z_1 + \delta y_1, \\ \dot{z_1} &= x_1 y_1 - \beta z_1. \end{aligned} \tag{14}$$

Drive 2: Lorenz system [41] is given by

$$\begin{aligned} \dot{x_2} &= \sigma(y_2 - x_2), \\ \dot{y_2} &= \rho x_2 - x_2 z_2 - y_2, \\ \dot{z_2} &= x_2 y_2 - \gamma z_2. \end{aligned} \tag{15}$$

So the corresponding response systems are as follows:

Response 1:

$$\begin{aligned} \dot{x_3} &= \hat{\alpha}(y_3 - x_3) + u_1, \\ \dot{y_3} &= -x_3 z_3 + \hat{\delta} y_3 + u_2, \\ \dot{z_3} &= x_3 y_3 - \hat{\beta} z_3 + u_3. \end{aligned} \tag{16}$$

Response 2:

$$\begin{aligned} \dot{x_4} &= \hat{\sigma}(y_4 - x_4) + u_4, \\ \dot{y_4} &= \hat{\rho} x_4 - x_4 z_4 - y_4 + u_5, \\ \dot{z_4} &= x_4 y_4 - \hat{\gamma} z_4 + u_6, \end{aligned} \tag{17}$$

where $\alpha, \delta, \beta, \sigma, \rho, \gamma$, are unknown system parameters, $\hat{\alpha}, \hat{\delta}, \hat{\beta}, \hat{\sigma}, \hat{\rho}, \hat{\gamma}$ are the estimates of $\alpha, \delta, \beta, \sigma, \rho, \gamma$, respectively, and $U = (u_1, u_2, u_3, u_4, u_5, u_6)^T$ is the controller function to be determined. The error dynamical system can be written as

$$
\begin{aligned}
\dot{e_1} &= \hat{\alpha}((y_3 - x_3) - h_1(t)(y_1 - x_1)) + h_1(t)\tilde{\alpha}(y_1 - x_1) - \dot{h}_1(t)x_1 + u_1, \qquad (18)\\
\dot{e_2} &= -x_3z_3 + \hat{\delta}(y_3 - h_2(t)y_1) + h_2(t)(x_1z_1 + \tilde{\delta}y_1) - \dot{h}_2(t)y_1 + u_2,\\
\dot{e_3} &= x_3y_3 - \hat{\beta}(z_3 - h_3(t)z_1) - h_3(t)(x_1y_1 + \tilde{\beta}z_1) - \dot{h}_3(t)z_1 + u_3,\\
\dot{e_4} &= \hat{\sigma}((y_4 - x_4) - h_4(t)(y_2 - x_2)) + h_4(t)\tilde{\sigma}(y_2 - x_2) - \dot{h}_4(t)x_2 + u_4,\\
\dot{e_5} &= \hat{\rho}(x_4 - h_5(t)x_2) - x_4z_4 - y_4 + h_5(t)(x_2z_2 + y_2 + \tilde{\rho}x_2) - \dot{h}_5(t)y_2 + u_5,\\
\dot{e_6} &= x_4y_4 - \hat{\gamma}(z_4 - h_6(t)z_2) - h_6(t)(x_2y_2 + \tilde{\gamma}z_2) - \dot{h}_6(t)z_2 + u_6,
\end{aligned}
$$

where $e_1 = x_3 - h_1(t)x_1$, $e_2 = y_3 - h_2(t)y_1$, $e_3 = z_3 - h_3(t)z_1$, $e_4 = x_4 - h_4(t)x_2$, $e_5 = y_4 - h_5(t)y_2$, $e_6 = z_4 - h_6(t)z_2$, and $\tilde{\alpha} = \hat{\alpha} - \alpha$, $\tilde{\delta} = \hat{\delta} - \delta$, $\tilde{\beta} = \hat{\beta} - \beta$, $\tilde{\sigma} = \hat{\sigma} - \sigma$, $\tilde{\rho} = \hat{\rho} - \rho$, $\tilde{\gamma} = \hat{\gamma} - \gamma$, respectively. Our goal is to find a suitable adaptive control law and parameter update rule equation so that pair of the two chaotic systems will approach projective dual synchronization for any initial conditions.

