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On the Hyers-Ulam Stability of Certain Partial Differential Equations of Second Order

Emel Biçer [1] and Cemil Tunç [2*]

[1] *Department of Mathematics, Faculty of Arts and Sciences, Bingöl University, 12000, Bingöl, Turkey*
[2] *Department of Mathematics, Faculty of Sciences, Yuzuncu Yil University, 65080, Van, Turkey*

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Abstract: In this paper, we obtain two new results on the Hyers-Ulam stability of the linear partial differential equation of second order with constant coefficients

$$Az_{xx} + (A + B)z_{xy} + Bz_{yy} + Az_x + Bz_y = 0$$

and the partial Euler differential equation of the form

$$x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} + mxz_x + myz_y - mz = 0.$$

Our findings make a contribution to the topic and complete those in the relevant literature.

Keywords: *partial differential equation; Hyers-Ulam stability; second order.*

Mathematics Subject Classification (2010): 39B82, 35B35, 35F05.

1 Introduction

The stability theory is an important research area in the qualitative analysis of differential equations and partial differential equations. It follows from the relevant literature that the investigation of the Hyers-Ulam and Hyers-Ulam-Rassias stability of equations with partial derivatives started recently. We should mention the earliest results on the topic or some results obtained for the linear partial differential equations of first or second order by Alsina and Ger [1], Cimpean and Popa [2], Gordji et al. [3], Hyers [4], Jung ([5], [6], [7], [8]), Li and Huang [9], Liu and Zhao [10], Lungu and Popa ([11], [12]), Rassias

* Corresponding author: mailto:cemtunc@yahoo.com

[13], Tunç and Biçer [14], Ulam [15] and the references therein. We shall now give the details of some works done on the topic. In 2009, Jung [8] investigated the Hyers-Ulam stability of linear partial differential equations of first order

$$au_x(x,y) + bu_y(x,y) + g(y)u(x,y) + h(y) = 0$$

and

$$au_x(x,y) + bu_y(x,y) + g(x)u(x,y) + h(x) = 0,$$

in the cases of $a \leq 0, b > 0$ and $a > 0$,$b \leq 0$, $(a, b \in \Re)$, respectively.

Later, in 2011, Gordji et al. [3] proved the Hyers-Ulam-Rassias stability of the following nonlinear partial differential equations

$$\begin{aligned}
&\gamma_x(x,t) = f(x,t,\gamma(x,t)),\\
&a\gamma_x(x,t) + b\gamma_t(x,t) = f(x,t,\gamma(x,t)),\\
&p(x,t)\gamma_{xx}(x,t) + q(x,t)\gamma_x(x,t) = f(x,t,\gamma(x,t))
\end{aligned}$$

and

$$p(x,t)\gamma_{xt}(x,t) + q(x,t)\gamma_t(x,t) + p_t(x,t)\gamma_x(x,t) - p_x(x,t)\gamma_t(x,t) = f(x,t,\gamma(x,t)),$$

respectively, by using Banach's contraction mapping principle.

After that, in 2012, Lungu and Popa [11] discussed the Hyers-Ulam stability of first order partial differential equation of the form

$$p(x,y)\frac{\partial u}{\partial x} + q(x,y)\frac{\partial u}{\partial y} = p(x,y)r(x)u + f(x,y).$$

Finally, in 2014, Li and Huang [9] proved the Hyers-Ulam stability of the first order linear partial differential equations in n-dimensional space of the form

$$\sum_{i=1}^{n} a_i x_{x_i}(x_1, x_2, ..., x_n) + g(x_j)u(x_1, x_2, ..., x_n) + h(x_j) = 0,$$

where $a_i \in \Re$ are arbitrarily given constants.

In this paper, we investigate the Hyers-Ulam stability of the partial differential equation of second order with constant coefficients

$$Az_{xx} + (A+B)z_{xy} + Bz_{yy} + Az_x + Bz_y = 0 \tag{1}$$

and the partial Euler differential equation

$$x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} + mxz_x + myz_y - mz = 0, \tag{2}$$

where $z = z(x,y)$ $(x,y) \in D$, $D = [a,b) \times \Re$, D is a subset of $\Re^2$ and A, B, m are real constants with $m > 0$ and $A > 0$. Let $\varepsilon > 0$ be a given number. Equation (1) is said to be stable in Hyers-Ulam sense if there exists $K > 0$ such that for every function $z : [a,b] \times \Re \to C$ satisfying

$$|Az_{xx} + (A+B)z_{xy} + Bz_{yy} + Az_x + Bz_y| < \varepsilon$$

for all $(x,y) \in D$ there exists a solution $z_0 : [a,b] \times \Re \to C$ of Eq. (1) with the property

$$|z(x,y) - z_0(x,y)| \leq K\varepsilon.$$

This work has been inspired basically by the papers of Gordji et al. [3], Jung [8], Li and Huang [9], Lungu and Popa [11], Vlasov [16], Vasundhara Devi [1] and those listed above. The results obtained here are different from those in the literature, new and original, and they have simple forms. They can be easily checked and applicable, and complete the previous ones in the literature. Hence the novelty and originality of the present paper.

