Nonlinear Dynamics and Systems Theory, **17** (2) (2017) 175–192

Capacity, Theorem of H. Brezis and F.E. Browder Type in Musielak–Orlicz–Sobolev Spaces and Application

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Received: July 9, 2016; Revised: April 9, 2017

Abstract: The second section of this paper is devoted to the study of the capacity theory in Musielak–Orlicz–Sobolev space, we study basic's properties, including monotonicity, countable subadditivity and several convergence results, we prove that each Musielak-Orlicz-Sobolev function has a quasi-continuous representative. In the third section, we generalize the Theorem of H. Brezis and F.E. Browder in the setting of Musielak–Orlicz–Sobolev space $W^mL_\varphi(\Omega)$, which extends the previous result of H. Brezis and F.E. Browder [10]. In the fourth section, we make an application to an unilateral problem.

Keywords: *Musielak-Orlicz-Sobolev spaces; capacity; theorem of H. Brezis and F.E. Browder; unilateral problem.*

Mathematics Subject Classification (2010): 31C15, 35M86, 35R35, 49J40.

1 Introduction

The theory of capacity and non-linear potential in the classical Lebesgue space $L^p(\Omega)$, was mainly studied by Maz'ya and Khavin in [17] and Meyers in [21]. These authors in their previous works have introduced the concept of capacity and non-linear potential in these spaces and provided very rich applications in functional analysis, harmonic analysis and in the theory of partial differential equations.

When we replace the spaces $L^p(\Omega)$ by the general one $L_A(\Omega)$ generated by an N-function, some fundamental properties are not satisfied, in particular, the reflexivity of

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spaces (obviously for an N-function which doses not satisfying the $\triangle_2$ condition). In this case, we found some works, in particular In [3] and [4].

When we replace $A(t)$ by some Musielak–Orlicz function $\varphi(x,t)$, the situation belong more difficult and the Musielak–Orlicz spaces obtained is $L_\varphi(\Omega)$ which has lost many interest functional properties. In this case, we refer the reader [13] and [18].

Thus, the first goal of this paper is to extend the theory of capacity in the setting of Musielak–Orlicz–Sobolev spaces $W^mL_\varphi(\Omega)$. Moreover, we generalize the Theorem 1 of [1], in the setting of Musielak–Orlicz–Sobolev spaces $W^mL_\varphi(\Omega)$, this generalisation is an extension of the corresponding result of H.Brezis and F.E.Browder(see [10] and [15]).

Now, let give and comment the following theorem:

Theorem 1.1 *Let Ω be an open subset of $\mathbb{R}^N$, $m \in \mathbb{N}$ and $1 < p, p' < +\infty$, such that $\frac{1}{p} + \frac{1}{p'} = 1$. Consider u in $W_0^{m,p}(\Omega)$, $u \geqslant 0$ a.e in Ω and T in $W_0^{-m,p'}(\Omega)$, such that $T = \mu + h$, where μ is a positive Radon measure and h an $L^1_{loc}(\Omega)$ function; Assume moreover that*

$$h(x)u(x) \geqslant -|\Phi(x)| \quad a.e \quad x \in \Omega, \ for\ some\ \Phi\ in\ L^1(\Omega).$$

Then:

$$hu \in L^1(\Omega),\ u \in L^1(\Omega, d\mu)\ and\ < T, u >= \int_\Omega u d\mu + \int_\Omega hudx. \tag{1}$$

This result is proved by L. Boccardo, D. Giachetti and F. Murat in [15], and extends previous Theorem of H. Brezis and F. Browder in [10], who considered the cases where either $\mu \equiv 0$ or $h \equiv 0$. the main tool in order to prove these results is the Hedberg's approximation (in $W_0^{m,p}(\Omega)$ norm) of function $u \in W_0^{m,p}(\Omega)$ by a sequence of functions $(u_n)_n$ which belong to $L^\infty(\Omega) \cap W_0^{m,p}(\Omega)$, have compact support in Ω and satisfy $u_n u \geqslant 0$, $|u_n| \leqslant u$ *a.e.* in Ω.

Note that an application of the previous theorem to study the following nonlinear variational inequality:

$$u \in K_\Phi,\ g(.\ ,\ u) \in L^1(\Omega),\ ug(.\ ,\ u) \in L^1(\Omega),$$

$$< Au, v-u > + \int_\Omega g(.,u)(v-u)dx \geqslant < f, v-u >,\ \forall v \in K_\Phi \cap L^\infty(\Omega), \tag{2}$$

where A is a pseudo-monotone operator acting on $W_0^{m,p}(\Omega)$, $f \in W^{-m,p'}(\Omega)$, $K_\Phi = \{v : v \in W_0^{m,p}(\Omega), v \geqslant \Psi\ a.e\ in\ \Omega\}$, $\Psi \in W_0^{m,p}(\Omega) \cap L^\infty(\Omega)$ and g satisfies the sign condition $sg(x,s) \geqslant 0$ but no growth restriction with respect to s.

Let us mention that a generalization of the Theorem1.1 and the problem (2) in the setting of Orlicz-Sobolev space $W^mL_A(\Omega)$ is studied by A.Benkirane in [1].

Hence, our second purpose is to extend the above Theorem1.1 in the general setting of Musielak–Orlicz–Sobolev space $W^mL_\varphi(\Omega)$ and also, we give an application of this generalized result in order to study the previous unilateral problem (2) in the Musielak–Orlicz–Sobolev space $W^mL_\varphi(\Omega)$.

2 Preliminary

2.1 Musielak–Orlicz function

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$, and let φ be a real-valued function defined in $\Omega \times \mathbb{R}^+$ and satisfying the following conditions:

a) $\varphi(x,.)$ is an N-function [convex; increasing; continuous; $\varphi(x,0) = 0$; $(\forall t > 0)$ $\varphi(x,t) > 0$; $\dfrac{\varphi(x,t)}{t} \to 0$ as $t \to 0$; $\dfrac{\varphi(x,t)}{t} \to \infty$ as $t \to \infty$].

b) $\varphi(.,t)$ is a measurable function.

A function $\varphi(x,t)$, which satisfies the conditions a) and b) is called a Musielak-Orlicz function. Equivalently, φ admits the representation: $\varphi(y,t) = \int_0^t a(y,\tau)d\tau$, for all $y \in \Omega$ and $t \geqslant 0$, where $a(y,.) : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, right continuous, with for all $y \in \Omega$: $a(y,0) = 0, a(y,t) > 0$ for $t > 0$ and $\lim_{t\to+\infty} a(y,t) = +\infty$. The function a(y , .) is called the derivative of $\varphi(y,.)$. The Musielak–Orlicz function φ is said to satisfy the Δ_2-condition if there exists $K \geqslant 2$ such that

$$\varphi(y,2t) \leqslant K\varphi(y,t), \quad for\ all\ y \in \Omega \quad and\ \ t \geqslant 0.$$

The smallest K is called the Δ_2-constant of φ. When the last inequality holds only for $t \geqslant$ *some* $t_0 > 0$ then φ is said to satisfy the Δ_2-condition near infinity.