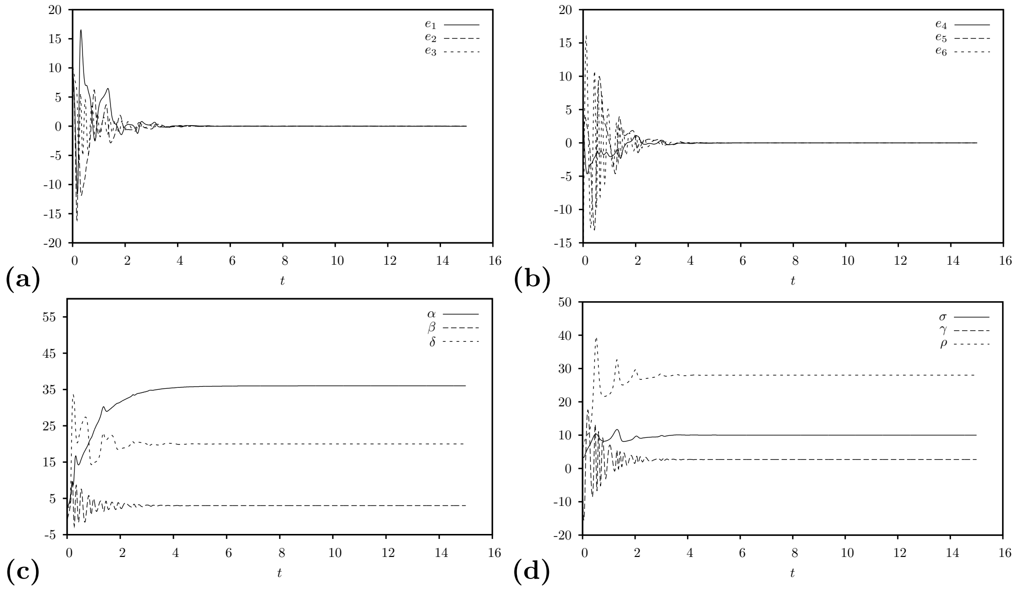

Figure 1: (a)–(b): Error signals between drive and response systems for Case I. (c)–(d): Estimated values for unknown parameters for Case I.

Theorem 2.1 *For given synchronization scaling function matrix* $h(t) = \mathrm{diag}(h_1(t), h_2(t), \ ..., h_6(t))$, *the function projective dual synchronization between the drive systems (14)-(15) and the response systems (16)–(17) will occur if the adaptive*

