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Multiplicity of Periodic Solutions for a Class of Second Order Hamiltonian Systems

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Abstract: In this paper, we study the multiplicity of periodic solutions for two classes of sublinear nonlinearity second order Hamiltonian systems by the use of minimax methods, in critical point theory. Our results improve and generalize those in some known literatures.

Keywords: *Hamiltonian system; periodic solutions; sublinear nonlinearity; saddle point theorem.*

Mathematics Subject Classification (2010): 34C37.

1 Introduction

Consider the following Hamiltonian system with unbounded nonlinearities

$$\begin{cases} \ddot{u}(t) + Au(t) - \nabla F(t,u(t)) = e(t), & a.e.\ t \in [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{HS}$$

where A is a $(N \times N)$-symmetric matrix, $e \in L^1(0,T;\mathbb{R}^N)$, $T > 0$, and $F : \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is a continuous function, T-periodic in the first variable and differentiable with respect to the second variable with continuous derivative $\nabla F(t,x) = \frac{\partial F}{\partial x}(t,x)$.

The study of the existence and multiplicity of periodic solutions of Hamiltonian systems plays a very important role to solve many problems of natural sciences such as chemistry, biology and physics. For physics problem, we can cite planetry systems and fluid dynamic problem.

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When $A = 0$ and $e(t) = 0$ for all $t \in \mathbb{R}$, problem (HS) is just the following second order Hamiltonian system

$$\begin{cases} \ddot{u}(t) = \nabla F(t, u(t)), \ \ a.e.\ t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases} \tag{1}$$

During the last decades, many authors studied the existence and multiplicity of periodic solutions for system (1) via critical point theory and variational methods, we refer the readers to [1]- [21] and references therein. Many solvability conditions are given such as the coercive condition (see [2]), the periodicity condition (see [18]), the convexity condition (see [4]) and the subadditive condition (see [13]).

For the case $A \neq 0$ and $e \neq 0$, Mawhin and Willem [5] proved that problem (HS) has at least one solution by using the saddle point theorem under the following bounded conditions: There exists $g \in L^1(0, T; \mathbb{R}^+)$ such that

$$|F(t, u)| \leq g(t), \ |\nabla F(t, u)| \leq g(t), \ \forall u \in \mathbb{R}^N, \ a.e.\ t \in [0, T]. \tag{2}$$

Precisely they obtained the following result.

Theorem 1.1 ([5], Theorem 4.9) *Suppose F satisfies (2) and the following assumptions:*
(C_1) $\dim N(A) = m \geq 1$ *and A has no eigenvalue of the form $k^2 w^2$ $(k \in \mathbb{N}^*)$, where $w = \frac{2\pi}{T}$,*
(C_2) $\int_0^T (e(t), \alpha_j) dt = 0 \ \ (1 \leq j \leq m)$ *where $(\alpha_1, \alpha_2, ..., \alpha_m)$ is a basis of $N(A)$.*
$(\tilde{F}_0)$ *There exists $T_j > 0$ such that $F(t, u + T_j \alpha_j) = F(t, u) \ \ (1 \leq j \leq m), \ \forall u \in \mathbb{R}^N, \ a.e.\ t \in [0, T]$.*
Then problem (HS) has at least one solution.

In 2006, Feng and Han [6] generalized Mawhin and Willem's result as follows:

Theorem 1.2 ([6], Theorem 2.1) *Suppose F satisfies (C_1), (C_2), $(\tilde{F}_0)$ and the following conditions: There exist $a, b \in L^1(0, T; \mathbb{R}^+), 0 \leq \alpha < 1$ such that*

$$|\nabla F(t, x)| \leq a(t)|x|^{\alpha} + b(t), \ \ \forall x \in \mathbb{R}^N, \ a.e. \ t \in [0, T]. \tag{3}$$

Then problem (HS) has at least one solution.

Theorem 1.3 ([6], Theorem 2.2) *Suppose F satisfies (C_1), (C_2), (3) and*

$$|u|^{-2\alpha} \int_0^T F(t, u) dt \to +\infty \ \ as \ \ |u| \to \infty, \ \ u \in N(A), \tag{4}$$

or

$$|u|^{-2\alpha} \int_0^T F(t, u) dt \to -\infty \ \ as \ \ |u| \to \infty, \ \ u \in N(A). \tag{5}$$

Then problem (HS) has at least one solution.

Theorem 1.4 ([6], Theorem 2.3) *Suppose F satisfies (C_1), (C_2), (3) (F_0) and*

$$|u|^{-2\alpha} \int_0^T F(t, u) dt \to +\infty \ \ as \ \ |u| \to \infty, \ \ u \in N(A) \ominus span(\alpha_1, ..., \alpha_r), \tag{6}$$

or

$$|u|^{-2\alpha}\int_0^T F(t,u)dt \to -\infty \ \ as \ \ |u|\to\infty, \ \ u\in N(A)\ominus span(\alpha_1,....,\alpha_r). \tag{7}$$

Then problem (HS) has at least $r+1$ *solutions in* H^1_T.

In 2012, Li Xiao [8] generalized Theorem 1.3. Precisely he proved that problem (HS) possesses at least one solution when the nonlinearity $\nabla F(t,u)$ may grow slightly slower than a control function $h(|u|)$ instead of $|u|^\alpha$.

A natural question is whether there exists a result which contains the corresponding results in [5], [6], [8] as a special case.

Motivated by [6] and [8], we give this question a positive answer by the minimax methods in critical point theory and we obtain some results (Theorems 1.5 and 1.6), unify and generalize Theorems 1.2, 1.3 and 1.4 in [6], and Theorems 1.4 and 1.5 in [8].

Our basic hypotheses on A and F are the following:

(C_1) $\dim N(A)=m\geq 1$ and A has no eigenvalue of the form k^2w^2 $(k\in\mathbb{N}^*)$, where $w=\frac{2\pi}{T}$,

(C_2) $\displaystyle\int_0^T (e(t),\alpha_j)dt=0$ $(1\leq j\leq m)$ where $(\alpha_1,\alpha_2,...,\alpha_m)$ is a basis of $N(A)$.

(F_0) There exists $0\leq r\leq m$, $T_j>0$ such that $F(t,u+T_j\alpha_j)=F(t,u)$ $(1\leq j\leq r)$ $\forall u\in\mathbb{R}^N$, *a.e.* $t\in[0,T]$.