2.2 Musielak–Orlicz spaces

Let φ be a Musielak–Orlicz function, we define the functional

$$\varrho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x,|u(x)|)dx,$$

where $u : \Omega \mapsto \mathbb{R}$ a Lebesgue measurable function. In the following the measurability of a function $u : \Omega \mapsto \mathbb{R}$ means the Lebesgue measurability.

The set

$$K_\varphi(\Omega) = \{u : \Omega \mapsto \mathbb{R},\ \text{measurable}/\varrho_{\varphi,\Omega}(u) < \infty\}$$

is called the Musielak–Orlicz class. The Musielak–Orlicz spaces $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently:

$$L_\varphi(\Omega) = \{u : \Omega \mapsto \mathbb{R},\ \text{measurable}/\varrho_{\varphi,\Omega}(\frac{u}{\lambda}) < +\infty,\ \text{for some}\ \lambda > 0\}.$$

$K\varphi(\Omega)$ is a convex subset of $L\varphi(\Omega)$. If $\Omega = \mathbb{R}^N$ then $L_\varphi(\mathbb{R}^N)$ is denoted by L_φ.

Let

$$\varphi^*(x,s) = sup\{st - \varphi(x,t) \ \ /t \geqslant 0\}.$$

That is, φ^* is the Musielak–Orlicz function complementary to φ in the sense of Young with respect to the variable s. For two complementary Musielak–Orlicz functions φ and φ^* the following inequality is called the Young inequality [20]

$$t.s \leqslant \varphi(x,t) + \varphi^*(x,s) \text{ for all s, t} \geqslant 0\ , x \in \Omega. \tag{3}$$

If $s = a(x,t)$, then

$$t.a(x,t) = \varphi(x,t) + \varphi^*(x,a(x,t)) \text{ for all t} \geqslant 0\ , x \in \Omega. \tag{4}$$

In the space $L_\varphi(\Omega)$ we define the following two norms:

$$||u||_{\varphi,\Omega} = \inf\{\lambda > 0 : \varrho_{\varphi,\Omega}(\frac{u}{\lambda}) \leqslant 1\}$$

which is called the Luxemburg norm and the so-called Orlicz norm by:

$$|||u|||_{\varphi,\Omega}= \sup_{||v||_{\varphi^*,\Omega}\leqslant 1} \int_\Omega |u(x)v(x)|dx,$$

where φ^* is the Musielak–Orlicz function complementary to φ. These two norms are equivalent [20].

For two complementary Musielak–Orlicz functions φ and φ^* let $u \in L_\varphi(\Omega)$ and $v \in L_{\varphi^*}(\Omega)$, we have the *Hölder* inequality [20]

$$|\int_\Omega u(x)v(x)dx| \leqslant ||u||_{\varphi,\Omega}\, |||v|||_{\varphi^*,\Omega}. \tag{5}$$

In $L_\varphi(\Omega)$ we have the relation with the norm and the modular:

$$|||u|||_{\varphi,\Omega} \leqslant \varrho_{\varphi,\Omega}(u) + 1, \tag{6}$$

$$||u||_{\varphi,\Omega} \leqslant \varrho_{\varphi,\Omega}(u) \ \text{, if } ||u||_{\varphi,\Omega} > 1, \tag{7}$$

$$||u||_{\varphi,\Omega} \geqslant \varrho_{\varphi,\Omega}(u) \ \text{, if } ||u||_{\varphi,\Omega} \leqslant 1. \tag{8}$$

If $\Omega = \mathbb{R}^N$ then $||u||_{\varphi,\mathbb{R}^N}$, $|||u|||_{\varphi,\mathbb{R}^N}$ and $\varrho_{\varphi,\mathbb{R}^N}(u)$ are denoted respectively by $||u||_\varphi$, $|||u|||_\varphi$ and $\varrho_\varphi(u)$ ($\forall u \in L_\varphi$).

We say that a sequence of function $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to+\infty} \varrho_{\varphi,\Omega}\left(\frac{u_n - u}{k}\right) = 0.$$

If φ satisfies the $\triangle_2$ condition, then modular convergence coincides with norm convergence. The closure in $L_\varphi(\Omega)$ of the set of bounded measurable functions with compact support in $\bar{\Omega}$ is denoted by $E_\varphi(\Omega)$ and it is a separable space. The equality $K_\varphi(\Omega) = E_\varphi(\Omega) = L_\varphi(\Omega)$ holds if and only if φ satisfies the $\triangle_2$ condition, for all t or for large t according to whether Ω has infinite measure or not. The dual of $E_\varphi(\Omega)$ can be identified with $L_{\varphi^*}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$ and the dual norm on $L_{\varphi^*}(\Omega)$ is equivalent to $||.||_{\varphi^*,\Omega}$. The space $L_\varphi(\Omega)$ is reflexive if and only if φ and φ^* satisfies the $\triangle_2$ condition, for all t or for large t according to whether Ω has infinite measure or not.

Lemma 2.1 *[12] Let φ be a Musielak-Orlicz function and f_n, f, g are measurable functions.*

(a) If $f_n \longrightarrow f$ almost everywhere, then $\varrho_{\varphi,\Omega}(f) \leqslant \liminf_{n\to+\infty} \varrho_{\varphi,\Omega}(f_n)$.

(b) If $|f_n| \nearrow |f|$ almost everywhere, then $\varrho_{\varphi,\Omega}(f) = \lim_{n\to+\infty} \varrho_{\varphi,\Omega}(f_n)$.

(c) If $f_n \longrightarrow f$ almost everywhere, $|f_n| \leqslant |g|$ almost everywhere, and $\varrho_{\varphi,\Omega}(\lambda g) < \infty$ for every $\lambda > 0$, then $f_n \to f$ strongly in $L_\varphi(\Omega)$.

Theorem 2.1 *[12] Let φ be a Musielak-Orlicz function.*

(a) $||f||_{\varphi,\Omega} = ||\ |f|\ ||_{\varphi,\Omega}$ for all $f \in L_\varphi(\Omega)$.

(b) If $f \in L_\varphi(\Omega)$, g a measurable function, and $0 \leqslant |g| \leqslant |f|$ almost everywhere, then:

$$g \in L_\varphi(\Omega) \ \ \textit{and} \ \ ||g||_{\varphi,\Omega} \leqslant ||f||_{\varphi,\Omega}\,.$$

(c) If $f_n \to f$ *almost everywhere, then:* $||f||_{\varphi,\Omega} \leqslant \liminf_{n\to+\infty} ||f_n||_{\varphi,\Omega}$.