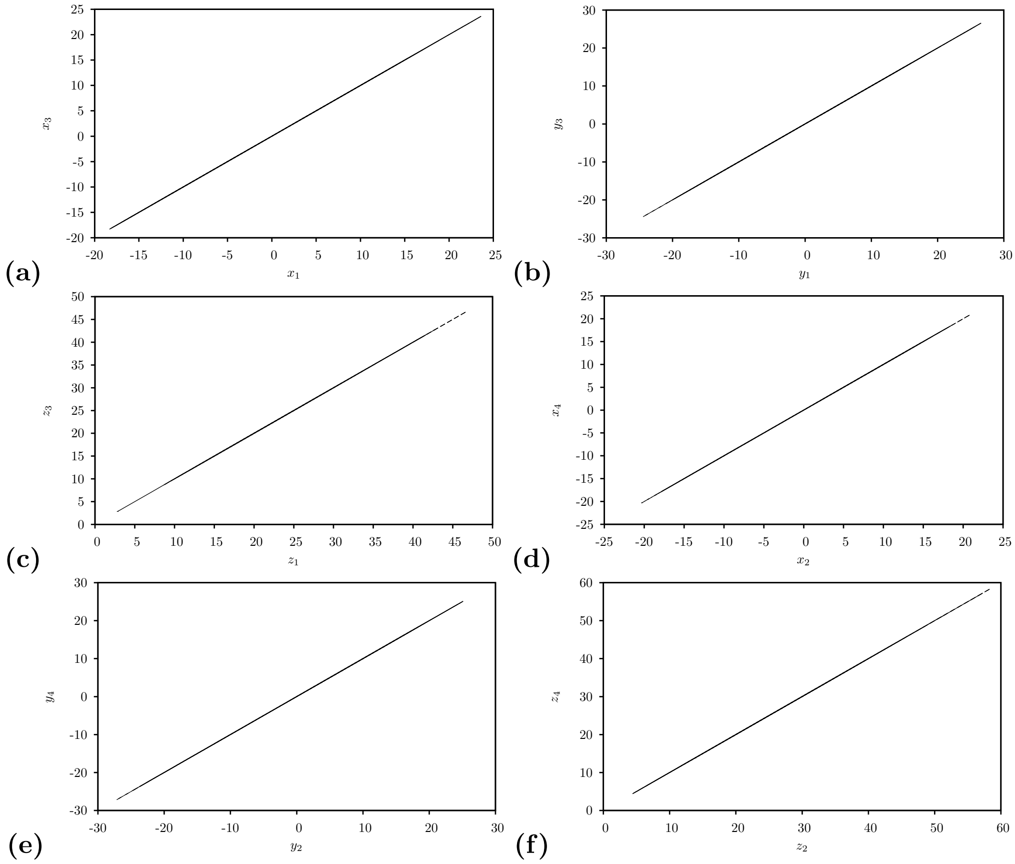

Figure 2: Signals x_1 versus x_3, y_1 versus y_3, and z_1 versus z_3 and signals x_2 versus x_4, y_4 versus y_4, and z_2 versus z_4 after dual–synchronization for Case I.

control law equation is designed as follows

$$\begin{aligned} u_1 &= -\hat{\alpha}((y_3 - x_3) - h_1(t)(y_1 - x_1)) + \dot{h}_1(t)x_1 - ke_1 - e_s, \\ u_2 &= x_3z_3 - h_2(t)x_1z_1 - \hat{\delta}(y_3 - h_2(t)y_1) + \dot{h}_2(t)y_1 - ke_2 - e_s, \\ u_3 &= h_3(t)x_1y_1 - x_3y_3 + \hat{\beta}(z_3 - h_3(t)z_1) + \dot{h}_3(t)z_1 - ke_3 - e_s, \\ u_4 &= -\hat{\sigma}((y_4 - x_4) - h_4(t)(y_2 - x_2)) + \dot{h}_4(t)x_2 - ke_4 - e_s, \\ u_5 &= x_4z_4 + y_4 - \hat{\rho}(x_4 - h_5(t)x_2) - h_5(t)(x_2z_2 + y_2) + \dot{h}_5(t)y_2 - ke_5 - e_s, \\ u_6 &= \hat{\gamma}(z_4 - h_6(t)z_2) + h_6(t)x_2y_2 - x_4y_4 + \dot{h}_6(t)z_2 - ke_6 - e_s, \end{aligned} \tag{19}$$

where

$$e_s = a_1e_1 + a_2e_2 + a_3e_3 + b_1e_4 + b_2e_5 + b_3e_6 \tag{20}$$

denotes the linear coupling of the drive and response systems and the adaptive parameter