2 Hyers-Ulam Stability

In this section, we give two theorems and two examples to show the Hyers-Ulam stability of equation (1) and equation (2).Our first Hyers-Ulam stability result is the following theorem.

Theorem 1. *Let ε be a positive constant. If the function z satisfies the differential inequality*

$$|Az_{xx} + (A+B)z_{xy} + Bz_{yy} + Az_x + Bz_y| < \varepsilon \tag{3}$$

for all $(x,y) \in D$, then there exists a solution $z_0 : D \to \Re$ of equation (1) such that

$$|z(x,y) - z_0(x,y)| \leq K\varepsilon, K > 0, K \in \Re.$$

Proof. Let $u(x,y) = Az_x + Bz_y$ for any $(x,y) \in D$. Then, it follows that

$$|u_x + u_y + u| = |Az_{xx} + (A+B)z_{xy} + Bz_{yy} + Az_x + Bz_y|$$

so that

$$|u_x + u_y + u| \leq \varepsilon.$$

Consider the change of coordinates

$$\zeta = x,$$
$$\eta = y - x.$$

Then, we have

$$|u_x + u_y + u| = |u_\zeta + u| < \varepsilon. \tag{4}$$

It is clear from (4) that

$$-\varepsilon \leq u_\zeta + u \leq \varepsilon.$$

Multiplying the above estimate by the function $exp(\zeta - a)$, we have

$$-\varepsilon e^{\zeta - a} \leq u_\zeta e^{\zeta - a} + ue^{\zeta - a} \leq \varepsilon e^{\zeta - a}.$$

Let $c \in [a,b]$. For any $\zeta \in [a,b]$ integrating the above inequality from c to ζ, we obtain

$$\int_c^\zeta -\varepsilon e^{s-a} ds \leq \int_c^\zeta \frac{\partial}{\partial s}[u(s,\eta)e^{s-a}]ds \leq \int_c^\zeta \varepsilon e^{s-a} ds.$$

Then

$$-\varepsilon e^{\zeta - a} \leq u(\zeta,\eta)e^{\zeta - a} - (u(c,\eta) + \varepsilon)e^{c-a} + f(\eta) \leq \varepsilon e^{\zeta - a}.$$

Hence, it is clear that

$$-\varepsilon \le u(\zeta,\eta) - (u(c,\eta)+\varepsilon)e^{c-\zeta} + f(\eta)e^{-(\zeta-a)} \le \varepsilon.$$

Let

$$v(\zeta,\eta) = (u(c,\eta)+\varepsilon)e^{c-\zeta} - f(\eta)e^{-(\zeta-a)}.$$

Then $v(\zeta,\eta)$ satisfies $v_\zeta + v = 0$ and $|u(\zeta,\eta) - v(\zeta,\eta)| \le \varepsilon$, respectively.

Taking into account the change of coordinates, we can write

$$|u(x,y) - v(x,y)| \le \varepsilon.$$

Since $u(x,y) = Az_x + Bz_y$, we have

$$-\varepsilon \le Az_x + Bz_y - v(x,y) \le \varepsilon.$$

Consider the change of coordinates

$$r = x,$$
$$s = Ay - Bx.$$

Hence

$$Az_x + Bz_y - v(x,y) = Az_r - v(r,s).$$

From this, it follows that

$$-\varepsilon \le Az_r - v(r,s) \le \varepsilon.$$

Multiplying the above estimate by $\frac{1}{A}$, $(A \ne 0)$, we obtain

$$-\frac{\varepsilon}{A} \le z_r - \frac{v(r,s)}{A} \le \frac{\varepsilon}{A}.$$

Select $k \in [a,b]$. For any $r \in [k,b]$ with $r > 2k$, integrating the above inequality from k to r, we have

$$-\frac{\varepsilon}{A}(r-k) \le z(r,s) - z(k,s) - \int_k^r \frac{v(u,s)}{A}du \le \frac{\varepsilon}{A}(r-k).$$