(F_1) There exist constants $C_0\geq 0$, $K_1>0$, $K_2>0$, $\alpha\in[0,1[$, $a\in L^1(0,T;\mathbb{R}^+)$, $b\in L^1(0,T;\mathbb{R}^+)$ and a function $h\in C(\mathbb{R}^+,\mathbb{R}^+)$ with the properties:

(i) $h(s)\leq h(t)$ $\qquad\forall s\leq t, s,t\in\mathbb{R}^+$,

(ii) $h(s+t)\leq C_0(h(t)+h(s))$ $\qquad\forall s,t\in\mathbb{R}^+$,

(iii) $0\leq h(t)\leq K_1t^\alpha+k_2$ $\qquad\forall t\in\mathbb{R}^+$,

(iv) $h(t)\to+\infty$ $\qquad$ as $t\to+\infty$,

such that

$$|\nabla F(t,x)|\leq a(t)h(|x|)+b(t)$$

for all $x\in\mathbb{R}^N$ and a.e. $t\in[0,T]$,

(F_1') There exist constants $C_0^*\geq 0$, $C^*>0$ and a function $h^*\in C(\mathbb{R}^+,\mathbb{R}^+)$ with the properties:

(i) $h^*(s)\leq h^*(t)+C_0^*$ $\qquad\forall s\leq t, s,t\in\mathbb{R}^+$,

(ii) $h^*(s+t)\leq C^*(h^*(t)+h^*(s))$ $\qquad\forall s,t\in\mathbb{R}^+$,

(iii) $th^*(t)-2H^*(t)\to-\infty$ $\qquad$ as $t\to+\infty$,

(iv) $\frac{H^*(t)}{t^2}\to 0$ $\qquad$ as $t\to+\infty$,

where $H^*(t)=\displaystyle\int_0^t h^*(s)ds$. Moreover, there exist $f\in L^1(0,T;\mathbb{R}^+)$ and $g\in L^1(0,T;\mathbb{R}^+)$ such that

$$|\nabla F(t,x)|\leq f(t)h^*(|x|)+g(t)$$

for all $x\in\mathbb{R}^N$ and a.e. $t\in[0,T]$.

Now we state our main results.

Theorem 1.5 *Suppose that conditions (C_1), (C_2), (F_0), (F_1) and the following assumption hold*
(F_2)

$$(i)\ \lim_{|x|\to+\infty}\frac{1}{h^2(|x|)}\int_0^T F(t,x)dt=-\infty,\quad x\in N(A)\ominus span(\alpha_1,....,\alpha_r),$$

or

$$(ii)\ \lim_{|x|\to+\infty}\frac{1}{h^2(|x|)}\int_0^T F(t,x)dt=+\infty,\quad x\in N(A)\ominus span(\alpha_1,....,\alpha_r),$$

then problem (HS) has at least $r+1$ T-periodic solutions in H_T^1.

Theorem 1.6 *Suppose that conditions (C_1), (C_2), (F_0), (F_1') and the following assumption hold*
(F_2')

$$(i)\ \lim_{|x|\to+\infty}\frac{1}{H^*(|x|)}\int_0^T F(t,x)dt=-\infty,\quad x\in N(A)\ominus span(\alpha_1,....,\alpha_r),$$

or

$$(ii)\ \lim_{|x|\to+\infty}\frac{1}{H^*(|x|)}\int_0^T F(t,x)dt=+\infty,\quad x\in N(A)\ominus span(\alpha_1,....,\alpha_r),$$

then problem (HS) has at least $r+1$ T-periodic solutions in H_T^1.

Example 1.1 Let

$$A=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}.$$

Then $\dim N(A)=2$ and N(A)=span$\{\alpha_1,\alpha_2\}$, where $\alpha_1=(0,1,0),\alpha_2=(0,0,1)$. So (C_1) holds.

Let

$$\begin{aligned}F(t,x)&=(0.4T-t)\ln^{\frac{3}{2}}[98+x_1^2+\sin^2(x_2)+\cos^2(x_3)]\\&+d(t)\ln[100+x_1^2+\sin^2(x_2)+\cos^2(x_3)]\end{aligned}\tag{8}$$

for all $x=(x_1,x_2,x_3)\in\mathbb{R}^3,\ t\in[0,T]$, where $d\in C([0,T];\mathbb{R}^+)$. We have

$$F(t,x+\pi\alpha_j)=F(t,x),\quad j=1,2.$$

Let e satisfy $\int_0^T e(t)dt=0$, then $\int_0^T(e(t),\alpha_j)dt=0,\ j=1,2$ and

$$|\nabla F(t,x)|\le 3|0.4T-t|\ln^{\frac{1}{2}}(100+|x|^2)+d(t).$$

Let $h(t)=\ln^{\frac{1}{2}}(100+|t|^2)$. Similar to the argument in [17], we know that (F_1) holds. Moreover,

$$\lim_{|x|\to+\infty}\frac{1}{h^2(|x|)}\int_0^T F(t,x)dt=-\infty.$$

Hence, $(F_2)i)$ holds and then by Theorem 1.5 , problem (HS) has at least three solutions. On the other hand, for any $\alpha \in (0,1)$,

$$\lim_{|x|\to+\infty} \frac{1}{|x|^{2\alpha}} \int_0^T F(t,x)dt = 0,$$

so (8) does not satisfy Theorem 1.3 in [6].

Example 1.2 Consider the function
$F(t,x) = (\frac{2}{3}T - t)\ln(100 + |x|^2) + l(t)\sqrt{100 + |x|^2}, \quad where\ l \in C([0,T],\mathbb{R}^+).$
It is easy to see that $|\nabla F(t,x)| \le 2\left|\frac{2}{3}T - t\right| \frac{|x|}{100+|x|^2} + l(t)$ for all $x \in \mathbb{R}^3$ and $t \in [0,T]$. Let $h^*(t) = \frac{t}{100+t^2}, H^*(t) = \int_0^t \frac{s}{100+s^2}ds, C_0^* = 2, C^* = 1, f(t) = 2\left|\frac{2}{3}T - t\right|$ *and* $g(t) = l(t)$, we infer
(i) $h^*(s) \le h^*(t) + 2 \qquad \forall s \le t, s,t \in \mathbb{R}^+,$
(ii) $h^*(s+t) = \frac{s+t}{100+(s+t)^2} \le (h^*(t) + h^*(s)) \qquad \forall s,t \in \mathbb{R}^+,$
(iii) $th^*(t) - 2H^*(t) = \frac{t^2}{100+t^2} - 2\left[\frac{1}{2}\ln(100+t^2) - \frac{1}{2}\ln(100)\right] \to -\infty$ as $t \to +\infty,$
(iv) $\frac{H^*(t)}{t^2} = \frac{\int_0^t \frac{s}{100+s^2}ds}{t^2} \to 0 \qquad$ as $t \to +\infty.$

Let e satisfy $\int_0^T e(t)dt = 0$, then $\int_0^T (e(t),\alpha_j)dt = 0,\ j = 1,2,$ we have $\lim_{|x|\to+\infty} \frac{1}{H^*(|x|)} \int_0^T F(t,x)dt \to +\infty$. So, by Theorem 1.6, problem (HS) has at least one solution in H_T^1.

Remark 1.1 Unlike the control functions in (F_1), where $h(t)$ is nondecreasing, here control function $h^*(t) = \frac{t}{100+t^2}$ is bounded but not increasing.