(d) If $|f_n| \nearrow |f|$ *almost everywhere, with* $f_n \in L_\varphi(\Omega)$ *and* $\sup_n ||f_n||_{\varphi,\Omega} < \infty$ *then:*

$$f \in L_\varphi(\Omega) \quad and \quad ||f_n||_{\varphi,\Omega} \nearrow ||f||_{\varphi,\Omega} .$$

2.3 Musielak–Orlicz–Sobolev spaces

For any fixed non-negative integer m we define

$$W^m L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) : \forall\ |\alpha| \leqslant m,\ D^\alpha u \in L_\varphi(\Omega)\},$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ with non-negative integer α_i, $|\alpha| = |\alpha_1| + |\alpha_2| + ...|\alpha_n|$ and $D^\alpha u = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} .....\partial x_n^{\alpha_n}}$ denote the distributional derivatives of u. The $W^m L_\varphi(\Omega)$ is called the Musielak–Orlicz–Sobolev space.

For $u \in W^m L_\varphi(\Omega)$ let:

$$\bar{\varrho}^m_{\varphi,\Omega}(u) = \sum_{|\alpha|\leqslant m} \varrho_{\varphi,\Omega}(D^\alpha u)$$

and

$$||u||^m_{\varphi,\Omega} = inf\{\lambda > 0 : \bar{\varrho}_{\varphi,\Omega}(\frac{u}{\lambda}) \leqslant 1\}.$$

These functionals are a convex modular and a norm on $W^m L_\varphi(\Omega)$, respectively, and the pair $(W^m L_\varphi(\Omega),||u||^m_{\varphi,\Omega})$ is a Banach space if φ satisfies the following condition [20]:

$$(\exists c > 0) : \inf_{x\in\Omega} \varphi(x, 1) \geqslant c. \tag{9}$$

We say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that:

$$\lim_{n\to+\infty} \bar{\varrho}^m_{\varphi,\Omega}(\frac{u_n - u}{k}) = 0.$$

If $\Omega = \mathbb{R}^N$ then $W^m L_\varphi(\Omega)$, $\bar{\varrho}^m_{\varphi,\Omega}(u)$ and $||u||^m_{\varphi,\Omega}$ are denoted respectively by $W^m L_\varphi$, $\bar{\varrho}^m_\varphi(u)$ and $||u||^m_\varphi$, $\forall u \in W^m L_\varphi$.

Theorem 2.2 *[7] Let* φ *and* φ^* *be two complementary Musielak–Orlicz functions such that* φ *satisfies the conditions (9) and there exists a constant* $A > 0$ *such that for all* $x, y \in \Omega :\ |x - y| \leqslant \frac{1}{2}$ *we have:*

$$\frac{\varphi(x, t)}{\varphi(y, t)} \leqslant t^{\dfrac{A}{log(\dfrac{1}{|x-y|})}} \tag{10}$$

for all $t \geq 1$*. If* $D \subset \Omega$ *is a bounded measurable set, then* $\int_D \varphi(x, 1)dx < \infty$. φ^* *satisfies the following condition :*

$$\exists C > 0 \ :\ \ \varphi^*(x, 1) \leqslant C \ almost\ everywhere\ in\ \ \Omega. \tag{11}$$

Under the previous conditions, $D(\overline{\Omega})$ *is dense in* $W^m L_\varphi(\Omega)$ *with respect to the modular topology.*

Theorem 2.3 *[7] Let φ be a Musielak–Orlicz functions which satisfies the assumptions of theorem2.2, with $\Omega = \mathbb{R}^N$. Then $D(\mathbb{R}^N)$ is dense in $W^m L_\varphi(\mathbb{R}^N)$ with respect to the modular topology.*

2.4 Capacity

Definition 2.1 Let T the classe of Borel sets in $\mathbb{R}^N$, and a function $C : T \to [0, +\infty]$.
1) C is called capacity if the following axioms are satisfied:
i) $C(\emptyset) = 0$.
ii) $X \subset Y \Rightarrow C(X) \leqslant C(Y)$, for all X and Y in T.
iii) For all sequences $(X_n) \subset T$:

$$C(\bigcup_n X_n) \leqslant \sum_n C(X_n).$$

2) C is called outer capacity if for all $X \in T$:

$$C(X) = \inf\{C(O) : O \supset X, \quad O \text{ is } \text{ open}\}.$$

3) C is called an interior capacity if for all $X \subset T$:

$$C(X) = \sup\{C(K) : K \subset X, \quad K \text{ is compact}\}.$$

4) A property, that holds true except perhaps on a set of capacity zero, is said to be true C-quasi-everywhere, (abbreviated C-q.e).

5) f *and* (f_n) are real-valued finite functions C-q.e. We say that (f_n) converges to f in C-capacity if:

$$\forall \varepsilon > 0, \quad \lim_{n \to +\infty} C(\{x : \ |f_n(x) - f(x)| > \varepsilon\}) = 0.$$

6) f *and* (f_n) are real-valued function finite C-q.e. We say that (f_n) converges to f C-quasi- uniformly, (abbreviated C-q.u) if

$$(\forall \varepsilon > 0), (\exists\ X \in T) \ : \ C(X) < \varepsilon \ \text{ and } (f_n) \ \text{ converges to } f \text{ uniformly on } X^c.$$

3 The Main Results

3.1 Preliminary lemma

Lemma 3.1 *Let φ be a Musielak-Orlicz function which satisfies the condition (9). If $u,\ v \in W^m L_\varphi(\Omega)$, then $\max\{u, v\}$ and $min\{u, v\}$ are in $W^m L_\varphi(\Omega)$ with $\forall\ |\alpha| \leqslant m$:*

$$D^\alpha \max\{u, v\}(x) = \begin{cases} D^\alpha u(x), & \text{for almost every } x \in \{u \geqslant v\}; \\ D^\alpha v(x), & \text{for almost every } x \in \{v \geqslant u\}; \end{cases}$$

and

$$D^\alpha \min\{u, v\}(x) = \begin{cases} D^\alpha u(x), & \text{for almost every } x \in \{u \leqslant v\}; \\ D^\alpha v(x), & \text{for almost every } x \in \{v \leqslant u\}. \end{cases}$$

In particular, $|u|$ belongs to $W^m L_\varphi(\Omega)$.

Proof. It suffices to prove the assertions for $\max\{u,v\}$ since $\min\{u,v\} = -\max\{-u,-v\}$. We have $\max\{u,v\} \leqslant |u| + |v|$ almost everywhere in Ω, and $(|u| + |v|) \in L_\varphi(\Omega)$, then by Theorem 2.1 we obtain $\max\{u,v\} \in L_\varphi(\Omega)$.