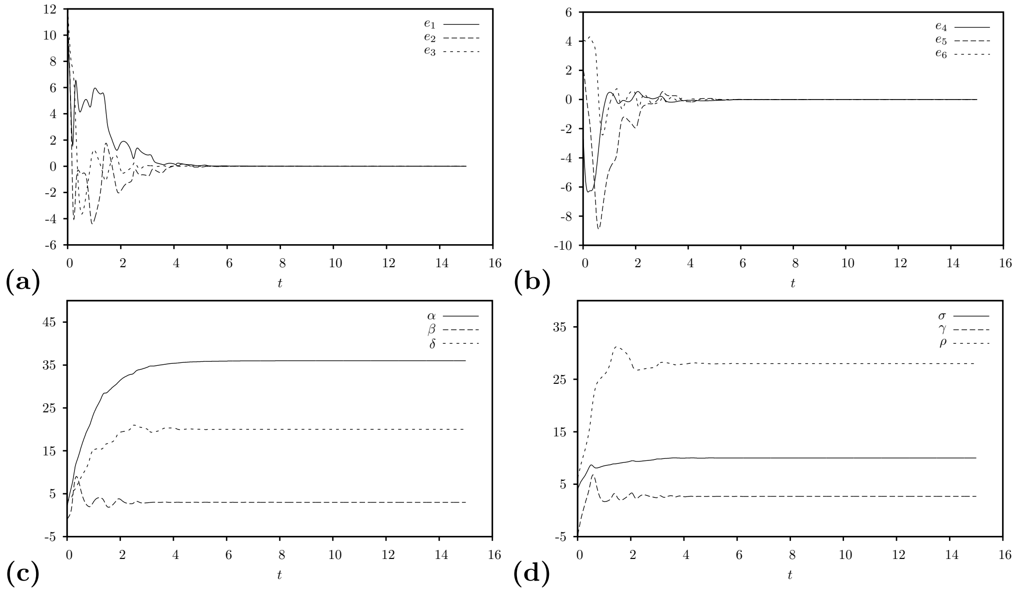

Figure 3: (a)–(b): Error signals between drive and response systems for Case II. (c)–(d): Estimated values for unknown parameters for Case II.

update laws are chosen as

$$\begin{aligned} \dot{\hat{\alpha}} &= -h_1(t)(y_1 - x_1)e_1, \quad \dot{\hat{\delta}} = -h_2(t)y_1e_2, \\ \dot{\hat{\beta}} &= h_3(t)z_1e_3, \quad \dot{\hat{\sigma}} = -h_4(t)(y_2 - x_2)e_4, \\ \dot{\hat{\rho}} &= -h_5(t)x_2e_5, \quad \dot{\hat{\gamma}} = h_6(t)z_2e_6. \end{aligned} \tag{21}$$

Proof. Substituting (19) into (18) leads to the following error system

$$\begin{aligned} \dot{e_1} &= h_1(t)\tilde{\alpha}(y_1 - x_1) - ke_1 - e_s, \quad \dot{e_2} = h_2(t)\tilde{\delta}y_1 - ke_2 - e_s, \\ \dot{e_3} &= -h_3(t)\tilde{\beta}z_3 - ke_3 - e_s, \quad \dot{e_4} = h_4(t)\tilde{\sigma}(y_2 - x_2) - ke_4 - e_s, \\ \dot{e_5} &= h_5(t)\tilde{\rho}x_2 - ke_5 - e_s, \quad \dot{e_6} = -h_6(t)\tilde{\gamma}z_2 - ke_6 - e_s. \end{aligned} \tag{22}$$

Construct a Lyapunov function of the form:

$$V = \frac{1}{2}(e^Te + \tilde{\alpha}^2 + \tilde{\delta}^2 + \tilde{\beta}^2 + \tilde{\sigma}^2 + \tilde{\rho}^2 + \tilde{\gamma}^2). \tag{23}$$

Inserting (20), (21) and (22) into the time derivative of V leads to

$$\begin{aligned} \dot{V} &= e^T\dot{e} + \tilde{\alpha}\dot{\tilde{\alpha}} + \tilde{\delta}\dot{\tilde{\delta}} + \tilde{\beta}\dot{\tilde{\beta}} + \tilde{\sigma}\dot{\tilde{\sigma}} + \tilde{\rho}\dot{\tilde{\rho}} + \tilde{\gamma}\dot{\tilde{\gamma}} \qquad (24) \\ &= (h_1(t)\tilde{\alpha}(y_1 - x_1) - ke_1 - e_s)e_1 + (h_2(t)\tilde{\delta}y_1 - ke_2 - e_s)e_2 - (h_3(t)\tilde{\beta}z_3 + e_3 + e_s)e_3 \\ &\quad + (h_4(t)\tilde{\sigma}(y_2 - x_2) - ke_4 - e_s)e_4 + (h_5(t)\tilde{\rho}x_2 - ke_5 - e_s)e_5 - (h_6(t)\tilde{\gamma}z_2 + ke_6 + e_s)e_6 \\ &\quad - \tilde{\alpha}(h_1(t)(y_1 - x_1)e_1) - \tilde{\delta}(h_2(t)y_1e_2) + \tilde{\beta}(h_3(t)z_1e_3) - \tilde{\sigma}(h_4(t)(y_2 - x_2)e_4) \\ &\quad - \tilde{\rho}(h_5(t)x_2e_5) + \tilde{\gamma}(h_6(t)z_2e_6) \end{aligned}$$