Remark 1.2 (i) Theorem 1.5. is a generalization of the main results in [[15], Theorems 2 and 3] and in [[6], Theorems 2.1, 2.2, 2.3]. Obviously , our theorems, as $r = m$, contain Theorems 1.4 and 1.5 in [8].
(ii) If we let $h(t) = t^\alpha$, it is easy to see that (F_1) generalizes (3).

2 Preliminaries.

Let
$H_T^1 = \left\{u : \mathbb{R} \to \mathbb{R}^N/\ u\ is\ absolutely\ continuous, u(t) = u(t+T), \dot{u} \in L^2(0,T;\mathbb{R}^N)\right\}.$
Then H_T^1 is a Hilbert space with the inner product

$$< u,v >= \int_0^T \left[(u(t),v(t)) + (\dot{u}(t),\dot{v}(t))\right] dt$$

and the associated norm

$$\|u\| = \left(\int_0^T \left[|u(t)|^2 + |\dot{u}(t)|^2\right] dt\right)^{\frac{1}{2}}$$

for each $u,v \in H_T^1$. Let

$$\bar{u} = \frac{1}{T}\int_0^T u(t)dt, \quad \tilde{u}(t) = u(t) - \bar{u}.$$

Then one has

$$\int_0^T |\tilde{u}(t)|^2 dt \le \frac{T^2}{4\pi^2} \int_0^T |\dot{u}(t)|^2 dt \ , \ (Wirtinger's\ inequality)$$

and

$$\|\tilde{u}\|_\infty^2 \le \frac{T}{12} \int_0^T |\dot{u}(t)|^2 dt \ , \ (Sobolev's\ inequality).$$

(see Proposition 1.3 in [5]) which implies that

$$\|u\|_\infty \le C \|u\| \tag{9}$$

for some $C > 0$ and all $u \in H_T^1$, where $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$. It is well known that the functional φ defined on H_T^1 by

$$\varphi(u) = \frac{1}{2}\int_0^T |\dot{u}|^2 dt - \frac{1}{2}\int_0^T (A(t)u(t), u(t))dt + \int_0^T F(t, u(t))dt + \int_0^T (e(t), u(t))dt$$

is continuously differentiable and its critical points are the solutions of problem (HS). Moreover, one has

$$< \varphi'(u), v >= \int_0^T [(\dot{u}(t), \dot{v}(t)) - (A(t)u(t), v(t)) + (\nabla F(t, u(t)), v(t)) + (e(t), v(t)]dt$$

for $u, v \in H_T^1$. Let

$$q(u) = \frac{1}{2}\int_0^T \Big(|\dot{u}|^2 - (A(t)u(t), u(t))\Big)\, dt.$$

It is easy to see that

$$q(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_0^T ((A(t) + I)u(t), u(t))dt = \frac{1}{2} < (I - K)u, u >,$$

where $K : H_T^1 \to H_T^1$ is the self-adjoint operator defined, using Riesz representation theorem, by

$$\int_0^T ((A(t) + I)u(t), v(t))dt =< (Ku, v) >, \forall u, v \in H_T^1.$$

The compact embedding of H_T^1 into $C(0, T; \mathbb{R}^N)$ implies that K is compact. By classical spectral theory, we can decompose H_T^1 into the orthogonal sum of invariant subspaces for $I - K$

$$H_T^1 = H^- \oplus H^0 \oplus H^+,$$

where $H^0 = Ker(I - K)$ and H^-, H^+ are such that, for some $\delta > 0$,

$$q(u) \le -\frac{\delta}{2}\|u\|^2 \quad if\ u \in H^-, \tag{10}$$

$$q(u) \ge \frac{\delta}{2}\|u\|^2 \quad if\ u \in H^+. \tag{11}$$

Moreover, by (C_1), it is well known that $H^0 = Ker(I - K) = N(A)$ (see [5]).

In the proofs, we mainly use the following generalized saddle point theorem from [9].

Theorem 2.1 *Let X be a Banach space and have a decomposition: $X = W + Z$ where W and Z are two subspaces of X with $\dim Z < +\infty$. Let V be a finite-dimensional, compact C^2-manifold without boundary. Let $f : X \times V \to \mathbb{R}$ be a C^1-function and satisfy the (PS) condition. Suppose that f satisfies*
$\inf_{u \in W \times X} f(u) \geq \alpha, \quad \sup_{u \in S \times X} f(u) \leq \beta < \alpha$, *where $S = \partial D, D = \{u \in Z \; / \; \|u\| \leq R\}$ and R, α, β are constants. Then the function f has at least cuplength(V)+1 critical points.*

Let $PH^0 = span(\alpha_1, ..., \alpha_r), QH^0 = N(A) \ominus PH^0 = span(\alpha_{r+1}, ..., \alpha_m)$. Then $u = u^- + u^+ + Pu^0 + Qu^0$, where $Pu^0 = \sum_{j=1}^{r} c_j \alpha_j$. Let $G = \{\sum_{j=1}^{r} k_j T_j \alpha_j / \; k_j \in \mathbb{N}\}$. Use the canonical mapping $\pi : H_T^1 \to H_T^1/G$. Let $H_T^1/G = X \times V = (W \oplus Z) \times V$, $W = H^+$, $Z = H^- \oplus QH^0$, $V = PH^0/G$. It is easy to see that $\dim Z < +\infty, \dim V < +\infty$, and V is a compact C^2-manifold without boundary as it is diffeomorphic to the r-torus T^r. Element in V can be represented as $P\hat{u}^0 = \sum_{j=1}^{r} \hat{c}_j \alpha_j$, where $\hat{c}_j = c_j - k_j T_j$ $(0 \leq \hat{c}_j < T_j)$.

Let $u = u^- + u^+ + P\hat{u}^0 + Qu^0$. Define the functional ψ on H_T^1/G by $\psi(\pi(u)) = \varphi(u)$. As $F(t, u + T_j \alpha_j) = F(t, u)$ $(1 \leq j \leq r)$, we can see that ψ is well-defined, and ψ is continuously differentiable on H_T^1/G.

3 Proof of the Main Results.

Proof of Theorem 1.5.

For the sake of convenience, we will denote various positive constants as C_i, $i = 1, 2, ...$ We only prove the case where $(F_2)(i)$ holds. The other case can be similarly given.

Lemma 3.1 *[Lemma 3.1, [8]] Assume that (F_1) holds. Then for any (PS) sequence $(u_n) \subset H_T^1$ of the functional φ, we have*

$$\|\tilde{u}_n\|^2 \leq C_1 h^2(|u_n^0|) + C_1, \tag{12}$$

where $u_n = u_n^+ + u_n^- + u_n^0$ and $\tilde{u}_n = u_n^+ + u_n^-$.