On the other hand we have $|D^\alpha \max(u,v)| \leqslant |D^\alpha u| + |D^\alpha v|$ almost everywhere in Ω, and $(|D^\alpha u| + |D^\alpha v|) \in L_\varphi(\Omega)$, then by Theorem 2.1 we obtain $D^\alpha \max\{u,v\} \in L_\varphi(\Omega)$.

Thus

$$\max\{u,v\} \in W^m L_\varphi(\Omega).$$

For $|u| \in W^m L_\varphi(\Omega)$ it suffices to note that $|u| = \max\{u,0\} - \min\{u,0\}$.

3.2 Capacity in Musielak–Orlicz–Sobolev space

In this section, $\Omega = \mathbb{R}^N$ and φ is a Musielak-Orlicz function which satisfies the condition (9).

Definition 3.1 The Sobolev φ-capacity of the set, $E \subset \mathbb{R}^N$ is defined by :

$$C_\varphi(E) = \inf_{u \in A_\varphi(E)} \bar{\rho}_{m,\varphi}(u),$$

where

$$A_\varphi(E) = \{u \in W^m L_\varphi \ : \ u \geqslant 1 \textit{ on an open set containing } E \textit{ and } u \geqslant 0\}.$$

If $A_\varphi(E) = \emptyset$ we set $C_\varphi(E) = \infty$. Functions belonging to $A_\varphi(E)$ are called admissible functions for E.

Remark 3.1 In the definition of the capacity C_φ, we can restrict ourselves to those admissible functions u for which, $0 \leqslant u \leqslant 1$. Indeed, if $A_\varphi^{'}(E) = \{u \in A_\varphi(E) \ : \ 0 \leqslant u \leqslant 1\}$, then $A_\varphi^{'}(E) \subset A_\varphi(E)$ implies

$$C_\varphi(E) \leqslant \inf_{u \in A_\varphi^{'}(E)} \bar{\rho}_{m,\varphi}(u).$$

For the reverse inequality, let $\varepsilon > 0$ and take $u \in A_\varphi(E)$ such that

$$\bar{\rho}_{m,\varphi}(u) \leqslant C_\varphi(E) + \varepsilon.$$

Then by Lemma 3.1 , we have $v = max(0, min(u,1))$ belongs to $A_\varphi^{'}(E)$.

Therefore,

$$\inf_{\omega \in A_\varphi^{'}(E)} \bar{\rho}_{m,\varphi}(\omega) \leqslant \bar{\rho}_{m,\varphi}(v) \leqslant \bar{\rho}_{m,\varphi}(u) \leqslant C_\varphi(E) + \varepsilon.$$

Letting $\varepsilon \longrightarrow 0$, we obtain

$$\inf_{\omega \in A_\varphi^{'}(E)} \bar{\rho}_{m,\varphi}(\omega) \leqslant C_\varphi(E).$$

This completes the proof.

Theorem 3.1 *Let $E \subset \mathbb{R}^N$. If there exists $f \in W^m L_\varphi$ such that $f = +\infty$ on E, then $C_\varphi(E) = 0$.*

Proof. If there exists $f \in W^m L_\varphi$ such that $f = +\infty$ on E, then $f \geqslant \alpha$ onE for all $\alpha > 0$. Therefore, $\forall\ \alpha > 0 \ \ : C_\varphi(E) \leqslant \bar{\rho}_{m,\varphi}(\frac{f}{\alpha})$.

Let $\alpha > 1$, we have $\bar{\rho}_{m,\varphi}(\frac{f}{\alpha}) \leqslant \frac{1}{\alpha}\bar{\rho}_{m,\varphi}(f)$, then $0 \leqslant C_\varphi(E) \leqslant \frac{1}{\alpha}\bar{\rho}_{m,\varphi}(f)$.

Letting $\alpha \longrightarrow +\infty$, we obtain $C_\varphi(E) = 0$.

Theorem 3.2 *Let us consider the following propositions:*
i) $f_n \longrightarrow f$ *in* $W^m L_\varphi$.
ii) $f_n \longrightarrow f$ *in* $C_\varphi -$ *capacity.*
iii) there is a subsequence (f_{n_j}) *such that :* $f_{n_j} \longrightarrow f, \quad C_\varphi - q.u.$
iv) $f_{n_j} \longrightarrow f, \quad C_\varphi - q.e.$
We have i) $\Rightarrow$ *ii)* $\Rightarrow$ *iii)* $\Rightarrow$ *iv).*

Proof. Let show that $i) \Rightarrow ii)$. By Theorem 3.1 we have f and f_n are finite for every n; $C_\varphi - q.e.$

Let $\varepsilon > 0$, we have

$$C_\varphi(\{x \ \ : \ \ |f_n - f|(x) > \varepsilon\}) \leqslant \bar{\rho}_{m,\varphi}(\frac{f_n - f}{\varepsilon}).$$

Since $f_n \longrightarrow f$ *in* $W^m L_\varphi(\Omega)$,

$$(\forall \varepsilon > 0): \ \ \bar{\rho}_{m,\varphi}(\frac{f_n - f}{\varepsilon}) \longrightarrow 0.$$

Therefore,

$$\lim_{n\to+\infty} C_\varphi(\{x \ \ : \ \ |f_n - f|(x) > \varepsilon\}) = 0.$$

Let show that $ii) \Rightarrow \ iii)$. Let $\varepsilon > 0 \ \ \exists\, f_{n_j}$ such that $C_\varphi(\{x \ : \ |f_{n_j} - f|(x) > 2^{-j}\}) < \varepsilon.2^{-j}$.

We put

$$E_j = \{x \ : \ |f_{n_j} - f|(x) > 2^{-j}\} \quad \text{and} \quad G_m = \bigcup_{j\geqslant m} E_j,$$

we have $C_\varphi(G_m) \leqslant \sum_{j\geqslant m} \varepsilon.2^{-j} < \varepsilon$.

On the other hand,

$$(\forall x \in (G_m)^c) \ : \ |f_{n_j} - f|(x) \leqslant 2^{-j}, (\forall j \geqslant m).$$

Thus

$$f_{n_j} \longrightarrow f \ \ C_\varphi - q.u.$$

Let show that $iii) \Rightarrow \ iv)$. We have $\forall j \in \mathbb{N}, \exists X_j : C_\varphi(X_j) \leqslant \frac{1}{j}$ and $f_{n_j} \longrightarrow f$ *on* $(X_j)^c$.

We put $X = \bigcap_j X_j$, then $C_\varphi(X) = 0$ *and* $f_{n_j} \longrightarrow f$ *on* X^c.