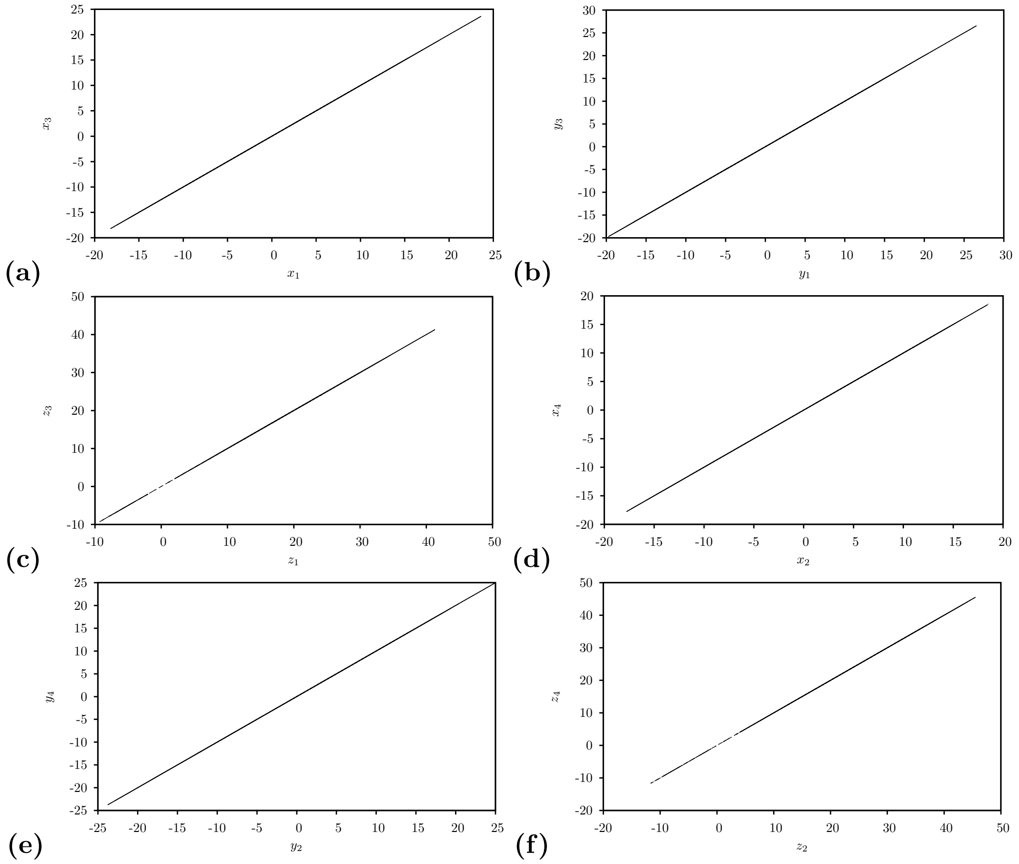

Figure 4: Signals x_1 versus x_3, y_1 versus y_3, and z_1 versus z_3 and signals x_2 versus x_4, y_4 versus y_4, and z_2 versus z_4 after dual–synchronization for Case II.

$$
\begin{aligned}
= \; - \; & [(k+a_1)e_1^2 + (a_1+a_2)e_1e_2 + (a_1+a_3)e_1e_3 + (a_1+b_1)e_1e_4 + (a_1+b_2)e_1e_5 \\
& +(a_1+b_3)e_1e_6 + (k+a_2)e_2^2 + (a_2+a_3)e_2e_3 + (a_2+b_1)e_2e_4 + (a_2+b_2)e_2e_5 \\
& +(a_2+b_3)e_2e_6 + (k+a_3)e_3^2 + (a_3+b_1)e_3e_4 + (a_3+b_2)e_3e_5 + (a_3+b_3)e_3e_6 \\
& +(k+b_1)e_4^2 + (b_1+b_2)e_4e_5 + (b_1+b_3)e_4e_6 + (k+b_2)e_5^2 + (b_2+b_3)e_5e_6 + (k+b_3)e_6^2] \\
= \; & -e^T Pe < 0,
\end{aligned}
$$