Lemma 3.2 *Suppose that (F_1) and $(F_2)(i)$ hold, Then every (PS) sequence $(u_n) \subset H_T^1$ such that (Pu_n^0) is bounded contains a convergent subsequence.*

Proof. By (12), we have

$$\|\tilde{u}_n\|^2 \quad \leq C_1 h^2(|u_n^0|) + C_1.$$

As (Pu_n^0) is bounded, we have the inequality

$$\|\tilde{u}_n\|^2 \leq C_2 h^2(|Qu_n^0|) + C_2. \tag{13}$$

It follows from (9), (F_1), (13), the mean value theorem and Young's inequality that

$$
\begin{aligned}
&\left|\int_0^T \left(F(t,u_n(t)) - F(t,Qu_n^0)\right) dt\right| \\
= &\left|\int_0^T \int_0^1 \left(\nabla F(t, Qu_n^0 + s(\tilde{u}_n(t)+Pu_n^0)), \tilde{u}_n(t)+Pu_n^0\right) dsdt\right| \\
\leq &\int_0^T \int_0^1 \left|\nabla F(t, Qu_n^0 + s(\tilde{u}_n(t)+Pu_n^0))\right| \left|\tilde{u}_n(t)+Pu_n^0\right| dsdt \\
\leq &\int_0^T \int_0^1 \left(a(t)h(\left|Qu_n^0 + s(\tilde{u}_n(t)+Pu_n^0)\right|) + b(t)\right) \left|\tilde{u}_n(t)+Pu_n^0\right| dsdt \\
\leq &\int_0^T \left[C_0(C_0+1)a(t)\left(h(|Qu_n^0|) + h(\|\tilde{u}_n\|_\infty) + h(|Pu_n^0|)\right)\right] \left(\|\tilde{u}_n\|_\infty + |Pu_n^0|\right) dt \\
+ &\int_0^T b(t)\left(\|\tilde{u}_n\|_\infty + |Pu_n^0|\right) dt \\
\leq & C_3\|\tilde{u}_n\|_\infty h(\|\tilde{u}_n\|_\infty) + C_3\|\tilde{u}_n\|_\infty h(|Qu_n^0|) + C_4\|\tilde{u}_n\|_\infty + C_5 h(|Qu_n^0|) \\
+ & C_5 h(\|\tilde{u}_n\|_\infty) + C_6 \\
\leq & C_3\|\tilde{u}_n\|_\infty (K_1\|\tilde{u}_n\|_\infty^\alpha + K_2) + C_3\|\tilde{u}_n\|_\infty h(\left|Qu_n^0\right|) + C_4\|\tilde{u}_n\|_\infty \\
+ & C_5 h(|Qu_n^0|) + C_5(K_1\|\tilde{u}_n\|_\infty^\alpha + K_2) + C_6 \\
\leq & C_7\|\tilde{u}_n\|^{\alpha+1} + C_8\|\tilde{u}_n\|^\alpha + C_9\|\tilde{u_n}\| \\
+ & C_{10}\|\tilde{u}_n\| h(|Qu_n^0|) + C_5 h(|Qu_n^0|) + C_{11} \\
\leq & C_{12}\|\tilde{u}_n\|^2 + C_{13}h^2(\left|Qu_n^0\right|) + C_{14} \\
\leq & C_{15}h^2(\left|Qu_n^0\right|) + C_{16}.
\end{aligned}
\tag{14}
$$

Hence, by (14) and the boundedness of $\varphi(u_n)$ we obtain

$$
\begin{aligned}
-C_{17} \leq\ & \varphi(u_n) = \frac{1}{2}((I-K)u_n, u_n) + \int_0^T \left(F(t,u_n(t)) - F(t,Qu_n^0)\right) dt \\
+ & \int_0^T F(t,Qu_n^0)dt + \int_0^T (e(t), u_n(t))dt \\
\leq\ & C_{18}\|\tilde{u}_n\|^2 + C_{15}h^2(\left|Qu_n^0\right|) + C_{16} + \int_0^T F(t,Qu_n^0)dt + C_{19}\|\tilde{u}_n\| \\
\leq\ & C_{20}h^2(\left|Qu_n^0\right|) + \int_0^T F(t,Qu_n^0)dt + C_{21} \\
=\ & h^2(|Qu_n^0|)\left(C_{20} + \frac{1}{h^2(|Qu_n^0|)}\int_0^T F(t,Qu_n^0)dt\right) + C_{21}.
\end{aligned}
\tag{15}
$$

It follows from $(F_2)(i)$ and (15) that (Qu_n^0) is bounded. Combining (13) and the boundedness of (Pu_n^0), we obtain that (u_n) is bounded. Arguing as in [Proposition 4.1, [5]] we conclude that (u_n) contains a convergent subsequence. Thus we complete the proof.

Now we are ready to prove Theorem 1.5. First, we prove that ψ satisfies the (PS) condition. Let $(u_n) \subset H_T^1$ be a (PS) sequence of ψ, that is $(\psi(\pi(u_n)))$ is bounded and $\psi'(\pi(u_n)) \to 0$.

We have $q(u) = \frac{1}{2}((I-K)u, u)$ so $q'(u) = (I-K)u$ and since $u_k - \hat{u}_k = \sum_{j=1}^{r} k_j T_j \alpha_j \in$

$N(I-K)$, we obtain that $q(u_n)=q(\hat{u}_n)$ and $q'(u_n)=q'(\hat{u}_n)$. Moreover, by conditions (F_0) and (C_2), we have $F(t,u_n(t))=F(t,\hat{u}_n(t)+\sum_{j=1}^{r}k_jT_j\alpha_j)=F(t,\hat{u}_n(t)$ and

$$\int_0^T(e(t),u_n(t))dt=\int_0^T(e(t),\hat{u}_n(t)+\sum_{j=1}^{r}k_jT_i\alpha_j)dt=\int_0^T(e(t),\hat{u}_n(t))dt.$$

Hence, we obtain that $\varphi(u_n)=\varphi(\hat{u}_n)$. Consequently $\psi(\pi(u_n))=\psi(\pi(\hat{u}_n))$. It follows from (F_0) that $\nabla F(t,u+T_j\alpha_j)=\nabla F(t,u)\quad(1\leq j\leq r)$. Hence $\varphi'(u_n)=\varphi'(\hat{u}_n)$, namely, $\psi'(\pi(u_n))=\psi'(\pi(\hat{u}_n))$. As $(P\hat{u}_n)$ is bounded, we obtain by Lemma 3.2 that $(\hat{u}_n)$ contains a convergent subsequence: $\hat{u}_{n_k}\to\hat{u}$. Then

$$\lim_{k\to+\infty}\psi(\pi(u_{n_k}))=\lim_{k\to+\infty}\psi(\pi(\hat{u}_{n_k}))=\psi(\pi(\hat{u})),$$
$$\lim_{k\to+\infty}\psi'(\pi(u_{n_k}))=\lim_{k\to+\infty}\psi'(\pi(\hat{u}_{n_k}))=\psi'(\pi(\hat{u})).$$

Hence ψ satisfies the (PS) condition.

