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# Analysis of a Set of Trajectories of Generalized Standard Systems: Averaging Technique

*Dedicated to the Memory of Yu. A. Mitropolsky*

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**Abstract:** For standard form nonlinear equations with Hukuhara derivative, estimates of deviation of a set of exact solutions from the averaged ones are established and the deviation of a set of trajectories of averaged equations from the equilibrium state is specified in terms of pseudo-linear integral inequalities. Sets of affine systems and problems of approximate integrations and stability over finite interval are considered as applications.

**Keywords:** *set of standard equations; estimates of set of solutions; set of affine systems; finite-time stability.*

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## Introduction

The averaging technique developed in the framework of nonlinear mechanics is a powerful tool widely used for the analysis of nonlinear systems found in various applied investigations (see [1–3] and bibliography therein).

The differential equation with Hukuhara derivative was first considered in [4], where its solution was shown to be a multivalued mapping. In the following papers a lot of authors (see [5, 6] and bibliography therein) established conditions for existence, uniqueness and convergence of successive approximate solutions and many other results. Moreover,

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new estimates of solutions were given and a principle of comparison with the matrix Lyapunov function was presented (see [7, 8]). Also, theorems on stability and boundedness of sets of trajectories of the equations of this type were proved [9].

The present paper deals with the sets of generalized standard systems of equations with Hukuhara derivative (see [6] and bibliography therein) and treats some problems of qualitative behaviour of set of trajectories of averaged equations.

The paper is arranged as follows. In Section 2 the problem for a set of standard systems of differential equations with Hukuhara derivative is stated. In Section 3 an estimate of distance between the set of solutions to the initial and the averaged system is found. In Section 4 the application of Theorem 1 to the analysis of set of solutions to quasilinear equations is considered. In Section 5 an estimate of deviation of the set of solutions to the averaged system from the equilibrium state is obtained. In Section 6 an estimate of deviation of the set of solutions to the affine system from zero is given. In Section 7 concluding remarks are presented and problems for further investigations are discussed.

## 1 Designations and Definitions

We shall recall designations and definition needed for further presentation of results (see [5] and bibliography therein).

Let $E$ be a real Banach space with the norm $\|\cdot\|$ and $2^E$ be a totality of all nonempty bounded subsets of the space $E$ with a Hausdorff pseudometric

$$D[A,B] = \max\Big\{\sup_{x\in B} d(x,A), \sup_{y\in A} d(y,B)\Big\}, \tag{1}$$

where $A, B \in 2^E$, $d(x,A) = \inf[d(x,y)\colon y \in A]$, $d(x,B) = \inf[d(y,x)\colon x \in B]$.

We designate by $K(E)$, $(K_c(E))$ a totality of all nonempty compact (convex) subsets of $E$ which are considered as subspaces $2^E$. Note that on $K(E)$ and $(K_c(E))$ the topology of the space $2^E$ is induced by the Hausdorff pseudometric (1). Besides, $K(E)$ and $(K_c(E))$ are complete and separable metric spaces if $E$ is separable.

Let $A, B \in K_c(E)$. The set $C \in K_c(E)$ satisfying the relation $A = B + C$ is a Hukuhara difference of the sets $A$ and $B$. The mapping $F\colon [0,a] \to K_c(E)$ possesses the Hukuhara derivative $D_H F(t_0)$ at the point $t_0 \in J = [0,*]$, $a > 0$, if there exists $D_H F(t_0) \in K_C(E)$ such that the limits

$$\lim\big\{[F(t_0+\tau) - F(t_0)]\tau^{-1}\colon \tau \to 0^+\big\} \text{ and } \lim\big\{[F(t_0) - F(t_0-\tau)]\tau^{-1}\colon \tau \to 0^+\big\} \tag{2}$$

exist in the topology of the space $K_C(E)$ and the both limits are equal to $D_H F(t_0)$.

It is known (see [4]) that if

$$F(t) = X_0 + \int_0^t \Phi(s)\,ds, \quad X_0 \in K_C(E), \tag{3}$$

where $\Phi\colon JK_C(E)$ is an integrable function in the Bohner sense, then $D_H F(t_0)$ exists almost everywhere on $J$ and

$$D_H F(t) = \Phi(t) \ \text{ almost everywhere on } \ J. \tag{4}$$

Recall that the Hausdorff pseudometric (1) satisfies the following relations

$$D[U+W, V+W] = D[U,V],$$
$$D[\lambda U, \lambda V] = \lambda D[U,V],$$
$$D[U,V] \le D[U,W] + D[W,V]$$

for all $U, V, W \in K_c(E)$ and $\lambda \in \mathbb{R}_+$. Besides, for $U \in K_c(E)$ we admit that $D[U, \Theta_0] = \|U\| = \sup\{\|u\|;\ \ u \in U\}$, where $\Theta_0$ is a zero element in $E$.