where $e = [|e_1|, |e_2|, |e_3|, |e_4|, |e_5|, |e_6|]$ and P is real symmetric matrix. From the Lyapunov theorem of stability [38], it is simple to point out that the zero equilibrium point $(e_i = 0, i = 1, ..., 6)$ of the error dynamical system (18) is asymptotically stable if the real symmetric matrix P is positive definite. According to Sylvester's theorem [39], P is positive definite if and only if $\Delta_i > 0, i = 1, 2, ..., 6$, where Δ_i represents the ith order sequential sub determinant of matrix. That is, we should choose the appropriate coupled parameters. Then, we realize the function projective dual synchronization between a pair of Lü systems and a pair of Lorenz systems. This completes the proof.

2.1 Numerical simulation and results for function projective dual synchronization

In the present section, the numerical simulations for the function projective dual synchronization of a pair of chaotic systems are studied. The true values of the unknown parameter of the systems ((14)–(15)) are taken as $\alpha = 36, \delta = 20, \beta = 3$, and $\sigma = 10, \rho = 28, \gamma = 8/3$, so both systems exhibit chaotic behavior. The initial values of the estimated unknown parameter vectors of the systems are taken as $\alpha(0) = 2, \delta(0) = -1, \beta(0) = 3$, and $\sigma(0) = 4, \rho(0) = -5, \gamma(0) = 6$. The initial conditions of the drive system (14) and the drive system (15) are taken as $x_1(0) = 1, y_1(0) = 2, z_1(0) = 3, x_2(0) = -9,\ y_2(0) = 5, z_2(0) = 30$, the initial conditions of the response system (16) and the response system (17) are taken as $x_3(0) = 11, y_3(0) = 12, z_3(0) = 13$ and $x_4(0) = -4, y_4(0) = 3, z_4(0) = 10$, respectively. The coupled parameters are valued as $a_i = (1, 1, 1), b_i = (1, 1, 1), i = 1, 2, 3$, for which condition P is positive definite. The real positive constants k is taken as 1.

Case I. Let the scaling function be $h_i(t) = 0.9 + \dfrac{t}{1+t^2}, i = (1, .2, ..., 6)$. The simulation results are shown through Fig. 1 (a)–(d), which shows that the dual synchronization errors converge asymptotically to zero and the estimated parameters $\hat{\alpha}, \hat{\delta}, \hat{\beta}$, and $\hat{\sigma}, \hat{\rho}, \hat{\gamma}$ converge to the original parameter $\alpha = 36, \delta = 20, \beta = 3$, and $\sigma = 10, \rho = 28, \gamma = 8/3$ as $t \to \infty$. Fig. 2 shows the signals after dual synchronization.

Case II. Let the scaling function be $h_i(t) = 0.2 + 0.5 \sin\left(\dfrac{\pi t}{10}\right)$. The simulation results are depicted through Fig. 3 (a)–(d), which shows that the dual synchronization errors converge asymptotically to zero and the estimated parameters $\hat{\alpha}, \hat{\delta}, \hat{\beta}$, and $\hat{\sigma}, \hat{\rho}, \hat{\gamma}$ converge to the original parameters $\alpha = 36, \delta = 20, \beta = 3$, and $\sigma = 10, \rho = 28, \gamma = 8/3$ as $t \to \infty$. Fig. 4 shows the signals after dual synchronization.

3 Conclusion

In the present paper, we have successfully demonstrated the function projective dual synchronization between a pair of chaotic systems using adaptive control method with uncertain parameters. The method is applied for the function projective dual synchronization between chaotic Lü and Lorenz systems. This clearly exhibits that the adaptive control method is effective and convenient to achieve the global dual synchronization of a pair of chaotic systems. Eventually some simulation results shown in corresponding figures have illustrated the effectiveness and feasibility of the proposed controller.

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References

[1] Chen, G. and Dong, X. *From Chaos to Order*. World Scientific, Singapore, 1998.

[2] Luo, A.C.J. A theory for synchronization of dynamical systems. *Commun. Nonlinear. Sci. Numer. Simulat.* **14** (5) (2009) 1901–1951.