In order to use the generalized saddle point theorem we only need to verify the following conditions:

$$(\psi_1)\quad \psi(\pi(u))\to+\infty,\quad as\ \ \|u\|\to+\infty\quad in\quad W\times V,$$

$$(\psi_2)\quad \psi(\pi(u))\to-\infty,\quad as\ \ \|u\|\to+\infty\quad in\quad Z\times V.$$

By (9), (F_1), the mean value theorem and the boundedness of $(P\hat{u}_n)$, we have $\forall\pi(u)\in W\times V,\ \ u=u^++Pu^0$,

$$\begin{aligned}
&\int_0^T(F(t,\hat{u}(t))-F(t,0))\,dt\\
=\ &\int_0^T\int_0^1\big(\nabla F(t,s(u^+(t)+P\hat{u}^0),u^+(t)+P\hat{u}^0\big)\,dsdt\\
\leq\ &\int_0^T\int_0^1\big|\nabla F(t,s(u^+(t)+P\hat{u}^0)\big|\,\big|u^+(t)+P\hat{u}^0\big|\,dsdt\\
\leq\ &\int_0^T\int_0^1\big(a(t)h(\big|u^+(t)+P\hat{u}^0\big|)+b(t)\big)\,\big|u^+(t)+P\hat{u}^0\big|\,dsdt\\
\leq\ &\int_0^T\big[C_0a(t)\left(h(\|u^+\|_\infty)+h(|P\hat{u}^0|)\right)+b(t)\big]\left(\|u^+\|_\infty+|P\hat{u}^0|\right)dt\\
\leq\ &\left(\|u^+\|_\infty+|P\hat{u}^0|\right)\left(C_0K_1\|u^+\|_\infty^\alpha\int_0^Ta(t)dt+C_0K_2\int_0^Ta(t)dt\right)\\
+\ &\left(\|u^+\|_\infty+|P\hat{u}^0|\right)h(|P\hat{u}^0|)C_0\int_0^Ta(t)dt+\left(\|u^+\|_\infty+|P\hat{u}^0|\right)\int_0^Tb(t)dt\\
\leq\ &C_{22}\left\|u^+\right\|_\infty^{\alpha+1}+C_{23}\left\|u^+\right\|_\infty^\alpha+C_{24}\left\|u^+\right\|_\infty+C_{25}\\
\leq\ &C_{26}\left\|u^+\right\|^{\alpha+1}+C_{27}\left\|u^+\right\|^\alpha+C_{28}\left\|u^+\right\|+C_{25}.
\end{aligned}\tag{16}$$

It follows from (11) and (16) that

$$\begin{aligned}
\psi(\pi(u)) &= \varphi(u) = \varphi(\hat{u}) \\
&= \frac{1}{2}\left((I-K)u^+, u^+\right) + \int_0^T \left(F(t,\hat{u}(t)) - F(t,0)\right) dt \\
&+ \int_0^T F(t,0)dt + \int_0^T (e(t),\hat{u}(t))dt \\
&\geq \frac{\delta}{2}\|u^+\|^2 - C_{26}\left\|u^+\right\|^{\alpha+1} - C_{27}\left\|u^+\right\|^{\alpha} - C_{29}\left\|u^+\right\| - C_{30}. \qquad (17)
\end{aligned}$$

Since $\alpha + 1 < 2$, then by (17), (ψ_1) is verified.
On the other hand, by (9), (F_1), the mean value theorem, the boundedness of $(P\hat{u}_n)$ and Young's inequality we obtain for $\pi(u) \in Z \times V$, $u = u^- + Qu^0 + Pu^0$,

$$\int_0^T \left(F(t,\hat{u}(t)) - F(t,Qu^0)\right) dt$$

$$\begin{aligned}
&= \int_0^T \int_0^1 \left(\nabla F(t, Qu^0 + s(u^-(t) + P\hat{u}^0)), u^-(t) + P\hat{u}^0\right) dsdt \\
&\leq \int_0^T \int_0^1 \left|\nabla F(t, Qu^0 + s(u^-(t) + P\hat{u}^0)\right| \left|u^-(t) + P\hat{u}^0\right| dsdt \\
&\leq \int_0^T \int_0^1 \left(a(t)h(\left|Qu^0 + s(u^-(t) + P\hat{u}^0)\right|) + b(t)\right) \left|u^-(t) + P\hat{u}^0\right| dsdt \\
&\leq \int_0^T C_0(C_0+1)a(t)\left(h(|Qu^0|) + h(\|u^-\|_\infty) + h(|P\hat{u}^0|)\right)\left(\|u^-\|_\infty + |P\hat{u}^0|\right) dt \\
&+ \int_0^T b(t)\left(\|u^-\|_\infty + |P\hat{u}^0|\right) dt \\
&\leq C_{31}\|u^-\|_\infty h(\|u^-\|_\infty) + C_{31}\|u^-\|_\infty h(|Qu^0|) \\
&+ C_{32}h(\|u^-\|_\infty) + C_{32}h(|Qu^0|) + C_{33}\|u^-\|_\infty + C_{34} \\
&\leq C_{31}\left\|u^-\right\|_\infty \left(K_1\left\|u^-\right\|_\infty^\alpha + K_2\right) + C_{31}\left\|u^-\right\|_\infty h(\left|Qu^0\right|) + C_{33}\|u^-\|_\infty \\
&+ C_{32}h(|Qu^0|) + C_{32}\left(K_1\left\|u^-\right\|_\infty^\alpha + K_2\right) + C_{34} \\
&\leq C_{35}\left\|u^-\right\|_\infty^{\alpha+1} + C_{36}\left\|u^-\right\|_\infty^\alpha + C_{37}\left\|u^-\right\|_\infty \\
&+ C_{31}\left\|u^-\right\|_\infty h(|Qu^0|) + C_{32}h(|Qu^0|) + C_{38} \\
&\leq C_{39}\left\|u^-\right\|^{\alpha+1} + C_{40}\left\|u^-\right\|^\alpha + C_{41}\left\|u^-\right\| \\
&+ C_{42}\left\|u^-\right\| h(|Qu^0|) + C_{32}h(|Qu^0|) + C_{38} \\
&\leq \varepsilon\left\|u^-\right\|^2 + C_{39}\left\|u^-\right\|^{\alpha+1} + C_{40}\left\|u^-\right\|^\alpha \\
&+ C_{41}\left\|u^-\right\| + C_{43}h^2(\left|Qu^0\right|) + C_{44} \qquad (18)
\end{aligned}$$

for any $\varepsilon > 0$. Hence, by (10) and (18) we obtain

$$\begin{aligned}
\psi(\pi(u)) &= \varphi(u) = \varphi(\hat{u}) \\
&= \frac{1}{2}\left((I-K)u^-, u^-\right) + \int_0^T \left(F(t,u(t)) - F(t,Qu^0)\right) dt \\
&+ \int_0^T F(t,Qu^0)dt + \int_0^T (e(t), u^-(t))dt \\
&\leq \frac{-\delta}{2}\|u^-\|^2 + \varepsilon \left\|u^-\right\|^2 + C_{39}\left\|u^-\right\|^{\alpha+1} \\
&+ C_{40}\left\|u^-\right\|^{\alpha} + C_{45}\left\|u^-\right\| + C_{43}h^2(|Qu^0|) + \int_0^T F(t,Qu^0)dt + C_{44} \\
&= (\frac{-\delta}{2} + \varepsilon)\left\|u^-\right\|^2 + C_{39}\left\|u^-\right\|^{\alpha+1} + C_{40}\left\|u^-\right\|^{\alpha} + C_{45}\left\|u^-\right\| \\
&+ h^2(|Qu^0|)\left(C_{43} + \frac{1}{h^2(|Qu^0|)}\int_0^T F(t,Qu^0)dt\right) + C_{44}. \qquad (19)
\end{aligned}$$