Then, it follows that

$$-\frac{\varepsilon}{A}r \le z(r,s) - z(k,s) - \int_k^r \frac{v(u,s)}{A}du - \frac{\varepsilon k}{A} \le \frac{\varepsilon}{A}(r-2k)$$

so that

$$-\frac{\varepsilon}{A}r \le z(r,s) - z(k,s) - \int_k^r \frac{v(u,s)}{A}du - \frac{\varepsilon k}{A} \le \frac{\varepsilon}{A}r.$$

Let

$$z_0(r,s) = z(k,s) + \int_k^r \frac{v(u,s)}{A}du + \frac{\varepsilon k}{A}.$$

Then $v(\zeta,\eta)$ satisfies

$$A(z_0)_r - v(r,s) = 0.$$

Hence, we can conclude that

$$|z(r,s) - z_0(r,s)| \leq \frac{\varepsilon r}{A}, K = \frac{r}{A}, A \neq 0.$$

This result completes the proof of Theorem 1.

Our second and last Hyers-Ulam stability result is the following theorem.

Theorem 2. *Let ε be a positive constant. If the function z satisfies the differential inequality*

$$|x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} + mxz_x + myz_y - mz(x,y)| \leq \varepsilon \tag{5}$$

for all $(x,y) \in D$, then there exists a solution $z_0 : D \to \Re$ of equation (2) such that

$$|z(x,y) - z_0(x,y)| \leq \frac{\varepsilon}{m} M, (m > 0, M > 0).$$

Proof. For any $(x,y) \in D$ let

$$g(x,y) = xz_x + yz_y + mz.$$

Then

$$xg_x(x,y) + yg_y(x,y) - g(x,y) = x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} + mxz_x + myz_y - mz.$$

Therefore, inequality (5) implies

$$|xg_x(x,y) + yg_y(x,y) - g(x,y)| \leq \varepsilon.$$

Consider the change of coordinates

$$\zeta = x,$$
$$\eta = \frac{y}{x},\ x \neq 0.$$

Then we have

$$|\zeta g_\zeta - g| \leq \varepsilon.$$

Assume that $\zeta > 0$. Making use of the former inequality, we arrive at

$$-\varepsilon \leq \zeta g_\zeta - g \leq \varepsilon.$$

Multiplying the above estimate by $\frac{a}{\zeta^2}$, we have

$$-\frac{\varepsilon a}{\zeta^2} \leq \frac{a}{\zeta} g_\zeta - \frac{a}{\zeta^2} g \leq \frac{\varepsilon a}{\zeta^2}.$$

Select $c_1 \in [a,b]$. For any $\zeta \in [c_1, b]$, $c_1 > 0$, integrating the above inequality from c_1 to ζ, we can write

$$\int_{c_1}^{\zeta} -\frac{\varepsilon a}{s^2} ds \leq \int_{c_1}^{\zeta} \frac{\partial}{\partial s}[\frac{a}{s} g(s,\eta)] ds \leq \int_{c_1}^{\zeta} \frac{\varepsilon a}{s^2} ds.$$

Hence

$$\frac{\varepsilon}{\zeta} - \frac{\varepsilon}{c_1} \leq \frac{1}{\zeta} g(\zeta,\eta) - \frac{1}{c_1} g(c_1,\eta) + f(\eta) \leq -\frac{\varepsilon}{\zeta} + \frac{\varepsilon}{c_1}.$$

From this, it is clear that

$$\frac{-\varepsilon}{c_1} \leq \frac{1}{\zeta}g(\zeta,\eta) - \frac{1}{c_1}g(c_1,\eta) + f(\eta) - \frac{\varepsilon}{\zeta} \leq \frac{\varepsilon}{c_1}.$$

Since $\zeta > 0$, if we multiply the above inequality by ζ, we get

$$-\frac{\varepsilon}{c_1}\zeta \leq g(\zeta,\eta) - \frac{\zeta}{c_1}g(c_1,\eta) + \zeta f(\eta) - \varepsilon \leq \frac{\varepsilon}{c_1}\zeta.$$

Let

$$v(\zeta,\eta) = \frac{\zeta}{c_1}g(c_1,\eta) - \zeta f(n) + \varepsilon.$$

Thus $v(\zeta,\eta)$ satisfies the following equation

$$\zeta v_\zeta - v = 0$$

and the inequality

$$|g(\zeta,\eta) - v(\zeta,\eta)| \leq M\varepsilon,$$

where $M = \frac{\zeta}{c_1}$. In view of the fact that

$$g(x,y) = xz_x + yz_y + mz_y,$$

it is clear that

$$-\varepsilon M \leq xz_x + yz_y + mz(x,y) - v(x,y) \leq M\varepsilon.$$

Consider the change of coordinates

$$r = x,$$
$$n = \frac{y}{x}, x \neq 0.$$

Then, from the previous inequality, we have

$$-\varepsilon M \leq rz_r + mz - v \leq M\varepsilon.$$

Multiplying the above estimate by the function $\frac{r^{m-1}}{a^m}$, $(r > 0, (\frac{r}{a})^m > 0)$, we get

$$-\varepsilon M\frac{r^{m-1}}{a^m} \leq \frac{r^m}{a^m}z_r + m\frac{r^{m-1}}{a^m}z - \frac{r^{m-1}}{a^m}v \leq \varepsilon M\frac{r^{m-1}}{a^m}.$$