Theorem 3.3 *Let* φ *be a Musielak-Orlicz function, uniformly convex that satisfies the* $\triangle_2$ *condition. If* $f_n,\ f \in W^m L_\varphi$ *such that* $f_n \rightharpoonup f$ *weakly in* $W^m L_\varphi$, *then*

$$\liminf(f_n)(x) \leqslant f(x) \leqslant \limsup f_n(x) \ \ C_\varphi - q.e.$$

Proof. $(W^mL_\varphi\ ,\ ||.||)$ is uniformly convex, therefore reflexive. By the Banach–Saks theorem, there is a subsequence denoted again by (f_n) such that the sequence $g_n = \frac{1}{n}\sum_{i=1}^{n} f_i$ converges to f strongly in W^mL_φ. By Theorem 3.2, there is a subsequence of (g_n) denoted again (g_n) such that

$$\lim_{n\to+\infty} g_n(x) = f(x) \quad C_\varphi - q.e.$$

On the other hand,

$$\liminf f_n(x) \leqslant \lim_{n\to+\infty} g_n(x).$$

Therefore,

$$\liminf_{n\to+\infty} f_n(x) \leqslant f(x) \quad C_\varphi - q.e.$$

For the second inequality, it suffices to replace f_n by $(-f_n)$ in the first inequality.

Theorem 3.4 *Let φ be a Musielak–Orlicz function, uniformly convex which satisfies the $\triangle_2$ condition. Let (X_n) be an increasing sequence of sets and $X = \bigcup_n X_n$. Then*

$$\lim_{n\to+\infty} C_\varphi(X_n) = C_\varphi(X).$$

Proof. We have $\lim_{n\to+\infty} C_\varphi(X_n) \leqslant C_\varphi(X)$. For the reverse inequality, if $\lim_{n\to+\infty} C_\varphi(X_n) = +\infty$, there is nothing to show.

Assuming that $\lim_{n\to+\infty} C_\varphi(X_n) < +\infty$, we have

$$\forall n \in \mathbb{N},\ \ \exists f_n \in W^mL_\varphi\ :\ \ f_n \geqslant 1 \ on\ \ X_n \ \ and \ \ \bar{\varrho}_{m,\varphi}(f_n) \leqslant C_\varphi(X_n) + \frac{1}{n}.$$

Now (f_n) is a bounded sequence in W^mL_φ, hence there exists a subsequence, which we denote again by (f_n), which converges weakly to a function $f \in W^mL_\varphi$. Thus

$$\bar{\rho}_{m,\varphi}(f) \leqslant \liminf_n \bar{\varrho}_{m,\varphi}(f_n).$$

On the other hand by Theorem 3.3, we have

$$\forall n \in \mathbb{N}\ : f \geqslant 1 \ \ on \ \ X_n \ \ , C_\varphi - q.e.$$

Therefore, $f \geqslant 1$ *on* X $C_\varphi - q.e.$

Let Y be a subset of X where $f \geqslant 1$, then $C_\varphi(X) = C_\varphi(Y)$. Thus,

$$\bar{\rho}_{m,\varphi}(f) \leqslant \lim_n (C_\varphi(X_n) + \frac{1}{n}).$$

Hence

$$C_\varphi(X) \leqslant \lim_n (C_\varphi(X_n).$$

Theorem 3.5 *Let φ be a Musielak–Orlicz function, uniformly convex which satisfies the $\triangle_2$ condition. C_φ is an outer capacity.*

Proof. It is obvious that $C_\varphi(\emptyset) = 0$ and $C_\varphi(X) \leqslant C_\varphi(Y)$ if $X \subset Y$.

To prove the countable sub-additivity, suppose that E_i , $i = 1, 2, \dots,$ subsets of $\mathbb{R}^N$, let $\varepsilon > 0$. We may assume that $\sum_i C_{k,\varphi}(X_i) < +\infty$, then

$$C_{k,\varphi}(X_i) < +\infty; \quad \forall i \in \mathbb{N}.$$

Next we choose $u_i \in A_\varphi(E_i)$ so that

$$\bar{\rho}_{m,\varphi}(u_i) \leqslant C_\varphi(E_i) + \varepsilon.2^{-i}; \quad \forall i \in \mathbb{N}.$$

Let $k \in \mathbb{N}$ and $v_k = \max_{1 \leqslant i \leqslant k} u_i$. By Lemma 3.1 we have $v_k \in A_\varphi(\bigcup_{i=1}^{k} E_i)$.

Thus,

$$\bar{\rho}_{m,\varphi}(v_k) \leqslant \sum_{i=1}^{k} \bar{\rho}_{m,\varphi}(u_i) \leqslant \sum_{i=1}^{k} (C_\varphi(E_i) + \varepsilon.2^{-i}) \leqslant \sum_{i=1}^{k} C_\varphi(E_i) + (\varepsilon(1 - (\frac{1}{2})^k).$$

Then,

$$C_\varphi(\bigcup_{i=1}^{k} E_i) \leqslant \sum_{i=1}^{k} C_\varphi(E_i) + \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain

$$C_\varphi(\bigcup_{i=1}^{k} E_i) \leqslant \sum_{i=1}^{k} C_\varphi(E_i) \leqslant \sum_{i=1}^{\infty} C_\varphi(E_i).$$

Since $(\bigcup_{i=1}^{k} E_i)$ increase to $(\bigcup_{i=1}^{\infty} E_i)$, by Theorem 3.4 we obtain:

$$C_\varphi(\bigcup_{i=1}^{\infty} E_i) \leqslant \sum_{i=1}^{\infty} C_\varphi(E_i).$$

It remains to prove that C_φ is outer. Indeed, by monotonicity we have:

$$(\forall\ E \subset \mathbb{R}^N\ )\ : C_\varphi(E) \leqslant \inf\{C_\varphi(O) : O \supset E, \ \ O \ is \ open\}.$$

For the reverse inequality, if $C_\varphi(E) = +\infty$, there is nothing to show.

Assume that $C_\varphi(E) < +\infty$, let $\varepsilon > 0$ and take $u \in A_\varphi(E)$ such that

$$\bar{\rho}_{m,\varphi}(u) \leqslant C_\varphi(E) + \varepsilon.$$

Since $u \in A_\varphi(E)$, there is an open set O containing E such that $u \geqslant 1$ *on* O, which implies that

$$C_\varphi(O) \leqslant \bar{\rho}_{m,\varphi}(u) \leqslant C_\varphi(E) + \varepsilon.$$

The inequality follows by letting $\varepsilon \to 0$.

Theorem 3.6 *Let (K_n) be a decreasing sequence of compacts and $K = \bigcap_n K_n$. Then,*

$$\lim_{n\to+\infty} C_\varphi(K_n) = C_\varphi(K).$$

Proof. First, we observe that $C_\varphi(K) \leqslant \lim_{n\to+\infty} C_\varphi(K_n)$. On the other hand, let O be an open set containing K. By the compactness of K, $K_i \subset O$ for all sufficiently large i. Therefore $\lim_{n\to+\infty} C_\varphi(K_n) \leqslant C_\varphi(O)$, and since C_φ is an outer capacity, we obtain the claim by taking infimum over all open set O containing K.