Fixing $\varepsilon < \frac{\delta}{2}$, by (19), $(F_2)(i)$ and since $\alpha + 1 < 2$, we obtain $\varphi(u) \to -\infty$ *as* $\|u\| \to +\infty$ *in* $Z \times V$. Thus (ψ_2) is verified. The proof is completed. □

Proof of Theorem 1.6.
We only prove the case where $(F_2')(i)$ holds. The other case can be similarly given.

Lemma 3.3 (Lemma 2.1, [19]) *Suppose that there exists a positive function* h^* *satisfying the conditions (i), (ii), (iv) of* (F_1')*, then we have the following estimates:*
(1) $0 < h^*(t) < \varepsilon t + C_0$ *for any* $\varepsilon > 0, C_0 > 0, t \in \mathbb{R}^+$,
(2) $\frac{h^{*2}(t)}{H^*(t)} \longrightarrow 0$ *as* $t \to +\infty$,
(3) $H^*(t) \longrightarrow +\infty$ *as* $t \to +\infty$.

Lemma 3.4 *Assume that* (F_1') *holds. Then for any (PS) sequence* $(u_n) \subset H_T^1$ *of the functional* φ*, we have*

$$\|\tilde{u}_n\|^2 \leq C_{45}h^{*2}(\left|u_n^0\right|) + C_{45}, \qquad (20)$$

where $u_n = u_n^+ + u_n^- + u_n^0$ *and* $\tilde{u}_n = u_n^+ + u_n^-$.

Proof. Assume that $(u_n) \subset H_T^1$ is a (PS) sequence for φ. Then

$$|\varphi(u_n)| \leq C_{46}, \quad |\varphi'(u_n)| \leq C_{46}, \forall n \in \mathbb{N}.$$

It follows from (F_1'), (9), Lemma 3.3 and Young's inequality that

$$
\begin{aligned}
&\left|\int_0^T \nabla F(t,u_n(t)), u_n^+(t)-u_n^-(t))dt\right| \\
\leq\ & \int_0^T |\nabla F(t,u_n(t))| \left|u_n^+(t)-u_n^-(t)\right| dt \\
\leq\ & \int_0^T f(t)h^*(\left|u_n^0+\tilde{u}_n(t)\right|)\left|u_n^+(t)-u_n^-(t)\right| dt + \int_0^T g(t)\left|u_n^+(t)-u_n^-(t)\right| dt \\
\leq\ & \left\|u_n^+-u_n^-\right\|_\infty \int_0^T f(t)[C_0^*+h^*(\left|u_n^0\right|+\|\tilde{u}_n\|_\infty)]dt + \left\|u_n^+-u_n^-\right\|_\infty \int_0^T g(t)dt \\
=\ & \left\|u_n^+-u_n^-\right\|_\infty h^*(\left|u_n^0\right|+\|\tilde{u}_n\|_\infty)\int_0^T f(t)dt + \left\|u_n^+-u_n^-\right\|_\infty \int_0^T (C_0^*f(t)+g(t))dt \\
\leq\ & C^*\left\|u_n^+-u_n^-\right\|_\infty h^*(\|\tilde{u}_n\|_\infty)\int_0^T f(t)dt + C^*\left\|u_n^+-u_n^-\right\|_\infty h^*(\left|u_n^0\right|)\int_0^T f(t)dt \\
+\ & \left\|u_n^+-u_n^-\right\|_\infty \int_0^T (C_0^*f(t)+g(t))dt \\
\leq\ & \varepsilon C^*\left\|u_n^+-u_n^-\right\|_\infty \|\tilde{u}_n\|_\infty \int_0^T f(t)dt + C_0C^*\left\|u_n^+-u_n^-\right\|_\infty \int_0^T f(t)dt \\
+\ & C^*\left\|u_n^+-u_n^-\right\|_\infty h^*(\left|u_n^0\right|)\int_0^T f(t)dt + \left\|u_n^+-u_n^-\right\|_\infty \int_0^T (C_0^*f(t)+g(t))dt \\
\leq\ & \varepsilon C_{47}\|\tilde{u}_n\|^2 + C_{48}h^*(\left|u_n^0\right|)\|\tilde{u}_n\| + C_{49}\|\tilde{u}_n\| \\
\leq\ & 3\varepsilon C_{47}\|\tilde{u}_n\|^2 + C_{50}(\varepsilon)h^{*2}(\left|u_n^0\right|) + C_{51}(\varepsilon)
\end{aligned}
\tag{21}
$$

for any $\varepsilon > 0$.

Thus, we have

$$
\begin{aligned}
C_{46}\left\|u_n^+-u_n^-\right\| &= C_{46}\|\tilde{u}_n\| \\
&\geq \left(\varphi'(u_n), u_n^+-u_n^-\right) \\
&= \left((I-K)u_n, u_n^+-u_n^-\right) + \int_0^T \left(\nabla F(t,u_n(t))+e(t), u_n^+(t)-u_n^-(t)\right) dt \\
&\geq \delta\|\tilde{u}_n\|^2 - 3\varepsilon C_{47}\|\tilde{u}_n\|^2 - C_{50}(\varepsilon)h^{*2}(\left|u_n^0\right|) - C_{51}(\varepsilon) \\
&- \left\|u_n^+-u_n^-\right\|_\infty \int_0^T |e(t)|\, dt \\
&\geq (\delta-3\varepsilon C_{47})\|\tilde{u}_n\|^2 - C_{50}(\varepsilon)h^{*2}(\left|u_n^0\right|) - C_{51}(\varepsilon) - C_{52}\|\tilde{u}_n\|.
\end{aligned}
$$

Hence, we obtain

$$
(\delta-5\varepsilon C_{47})\|\tilde{u}_n\|^2 \leq C_{50}h^{*2}(\left|u_n^0\right|) + C_{53}, \tag{22}
$$

if we fix $\varepsilon < \frac{\delta}{5C_{47}}$, then by (22) we have

$$
\|\tilde{u}_n\|^2 \leq C_{54}h^{*2}(\left|u_n^0\right|) + C_{55}.
$$

Take $C_{45} = \max\{C_{54}, C_{55}\}$, the proof is complete.