[3] Carroll, T. and Perora, L.M. Synchronization in chaotic systems. *Phys Rev. Lett.* **64** (8) (1990) 821–824.

[4] Ouannas, A. Al-Sawalha, M. M. A new approach to synchronize different dimensional chaotic maps using two scaling matrices. *Nonlinear Dynamics and Systems Theory* **15** (4) (2015) 400–408.

[5] Rosenblum, M. G., Pikovsky, A. S. and Kurths. J., Phase synchronization of chaotic oscillators. *Phys Rev. Lett.* **76** (11) (1996) 1804–1807.

[6] Lòpez-Mancilla1, D. and. Cruz-Hern, C. Output synchronization of chaotic systems: model-matching approach with application to secure communication. *Nonlinear Dynamics and Systems Theory* **5** (2) (2005) 141–156.

[7] Al-Sawalha, M.M. Adaptive synchronization and anti-synchronization of a chaotic Lorenz-Stenflo system with fully unknown parameters. *Chin. Phys. Lett.* **30** (7) (2013) 070502.

[8] Al-Sawalha, M.M and Noorani, M.S.M. Chaos anti-synchronization between two novel different hyperchaotic systems. *Chin. Phys. Lett.* **25** (8) (2008) 2743.

[9] Al-Sawalha, M.M and Noorani, M.S.M. Adaptive anti-synchronization of two identical and different hyperchaotic systems with uncertain parameters. *Commun. Nonlinear. Sci. Numer. Simulat.* **15** (4) 1036–1047 (2010).

[10] Al-Sawalha, M.M and Noorani, M.S.M. Anti-synchronization of chaotic systems with uncertain parameters via adaptive control. *Phys. Lett. A* **373** (32) 2852–2857 (2009).

[11] Chen, Y., Chen, X.X. and Gu, S.S. Lag synchronization of structurally nonequivalent chaotic systems with time delay. *Nonlinear Anal. Theory, Methods Appl.* **66** (9) (2007) 1929–1937.

[12] Shahverdiev, E.M., Sivaprakasam, S. and Shore, K.A. Lag synchronization in time-delayed systems. *Phys. Lett. A* **292** (6) (2002) 320–324.

[13] Rulkov, N.F., Sushchik, M.M. and Tsimring, L.S. Generalized synchronization of chaos in directionally coupled chaotic systems. *Phys. Rev. E* **51** (2) (1995) 980–994.

[14] Zhang, K., Wang, H. and Fang, H. Feedback control and hybrid projective synchronization of a fractional-order NewtonLeipnik system. *Commun. Nonlinear Sci. Numer. Simul.* **17** (1) (2012) 317–328.

[15] Chen, J., Jiao, L., Wu, J. and Wang, X. Projective synchronization with different scale factors in a driven-response complex network and its application in image encryption. *Nonlinear Analysis: Real World Applications* **11** (4) (2010) 3045–3058.

[16] Sudheer, K.S. and Sabir, M. Function projective synchronization in chaotic and hyperchaotic systems through open-plus-closed-loop coupling. *Chaos* **20** (2010) 013115.

[17] Dua, H., Zeng, Q., Wang, C. and Ling, M., Function projective synchronization in coupled chaotic systems. *Nonlinear Analysis: Real World Applications* **11** (2) (2010) 705–712.

[18] Xiaobing, Z., Lianglin X. and Xiaomei, C. Adaptive switched generalized function projective synchronization between two hyperchaotic systems with unknown parameters, *Entropy* **16** (1) (2014) 377–388.

[19] Mainieri, R. and Rehacek, J. Projective synchronization in three-dimensional chaotic systems. *Phys Rev Lett* **82** (15) (1999) 3042–3045.

[20] Wu, X. and Lu, Y. Generalized projective synchronization of the fractional-order Chen hyperchaotic system. *Nonlinear Dynamics* **57** (22) (2009) 25–53.

[21] Li, Z. and Xu, D. A secure communication scheme using projective chaos synchronization. *Chaos, Solitons & Fractals* **22** (2) (2004) 477–481.

[22] An, H.L. and Chen, Y. The function cascade synchronization method and applications, *Commun. Nonlinear Sci. Numer. Simul.* **13** (10) (2008) 2246–2255.