Theorem 3.7 *Let φ be a Musielak–Orlicz function.*

$$(\exists c > 0)(\forall X \subset \mathbb{R}^N): \quad |X| \leqslant c.C_\varphi(X),$$

where $|X|$ is the Lebesgue's measure of X.

Proof. Let $u \in A_\varphi(X)$, we have $u \geqslant 1$ on X and $\varrho_\varphi(u) \leqslant \bar{\rho}_{m,\varphi}(u)$. But $\varrho_\varphi(u) = \int_{\mathbb{R}^N} \varrho_\varphi(y, |u(y)|dy$, then

$$\varrho_\varphi(u) \geqslant \int_X \varrho_\varphi(y, |u(y)|dy \geqslant \int_X \varrho_\varphi(y, 1)dy.$$

By the inequality (9) there exists a constant $c > 0$ such that $\inf_{y\in\mathbb{R}^N} \varphi(y, 1) \geqslant c$. Therefore, $\varrho_\varphi(u) \geqslant c.|X|$. Thus,

$$c.|X| \leqslant \bar{\rho}_{m,\varphi}(u).$$

The claim follows by passing to inf on $u \in A_\varphi(X)$.

Corollary 3.1 *Let φ be a Musielak–Orlicz function. If $(f_n)_n$ is a sequence which converges to f in $W^m L_\varphi$, then there exists a subsequence of $(f_n)_n$ which converge to f almost everywhere.*

Proof. It is an immediate consequence of Theorem 3.2 and Theorem 3.7.

Theorem 3.8 *Let φ be a Musielak–Orlicz function which satisfies the condition $\triangle_2$ and the assumptions of Theorem 2.2. For each $f \in W^m L_\varphi$, there is a C_φ-quasicontinuous function $g \in W^m L_\varphi$ such that $f = g$ almost everywhere.*

Proof. Let $f \in W^m L_\varphi$. By Theorem 2.3, there exists a sequence (f_n) in $D(\mathbb{R}^N)$ such that $f_n \longrightarrow f$ in $W^m L_\varphi$. By Theorem 3.2, there exists a subsequence of (f_n) denoted again by (f_n) such that $f_n \longrightarrow f \ \ C_\varphi - q.u.$ The claim follows by Theorem 3.7.

Remark 3.2 By theorem 2.6 in [7], we have the same result if we replace $W^m L_\varphi$, by $W^m L_\varphi(\Omega)$, where Ω is a bounded Lipschitz domain in $\mathbb{R}^n$.

Theorem 3.9 *Let φ be a Musielak-Orlicz function, uniformly convex which satisfies the condition $\triangle_2$*

1) If O is an open set of $\mathbb{R}^N$ and $E \subset \mathbb{R}^N$ such that $|E| = 0$, then

$$C_\varphi(O) = C_\varphi(O - E).$$

2) Let u and v are C_φ-quasicontinuous functions in $\mathbb{R}^N$, we have
i) if $u = v$, almost everywhere in an open set $O \subset \mathbb{R}^N$, then

$$u = v \ C_\varphi - quasieverywhere \ in \ O,$$

ii) if , $u \leqslant v$, almost everywhere in an open set $O \subset \mathbb{R}^N$, then

$$u \leqslant v \ C_\varphi - quasieverywhere \ in \ O.$$

Proof. 1) It obvious that $C_\varphi(O) \geqslant C_\varphi(O - E)$. Let $u \in A_\varphi(O - E)$ thus $u \geqslant 1$ in an open containing $O - E$. Let the function f define as

$$\begin{cases} f(x) = u(x), & if \ x \in \mathbb{R}^N - E \\ f(x) = 1, & if \ x \in E. \end{cases}$$

We have $f \in A_\varphi(O)$ and $\bar{\rho}_{m,\varphi}(f) = \bar{\rho}_{m,\varphi}(u)$, thus

$$C_\varphi(O) \leqslant \bar{\rho}_{m,\varphi}(u),$$

and by passing to inf we get $C_\varphi(O) \leqslant C_\varphi(O - E)$.

2) Since C_φ is an outer capacity we get the results by [16].

Lemma 3.2 *Let Ω be a bounded Lipschitz domain in R^n, φ be a Musielak-Orlicz function which satisfies the condition (9), φ and φ^* satisfy the $\triangle_2$ condition and $m \in \mathbb{N}$. Consider $T \in W^{-m}L_{\varphi^*}(\Omega) \cap M(\Omega)$, where $M(\Omega)$ denote the set of Radon measures in Ω. If $X \subset \Omega$ is such that $C_\varphi(X) = 0$, then X is $|T|$ -measurable and $|T|(X) = 0$.*

Proof. It is the same as in [19] and [10].

3.3 Theorem of H. Brezis and F. Browder type in Musielak–Orlicz–Sobolev spaces

In this section we generalize the theorem of H. Brezis and F. Browder [10] in the setting of the Musielak–Orlicz–Sobolev spaces $W^mL_\varphi(\Omega)$.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and $m \in \mathbb{N}$. In this section we study the following question: let $w \in W_0^mL_\varphi\ (\Omega)$ and $T \in W^{-m}L_{\varphi^*}(\Omega)$ such that $T = \mu + h$, where μ lie in $M^+(\Omega)$ (the subset of positive Radon measures) and h lie $L^1_{loc}(\Omega)$; find sufficient conditions on the data in order for w to belong $L^1(\Omega; d\mu)$, for hw to belong to $L^1(\Omega)$ and finally to have:

$$< T, w >= \int_\Omega w d\mu + \int_\Omega hw dx.$$

This question was solved in [15] in the case of the classical Sobolev spaces, in [5] when $\mu = 0$ in the case of Orlicz–Sobolev spaces and in [1] in the case of Orlicz–Sobolev spaces.

Theorem 3.10 *Let φ be a Musielak–Orlicz function which satisfies the condition (9), φ and φ^* satisfy the $\triangle_2$ condition and $m \in \mathbb{N}$. Consider $w \in W_0^mL_\varphi(\Omega)$, $w \geqslant 0$ a.e in Ω and $T \in W^{-m}L_{\varphi^*}(\Omega)$ such that $T = \mu + h$, where μ lie in $M^+(\Omega)$ (the subset of positive Radon measures) and $h \in L^1_{loc}(\Omega)$, assume that:*

$$hw \geqslant -|\Phi| \ \ a.e \ in \ \Omega \ for \ some \ \ \Phi \ in \ L^1(\Omega). \tag{12}$$

Then:

$$hw \in L^1(\Omega), w \in L^1(\Omega; d\mu) \ and \ < T, w >= \int_\Omega w d\mu + \int_\Omega hw dx. \tag{13}$$

Remark 3.3 Note that $\mu(X)=0$ for all $X\subset\Omega$ such that $C_\varphi(X)=0$. Indeed by Lemma 3.2

$$|T|(X)=|\mu+h|(X)=0,$$

but

$$0\leqslant\mu(X)\leqslant|h|(X)+|\mu+h|(X)=0.$$

Let prove Theorem 3.10.