Lemma 3.5 *Suppose that* (F_1') *and* $(F_2')(i)$ *hold, Then every (PS) sequence* $(u_n) \subset H_T^1$ *such that* (Pu_n^0) *is bounded contains a convergent subsequence.*

Proof. By (20), we have

$$\|\tilde{u}_n\|^2 \leq C_{45} h^{*2}(|u_n^0|) + C_{45}.$$

As (Pu_n^0) is bounded, we have the inequality

$$\|\tilde{u}_n\|^2 \leq C_{56} h^{*2}(|Qu_n^0|) + C_{56}. \tag{23}$$

It follows from (9), (F_1'), (23), the mean value theorem and Young's inequality that
$\left|\int_0^T \left(F(t, u_n(t)) - F(t, Qu_n^0)\right) dt\right|$

$$\begin{aligned}
&= \left|\int_0^T \int_0^1 \left(\nabla F(t, Qu_n^0 + s(\tilde{u}_n(t) + P\hat{u}_n^0), \tilde{u}_n(t) + P\hat{u}_n^0\right) dsdt\right| \\
&\leq \int_0^T \int_0^1 \left|\nabla F(t, Qu_n^0 + s(\tilde{u}_n(t) + P\hat{u}_n^0)\right| \left|\tilde{u}_n(t) + P\hat{u}_n^0\right| dsdt \\
&\leq \int_0^T \int_0^1 \left(f(t) h^*(\left|Qu_n^0 + s(\tilde{u}_n(t) + P\hat{u}_n^0)\right|) + g(t)\right) \left|\tilde{u}_n(t) + P\hat{u}_n^0\right| dsdt \\
&\leq \int_0^T \left[f(t)\left(h^*(|Qu_n^0| + \|\tilde{u}_n\|_\infty + |P\hat{u}_n^0|) + C_0^*\right) + g(t)\right] \left(|\tilde{u}_n(t)| + |P\hat{u}_n^0|\right) dt \\
&\leq C^*(C^*+1)\left(h^*(|Qu_n^0|) + h^*(\|\tilde{u}_n\|_\infty) + h^*(|P\hat{u}_n^0|)\right) \left(\|\tilde{u}_n\|_\infty + |P\hat{u}_n^0|\right) \int_0^T f(t) dt \\
&+ \left(\|\tilde{u}_n\|_\infty + |P\hat{u}_n^0|\right) \int_0^T (g(t) + C_0^* f(t)) dt \\
&\leq C_{57} \|\tilde{u}_n\|_\infty h^*(\|\tilde{u}_n\|_\infty) + C_{58} \|\tilde{u}_n\|_\infty h^*(|Qu_n^0|) + C_{59} \|\tilde{u}_n\|_\infty + C_{60} h^*(|Qu_n^0|) \\
&+ C_{61} h(\|\tilde{u}_n\|_\infty) + C_{62} \\
&\leq C_{63} h^{*2}(|Qu_n^0|) + C_{64}.
\end{aligned} \tag{24}$$

It follows from the boundedness of $\varphi(u_n)$ and (24) that

$$\begin{aligned}
-C_{65} &\leq \varphi(u_n) \\
&= \frac{1}{2}((I-K)u_n, u_n) + \int_0^T \left(F(t, u_n(t)) - F(t, Qu_n^0)\right) dt \\
&+ \int_0^T F(t, Qu_n^0) dt + \int_0^T (e(t), u_n(t)) dt \\
&\leq C_{66} \|\tilde{u}_n\|^2 + C_{63} h^{*2}(|Qu_n^0|) + C_{64} + \int_0^T F(t, Qu_n^0) dt + C_{67} \|\tilde{u}_n\| \\
&\leq C_{68} h^{*2}(|Qu_n^0|) + \int_0^T F(t, Qu_n^0) dt + C_{69} \\
&= H^*(|Qu_n^0|) \left(\frac{C_{68} h^{*2}(|Qu_n^0|)}{H^*(|Qu_n^0|)} + \frac{1}{H^*(|Qu_n^0|)} \int_0^T F(t, Qu_n^0) dt\right) \\
&+ C_{69}.
\end{aligned} \tag{25}$$

Hence, by $(F_2')(i)$, (25) and Lemma 3.3 we deduce that (Qu_n^0) is bounded. Combining (20) and the boundedness of (Pu_n^0), we obtain that (u_n) is bounded. Arguing as in [Proposition 4.1, [5]] we conclude that (u_n) contains a convergent subsequence. We complete the proof.

Now we are ready to prove Theorem 1.6. First, we prove that ψ satisfies the (PS) condition. Let $(u_n) \subset H_T^1$ be a (PS) sequence of ψ, that is $(\psi(\pi(u_n)))$ is bounded and $\psi'(\pi(u_n)) \to 0$. We have $q(u) = \frac{1}{2}((I-K)u, u)$ so $q'(u) = (I-K)u$ and since $u_k - \hat{u}_k = \sum_{j=1}^{r} k_j T_j \alpha_j \in N(I-K)$, we obtain that $q(u_n) = q(\hat{u}_n)$ and $q'(u_n) = q'(\hat{u}_n)$. Therefore, by conditions (F_0) and (C_2), we have

$$F(t, u_n(t)) = F(t, \hat{u}_n(t) + \sum_{j=1}^{r} k_j T_j \alpha_j) = F(t, \hat{u}_n(t),$$

$$\int_0^T (e(t), u_n(t))dt = \int_0^T (e(t), \hat{u}_n(t) + \sum_{j=1}^{r} k_j T_i \alpha_j)dt = \int_0^T (e(t), \hat{u}_n(t))dt.$$

Hence, we obtain that $\varphi(u_n) = \varphi(\hat{u}_n)$. Consequently $\psi(\pi(u_n)) = \psi(\pi(\hat{u}_n))$. It follows from (F_1) that $\nabla F(t, u + T_j\alpha_j) = \nabla F(t, u) \quad (1 \le j \le r)$. Hence $\varphi'(u_n) = \varphi'(\hat{u}_n)$, namely, $\psi'(\pi(u_n)) = \psi'(\pi(\hat{u}_n))$. As $(P\hat{u}_n)$ is bounded, we obtain by Lemma 3.5 that $(\hat{u}_n)$ contains a convergent subsequence. Let $\hat{u}_{n_k} \to \hat{u}$.
Then

$$\lim_{k\to+\infty} \psi(\pi(u_{n_k})) = \lim_{k\to+\infty} \psi(\pi(\hat{u}_{n_k})) = \psi(\pi(\hat{u})),$$
$$\lim_{k\to+\infty} \psi'(\pi(u_{n_k})) = \lim_{k\to+\infty} \psi'(\pi(\hat{u}_{n_k})) = \psi'(\pi(\hat{u})).$$

It implies that ψ satisfies the (PS) condition.

In order to use the generalized saddle point theorem we only need to verify the following conditions:

(ψ_1) There exists $\alpha \in \mathbb{R}$ such that $\psi(\pi(u)) \ge \alpha, \quad$ on $\; W \times V$,

(ψ_2) There exists $\beta < \alpha$ such that $\psi(\pi(u)) \le \beta, \quad$ on $\; Z \times V$.