Select $k \in [a,b]$. For any $r \in [k,b]$ with $\frac{k^m}{ma^m} > 0$, integrating above inequality from k to r, we obtain

$$-\varepsilon(\frac{r^m}{ma^m} - \frac{k^m}{ma^m})M \leq \frac{r^m}{a^m}z(r,n) - \frac{k^m}{a^m}z(k,n) - \int_k^r \frac{s^{m-1}}{a^m}v(s,n)ds \leq \varepsilon(\frac{r^m}{ma^m} - \frac{k^m}{ma^m})M.$$

From the last inequality, it may be seen that

$$-\varepsilon M\frac{r^m}{ma^m} \leq \frac{r^m}{a^m}z(r,n) - \frac{k^m}{a^m}z(k,n) - \int_k^r \frac{s^{m-1}}{a^m}v(s,n)ds - \varepsilon\frac{k^m}{ma^m} \leq \varepsilon M\frac{r^m}{ma^m}$$

so that

$$-\frac{\varepsilon}{m}M \leq z(r,n) - \frac{k^m}{r^m}z(k,n) - r^{-m}\int_k^r s^{m-1}v(s,n)ds - \varepsilon\frac{k^m}{mr^m} \leq \frac{\varepsilon}{m}M.$$

Let

$$z_0(r,\eta) = \frac{k^m}{r^m}z(k,n) + r^{-m}\int_k^r s^{m-1}v(s,n)ds + \varepsilon\frac{k^m}{mr^m}.$$

Then

$$|z(r,s) - z_0(r,s)| \leq \frac{\varepsilon}{m}M.$$

This completes the proof of Theorem 2.

Example 1. We consider the following linear partial differential equation of second order with constant coefficients

$$z_{xx} + 2z_{xy} + z_{yy} + z_x + z_y = 0.$$

Let $s = y - x$ and $f(s) > 0$. It can be seen that $z(x,y) = (e^{-x} - 1)f(y - x)$ is a solution of this equation and

$$|z_{xx} + 2z_{xy} + z_{yy} + z_x + z_y| \leq \varepsilon.$$

Let $[a,b] = [0,\infty)$ and $k = 0$, $c = 2$, $r = \frac{5}{2}$. Then, from Theorem 1, we have

$$|z - z_0| \leq \frac{5}{2}\varepsilon$$

and

$$z_0(r,s) = z(k,s) + \int_k^r \frac{v(u,s)}{A}du + \frac{\varepsilon k}{A}.$$

Thus, we can write

$$\begin{aligned} z_0(r,s) = \int_0^r v(u,s)du &= \int_0^r [(u(c,s)+\varepsilon)e^{c-m} - f(s)e^{-(m-a)}]dm \\ &= -(u(c,s)+\varepsilon)e^{c-r} + (u(c,s)+\varepsilon)e^{c} + f(s)(e^{-r} - 1). \end{aligned}$$

At the end, we can conclude that $|z - z_0| \leq \varepsilon r$. This inequality shows that the result of Theorem 1 is true.

Example 2. Consider the partial Euler differential equation of the form

$$x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} + xz_x + yz_y - z = 0.$$

Then, it may be followed that

$$z(x,y) = xf\left(\frac{y}{2x}\right) + \frac{x}{2}g\left(\frac{y}{x}\right)$$

is a solution of the former equation, and we can find

$$|x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} + xz_x + yz_y - z| \leq \varepsilon.$$

For $k = 0$, from Theorem 2, we have

$$z_0 = \frac{1}{r}\int_0^r (\frac{s}{c_1}g(c_1,n) - sf(n) + \varepsilon)ds = \frac{r}{2c_1}g(c_1,n) - \frac{r}{2}f(n) + \frac{\varepsilon}{m}$$

and

$$|z - z_0| \leq \frac{\varepsilon}{m}.$$

Hence, we can conclude that the result of Theorem 2 is correct.

3 Conclusion

We consider a linear partial differential equation of second order with constant coefficients and a partial Euler differential equation of second order. We study the Hyers-Ulam stability of these equations. We give two examples to verify the obtained results and for illustrations. Our results are contributions to the topic and the related literature.

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