Proof. Let $w\in W_0^mL_\varphi(\Omega)$, the Lemma 2.4 of [9] yields the existence of a sequence w_n such that:
(i) $w_n\in W_0^mL_\varphi(\Omega)\cap L^\infty(\Omega)$,
(ii) supp w_n is compact,
(iii) $|w_n|\leqslant|w|$ a.e. in Ω,
(v) $w_n\longrightarrow w$ in $W_0^mL_\varphi(\Omega)$.
(vi) $w_nw\geqslant0$ a.e. in Ω.

Following the lines of [15], it is easy to deduce that

$$<\mu+h,w_n>=\int_\Omega w_nd\mu+\int_\Omega hw_ndx. \tag{14}$$

Since $w_n\longrightarrow w$ in $W_0^mL_\varphi(\Omega)$, by using the Theorem 3.2, Lemma3.2 and Remark 3.3 we have

$$w_n\longrightarrow w\quad \mu.a.e\ \ and\ \ a.e.\ \ in\ \ \Omega. \tag{15}$$

We recall that by Theorem 3.9 and Theorem 3.7, for any $v\in W^mL_\varphi(\Omega)$ one has

$$v\geqslant0\quad a.e.\ \ in\ \Omega\Leftrightarrow v\geqslant0\quad q.e.\ \ in\ \Omega.$$

This equivalence, Remark 3.3 and the fact ($w\geqslant0$ a.e. in Ω), imply

$$w_n\geqslant0\quad a.e.\ \ ,\ \ w_n\geqslant0\quad \mu.a.e.\quad and\ \ 0\leqslant w_n\leqslant w\ \ a.e.\ \ in\ \ \Omega. \tag{16}$$

On the other hand from $hw\geqslant-|\Phi|$ and $0\leqslant w_n\leqslant w\ \ a.e.\ \ in\ \ \Omega$ we have

$$hw_n\geqslant-|\Phi|\ a.e.in\ \ \Omega \tag{17}$$

Since $<\mu+h,w_n>$ is bounded, (14)and (16) imply $\int_\Omega hw_ndx\leqslant cst$; Similary (14) and (17) imply $\int_\Omega w_nd\mu\leqslant cst$.

By using (15), (16), (17) and Fatou's lemma we get $hw\in L^1(\Omega)$ and $w\in L^1(\Omega;d\mu)$. Using $0\leqslant w_n\leqslant w\quad \mu.a.e.\ \ in\ \ \Omega$ and $|hw_n|\leqslant|hw|$ a.e. in Ω, it is now easy to pass to the limit in (14); we use the convergence of w_n to w in $W_0^mL_\varphi$ (Ω) for the left hand side and Lebesgue's dominated convergence theorem in each term of the right hand side: we obtain

$$<T,w>=\int_\Omega wd\mu+\int_\Omega hwdx.$$

3.4 Application to unilateral problem

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and $m \in \mathbb{N}$. φ be a Musielak-Orlicz function which satisfies the condition (9), φ and φ^* satisfy the $\triangle_2$ condition.

We consider some right hand side $f \in W^{-m}L_{\varphi^*}(\Omega)$ and the convex set

$$K_\Phi = \{v \in W_0^m L_\varphi(\Omega),\ v \geqslant \Phi\ a.e\ in\ \Omega\},$$

where the obstacle Φ belong to $W_0^m L_\varphi(\Omega) \cap L^\infty(\Omega)$. Let a pseudo-monotone mapping S from $W_0^m L_\varphi(\Omega)$ into $W^{-m}L_{\varphi^*}(\Omega)$. which satisfies the following conditions:
(1) S is continuous from each finite-dimensional subspace of $W_0^m L_\varphi(\Omega)$ into $W^{-m}L_{\varphi^*}(\Omega)$ for the weak* topology.
(2) S maps bounded sets into bounded sets.
(3) S is coercive, i.e that for some v_0 in $K_\Phi \cap L^\infty(\Omega)$

$$\frac{< S(v), v - v_0 >}{||v||_{W_0^m L_\varphi(\Omega)}} \longrightarrow +\infty \quad as \quad ||v||_{W_0^m L_\varphi(\Omega)} \longrightarrow +\infty. \tag{18}$$

Consider finally a carathéodory function $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ witch satisfies :
(4) $s.g(x,s) \geqslant 0$, $\forall\ s \in \mathbb{R}\ \ and\ \ a.e\ \ in\ \ \Omega$,
(5) $h_t = sup_{|s| \leqslant t}|g(x,s)| \in L^1(\Omega)\ \forall t\ \geqslant 0$.

Theorem 3.11 *The variational inequality:*

$$u \in K_\Phi,\ g(.,u) \in L^1(\Omega),\ ug(.,u) \in L^1(\Omega)$$

$$< Su, v - u > + \int_\Omega g(.,u)(v-u)dx \geqslant \ < f, v-u >,\ \forall v \in K_\Phi \cap L^\infty(\Omega)$$

has at least one solution.

Proof. **First part** *Approximation and a priori istimates.*
Define $g_n(x,s) = \begin{cases} \chi_n(x)g(x,s) & if\ \ |g(x,s)| \leqslant n, \\ \chi_n(x)n\dfrac{g(x,s)}{|g(x,s)|} & if\ \ |g(x,s)| > n, \end{cases}$
where χ_n is the characteristic function of the set $\{x \in \Omega : |x| \leqslant n\}$
By by using the proposition 1 of [14] we have the approximate problem

$$\begin{cases} u_n \in K_\Phi, \\ < Su_n, v - u_n > + \displaystyle\int_\Omega g_n(.,u_n)(v-u_n)dx \geqslant \ < f, v - u_n >,\ \forall v \in K_\Phi \cap L^\infty(\Omega) \end{cases} \tag{19}$$

has at least one solution. Using $v = v_0$ as test function in (19) we get

$$< Su_n, u_n - v_0 > + \int_\Omega g_n(.,u_n)(u_n - v_0)dx \leqslant \ < f, u_n - v_0 > . \tag{20}$$

If (u_n) is not bonded in $W_0^m L_\varphi(\Omega)$ then by the assumptions **(3)** we have

$$(\forall A > 0)(\exists n_0 \in \mathbb{N}) : (\forall n \geqslant n_o)(\frac{< S(u_n), u_n - v_0 >}{||u_n||_{W_0^m L_\varphi(\Omega)}} > A). \tag{21}$$

Let $E_n = \{x \in \Omega : u_n(x) \geqslant 0\}$, by (20) and (21) we have for large n :
$A||u_n||_{W_0^mL_\varphi(\Omega)} + \int_{E_n} g_n(.,u_n)(u_n - v_0)dx + \int_{\Omega - E_n} g_n(.,u_n)u_n dx$
$\leqslant \int_{\Omega - E_n} g_n(.,u_n)v_0 dx + ||f||_{W^{-m}L_{\varphi^*}(\Omega)}||u_n||_{W_0^mL_\varphi(\Omega)} + ||f||_{W^{-m}L_{\varphi^*}(\Omega)}||v_0||_{W_0^mL_\varphi(\Omega)}$
Let $G_n = \{x \in \Omega : u_n(x) \geqslant v_o\}$ and $l = sup(|v_0|, |\Phi|)$.