It follows from (9), (F_1'), the mean value theorem and the boundedness of $(P\hat{u}_n)$, that $\forall \pi(u) \in W \times V, u = u^+ + Pu^0$,

$$\begin{aligned}
&\int_0^T (F(t, \hat{u}(t)) - F(t,0))\,dt \\
&= \int_0^T \int_0^1 \left(\nabla F(t, s(u^+(t) + P\hat{u}^0), u^+(t) + P\hat{u}^0\right) ds dt \\
&\le \int_0^T \int_0^1 \left|\nabla F(t, s(u^+(t) + P\hat{u}^0)\right| \left|u^+(t) + P\hat{u}^0\right| ds dt \\
&\le \int_0^T \int_0^1 \left(f(t) h^*(\left|s(u^+(t) + P\hat{u}^0)\right|) + g(t)\right) \left|u^+(t) + P\hat{u}^0\right| ds dt \\
&\le \int_0^T \left[f(t)\left(h^*(\|u^+\|_\infty + |P\hat{u}^0|) + C_0^*\right) + g(t)\right] \left(\|u^+\|_\infty + |P\hat{u}^0|\right) dt \\
&\le C^* \left(\|u^+\|_\infty + |P\hat{u}^0|\right) \left(h^*(\|u^+\|_\infty) + h^*(|P\hat{u}^0|)\right) \int_0^T f(t)dt
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \left(\|u^+\|_\infty + |P\hat{u}^0|\right) \int_0^T (g(t) + C_0^* f(t))dt \\
&\leq \quad \varepsilon C_{70} \left\|u^+\right\|_\infty h^*(\left\|u^+\right\|_\infty) + C_{71} h^*(\left\|u^+\right\|_\infty) + C_{72} \left\|u^+\right\|_\infty + C_{73} \\
&\leq \quad \varepsilon C_{70} \left\|u^+\right\|_\infty^2 + C_{74} \left\|u^+\right\|_\infty + C_{75} \\
&\leq \quad \varepsilon C_{76} \left\|u^+\right\|^2 + C_{77} \left\|u^+\right\| + C_{75} \qquad (26)
\end{aligned}
$$

for any $\varepsilon > 0$.
Hence, we deduce from (11) and (26) that

$$
\begin{aligned}
\psi(\pi(u)) &= \varphi(u) = \varphi(\hat{u}) \\
&= \frac{1}{2}\left((I-K)u^+, u^+\right) + \int_0^T \left(F(t,\hat{u}(t)) - F(t,0)\right) dt \\
&+ \int_0^T F(t,0)dt + \int_0^T (e(t), \hat{u}(t))dt \\
&\geq (\frac{\delta}{2} - \varepsilon C_{76}) \left\|u^+\right\|^2 - C_{80} \left\|u^+\right\| - C_{81}. \qquad (27)
\end{aligned}
$$

Choosing $\varepsilon < \frac{\delta}{2C_{76}}$, by (27) ψ is bounded below on $W \times V$, and (ψ_1) is verified.

On the other hand, by (9), (F_1'), the mean value theorem, the boundedness of $(P\hat{u}_n)$ and Young's inequality we have
$\forall \pi(u) \in Z \times V, \quad u = u^- + Qu^0 + Pu^0,$

$$
\begin{aligned}
&\int_0^T \left(F(t,\hat{u}(t)) - F(t,Qu^0)\right) dt \\
&= \int_0^T \int_0^1 \left(\nabla F(t, Qu^0 + s(u^-(t) + P\hat{u}^0), u^-(t) + P\hat{u}^0\right) dsdt \\
&\leq \int_0^T \int_0^1 \left|\nabla F(t, Qu^0 + s(u^-(t) + P\hat{u}^0)\right| \left|u^-(t) + P\hat{u}^0\right| dsdt \\
&\leq \int_0^T \int_0^1 \left[f(t) h^*(\left|Qu^0 + s(u^-(t) + P\hat{u}^0)\right|) + g(t)\right] \left|u^-(t) + P\hat{u}^0\right| dsdt \\
&\leq \int_0^T f(t) \left(h^*(|Qu^0| + \|u^-\|_\infty + |P\hat{u}^0|) + C_0^*\right) \left(\|u^-\|_\infty + |P\hat{u}^0|\right) dt \\
&+ \left(\|u^-\|_\infty + |P\hat{u}^0|\right) \int_0^T g(t)dt \\
&\leq C^*(C^*+1) \left(h^*(|Qu^0|) + h^*(\|u^-\|_\infty) + h^*(|P\hat{u}^0|)\right) \left(\|u^-\|_\infty + |P\hat{u}^0|\right) \int_0^T f(t)dt \\
&+ \left(\|u^-\|_\infty + |P\hat{u}^0|\right) \int_0^T (g(t) + C_0^* f(t))dt \\
&\leq C_{82}\|u^-\|_\infty h^*(\|u^-\|_\infty) + C_{82}\|u^-\|_\infty h^*(|Qu^0|) \\
&+ C_{83} h^*(\|u^-\|_\infty) + C_{83} h^*(|Qu^0|) + C_{84}\|u^-|_\infty + C_{85} \\
&\leq C_{86}\|u^-\|_\infty^2 + C_{86} h^{*2}(|Qu^0|) + C_{87}\|u^-\|_\infty + C_{88} \\
&\leq C_{89}\|u^-\|_\infty^2 + C_{90} h^*(|Qu^0|) + C_{91}\|u^-\|_\infty + C_{88} \\
&\leq C_{91} h^{*2}(\left|Qu^0\right|) + C_{92}. \qquad (28)
\end{aligned}
$$

Hence, by (10) and (28) we obtain

$$\begin{aligned}\psi(\pi(u)) &= \varphi(u)=\varphi(\hat{u})\\ &= \frac{1}{2}\left((I-K)u^-,u^-\right)+\int_0^T\left(F(t,u(t))-F(t,Qu^0)\right)dt\\ &+ \int_0^T F(t,Qu^0)dt+\int_0^T(e(t),u^-(t))dt\\ &\le \frac{-\delta}{2}\|u^-\|^2+C_{91}h^{*2}(|Qu^0|)+C_{92}+C_{93}\|u^-\|+\int_0^T F(t,Qu^0)dt\\ &= H^*(|Qu^0|)\left(\frac{C_{91}h^{*2}(|Qu^0|)}{H^*(|Qu^0|)}+\frac{1}{H^*(|Qu^0|)}\int_0^T F(t,Qu^0)dt\right)\\ &+ \frac{-\delta}{2}\left\|u^-\right\|^2+C_{93}\left\|u^-\right\|+C_{92}. \end{aligned} \tag{29}$$

Hence, by (29), $(F_2')(i)$ we obtain that $\varphi(u)\to-\infty \quad as \quad \|u\|\to+\infty \quad in \quad Z\times V$.
Thus, (ψ_2) is verified and we complete the proof.

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