[23] Chen, Y. and Li, X. Function projective synchronization between two identical chaotic systems. *Int. J. Modern Phys. C* **18** (5) (2007) 883–888.

[24] Chen, Y. An, H.L. and Li, Z.B. The function cascade synchronization approach with uncertain parameters or not for hyperchaotic systems. *Appl. Math. Comput.* **197** (1) (2008) 96–110.

[25] Olusola, O.I., Vincent, U.E., Njah, A.N. and Idowu, B.A. Global stability and synchronization criteria of linearly coupled gyroscope. *Nonlinear Dynamics and Systems Theory* **13** (3) (2013) 258–269.

[26] Tsimring, L. S. and Sushchik, M.M. Multiplexing chaotic signals using synchronization. *Phys. Lett. A* **213** (3) (1996) 155–166.

[27] Liu, Y. and Davids, P. Dual synchronization of chaos, *Phys. Rev. E* **61** (3) (2000) 2176–2179.

[28] Uchida, A., Kinugawa, S., Matsuura,T. and Yoshimori, S. Dual synchronization of chaos in one-way coupled microchip lasers, *Phys. Rev. E* **67** (2) (2003) 026220.

[29] Ning, D., Lu, J. and Han, X. Dual synchronization based on two different chaotic systems: Lorenz systems and Rossler systems. *J. Comput. Appl. Math.* **206** (2) (2007) 1046–1050.

[30] Hassan. S. and Mohammad. S. Dual synchronization of chaotic systems via time-varying gain proportional feedback. *Chaos, Solitons* & *Fractals* **38** (5) (2008) 1342–1348.

[31] Ghosh, D. and Roy C. A. Dual-anticipating, dual and dual-lag synchronization in modulated time-delayed systems. *Phys. Lett. A* **374** (34) (2010) 3425–3436.

[32] Ghosh, D. Projective-dual synchronization in delay dynamical systems with time-varying coupling delay. *Nonlinear Dynamics* **66** (4) (2011) 717–730.

[33] Jian, X., Zhen, Z.M. and Ye. H.Y. Dual synchronization of fractional-order chaotic systems via a linear controller. *The Scientific World Journal* **2013**, Article ID 159194, (2013) 1–6.

[34] Othman, A.A., Noorani, M.S.M. and Al-sawalha, M. M. Dual synchronization of chaotic and hyperchaotic systems. *J. Nonlinear Sci. Appl.* **9** (2016) 4666–4677.

[35] Othman, A.A., Noorani, M.S.M. and Al-sawalha, M. M. Adaptive dual synchronization of chaotic and hyperchaotic systems with fully uncertain parameter. *Optik* **127** (19) (2016) 7852–7864.

[36] Othman, A.A., Noorani, M.S.M. and Al-sawalha, M. M. Adaptive dual anti-synchronization of chaotic systems with fully uncertain parameters. *Optik* **127** (22) (2016) 10478–10489.

[37] Othman, A.A., Noorani, M.S.M. and Al-sawalha, M. M. Nonlinear feedback control for dual synchronization of chaotic systems. *AIP Conference Proceedings* **1784** (1) (2016) 050007.

[38] Lasalle, J. and Lefschtg, S. *Stability by Lyapunovs Direct Method with Application.* Academic Press, New York, 1961.

[39] Slotine, J.E. and Li, W. *Applied Nonlinear Control.* Prentice Hall, New Jersey (1991).

[40] Lü, J., Chen, G. and Zhang, S. A new chaotic attractor coined. *Int. J. Bifurcat. Chaos* **12** (3) (2002) 1001–1115.

[41] Lorenz, E. N. Deterministic nonperiodic flow. *J. Atmos. Sci.* **20** (1963), 130–141.

[42] J. Park. Adaptive synchronization of hyperchaotic Chen system with uncertain parameters. *Chaos, Solitons* & *Fractals* **26** (3)(2005) 959–964.

[43] Yuxia, L., Wallace, K. and Chen, G. Generating hyperchaos via state feedback control. *Int. J. Bifurcat. Chaos* **15** (3367) (2005) 3367–3375.