By the assumptions **(4)** and **(5)** we have

$$\int_{E_n \cap G_n} g_n(.,u_n)(u_n - v_0)dx \geqslant 0,$$

$$\int_{E_n \cap G_n^c} g_n(.,u_n)u_n dx \geqslant 0,$$

$$\int_{E_n \cap G_n^c} g_n(.,u_n)v_0 dx \leqslant \int_\Omega |h_{||l||_{L^\infty(\Omega)}} v_0|,$$

$$\int_{\Omega - E_n} g_n(.,u_n)u_n dx \geqslant 0,$$

$$\int_{\Omega - E_n} g_n(.,u_n)v_0 dx \leqslant \int_\Omega |h_{||\Phi||_{L^\infty(\Omega)}} v_0|.$$

Then we get

$$||u_n||_{W_0^mL_\varphi(\Omega)} \leqslant C_1, \ \forall n \geqslant n_0,$$

which is impossible, thus (u_n) is bounded in $W_0^mL_\varphi(\Omega)$.

It follows that there exists a subsequence, again denoted by u_n such that

$$u_n \rightharpoonup u, \quad weakly \ \ in \ \ W_0^mL_\varphi(\Omega) \ \ and \ \ a.e. \ \ in \ \ \Omega.$$

Thus

$$g_n(x, u_n(x)) \longrightarrow g(x, u(x)) \quad a.e. \ \ in \ \ \Omega.$$

From (20) we get

$$\int_\Omega g_n(.,u_n)(u_n - v_0)dx \leqslant C_2. \tag{22}$$

We shall prove

$$\int_\Omega |g_n(.,u_n)(u_n - v_0)|dx \leqslant C_3.$$

Indeed

$$\begin{aligned}
\int_\Omega |g_n(.,u_n)(u_n - v_0)|dx &= \int_{G_n} g_n(.,u_n)(u_n - v_0)dx - \int_{\Omega - G_n} g_n(.,u_n)(u_n - v_0)dx \\
&= -2\int_{\Omega - G_n} g_n(.,u_n)(u_n - v_0)dx + \int_\Omega g_n(.,u_n)(u_n - v_0)dx \\
&\leqslant C_2 + 2\int_{\Omega - G_n} g_n(.,u_n)v_0 dx \\
&\leqslant C_2 + 2\int_\Omega |h_{||b||_{L^\infty}} v_0|dx = C_3,
\end{aligned} \tag{23}$$

where $b = sup(|\Phi|, |v_0|)$.

In order to prove

$$g_n(., u_n) \longrightarrow g(., u) \quad in \quad L^1(\Omega), \tag{24}$$

let us observe that, for any $\delta > 0$,

$$|g_n(x, u_n(x))| \leqslant \sup_{|t| \leqslant \delta^{-1} + ||v_0||_{L^\infty}} |g(., t)| + \delta |g_n(x, u_n(x))(u_n(x) - v_0(x))|,$$

and there fore, fore any measurable set E in Ω we have

$$\int_E |g_n(., u_n)| dx \leqslant \int_E |h_{\frac{1}{\delta} + ||v_0||_{L^\infty}}| + \delta C_3.$$

By Vitali's theorem, we obtain (24).

Furthermore by (22) we have

$$\int_\Omega g_n(., u_n) u_n dx \leqslant C_2 + \int_\Omega g_n(., u_n) v_0 dx.$$

By Fatou's lemma and (24), we get

$$0 \leqslant \int_\Omega g(., u) u dx \leqslant C_2 + \int_\Omega g(., u) v_0 dx.$$

Thus

$$g(., u) u \in L^1(\Omega).$$

Second part : *Passing to the limit in (19)*

Let

$$\mu_n = S u_n - f + g_n(., u_n).$$

From (19) it is clear that $\mu_n \in M^+(\Omega)$. Since S maps bounded sets of $W_0^m L_\varphi(\Omega)$ in to bounded sets of $W^{-m} L_{\varphi^*}(\Omega)$, then we can assume for the same sequence that

$$S u_n \rightharpoonup \chi \quad weakly \quad in\ W^{-m} L_{\varphi^*}(\Omega),$$

which implies that

$$\mu_n \longrightarrow \mu \quad in \quad D^{'}(\Omega),$$

where

$$\mu = \chi - f + g(., u).$$

We put $w = u - \Phi,\ h = -g(., u) \ \ and \ \ T = \mu + h.$

The assumptions of theorem 3.10 are satisfied since $T = \chi - f \in W^{-m} L_{\varphi^*}(\Omega)$ and $h \in L^1(\Omega)$. Thus

$$\left\{ \begin{array}{l} u - \Phi \in L^1(\Omega; d\mu), \\ < \chi - f, u - \Phi >= \displaystyle\int_\Omega (u - \Phi) d\mu - \int_\Omega g(., u)(u - \Phi) dx. \end{array} \right. \tag{25}$$

Using $v = \Phi$ as test function in (19) we get

$$< S u_n, u_n > \quad \leqslant \quad < S u_n, \Phi > - < f, \Phi - u_n > + \int_\Omega g_n(., u_n)(\Phi - u_n),$$

which gives passing to the limit and then using (25)

$$\begin{cases} limsup_n < Su_n, u_n > & \leqslant \ < \chi, \Phi > - < f, \Phi - u > + \int_\Omega g(., u)(\Phi - u)dx, \\ & \leqslant \ < \chi, u > \quad + \quad \int_\Omega (\Phi - u)d\mu \quad \leqslant \quad < \chi, u >; \end{cases} \tag{26}$$

since, by theorem 3.9 we have

$$(\Phi - u) \leqslant 0 \quad \mu.a.e. \quad in \ \ \Omega. \tag{27}$$

Using (26) and since S is a pseudo-monotone operator, we obtain

$$\chi = Su \quad and \quad < Su_n, u_n > \longrightarrow < Su, u > .$$

It is now easy to pass to the limit in (19) for any fixed $v \in K_\Phi \cap L^\infty(\Omega)$.

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