## 2 Statement of the Problem

The set of systems of differential equations

$$D_H X = \mu F(t,X), \quad X(t_0) = X_0 \in K_c(E), \tag{5}$$

where $F \in C(\mathbb{R} \times K_C(E), K_c(E))$, and $\mu \in (0,1]$ is a small parameter, is called a generalized standard system. Together with equation (5) we shall consider a partially averaged differential equation (see [6])

$$D_H Y = \mu G(t,Y), \quad Y(t_0) = Y_0 \in K_c(E), \tag{6}$$

for which

$$\lim_{T\to\infty} \frac{1}{T} D\left[\int_0^T F(s,X)ds,\ \int_0^T G(s,Y)ds\right] = 0, \tag{7}$$

for $X, Y \in D^* \subset K_c(E)$.

We assume on the families of equations (5) and (6) as follows.

$H_1$. There exists a function $M(t,\cdot) > 0$, integrable on $J$, for all $t \in J$ such that

$$\mu D[G(t,X), G(t,Y)] \le M(t,\mu) D[X,Y]$$

for all $0 < \mu < \mu_1 \in (0,1]$;

$H_2$. There exist a function $f(t,\cdot) > 0$, integrable on $J$, $\lim f(t,\mu) = 0$ as $t \to \infty$, and $\alpha > 1$ such that

$$\mu D[F(t,X), G(t,Y)] \le f(t,\mu) D^\alpha[X,Y]$$

for all $(X,Y) \in D^* \subseteq K_c(E)$ and $0 < \mu < \mu_1 \in (0,1]$.

This paper is aimed to obtain estimate of deviation between solutions to the family of equations (5) and (6) and deviation of solutions to averaged equations (6) from the equilibrium state $\Theta_0 \in K_c(E)$.

## 3 Estimate of the Distance Between Sets of Solutions

We shall estimate deviations between the sets of solutions to the families of equations (5) and (6). Let us show that the following result holds true.

**Theorem 1** *In the domain* $Q = \{(t,X)\colon t \ge t_0 \ge 0, X \in D^* \subseteq K_c(E)\}$ *let the following conditions be satisfied:*

(1) *solution* $X(t)$ *of the initial problem for the family of equations* (5) *exists for all* $t \ge t_0$ *and* $0 < \mu < \mu_1,\ \mu_1 \in (0,1]$*;*

(2) *solution* $Y(t)$ *of the family of equations* (6) *with the initial condition* $Y_0 \in D^* \subset D$ *exists for all* $t \ge t_0$ *and* $0 < \mu < \mu_2,\ \mu_2 \in (0,1]$*;*

(3) *limit* (7) *exists uniformly with respect to* $X \in D$*;*

(4) *conditions of hypotheses* $H_1$ *and* $H_2$ *are satisfied;*

(5) *for all* $t \in J$ *and* $0 < \mu < \mu_0$ *the inequality*

$$1 - (\alpha - 1) D^{\alpha-1}[X_0, Y_0] \int\limits_0^t f(s,\mu) \exp\Big(2(\alpha-1) \int\limits_0^s M(\tau,\mu) d\tau\Big) ds > 0.$$

*is valid.*

*Then the deviation between the sets of solutions to equations (5) and (6) is estimated as*

$$D[X(t), Y(t)] \le$$

$$\le \frac{D[X_0, Y_0] \exp\Big(\int\limits_0^t M(s,\mu) ds\Big)}{\Big(1 - (\alpha-1) D^{\alpha-1}[X_0, Y_0] \int\limits_0^t f(s,\mu) \exp\Big(2(\alpha-1)\int\limits_0^s M(\tau,\mu) d\tau\Big) ds\Big)^{\frac{1}{\alpha-1}}} \tag{8}$$

*for all* $t \in J$ *and* $0 < \mu < \mu_0,\ \mu_0 = \min(\mu_1, \mu_2)$.

***Proof.*** We represent the families of equations (5) and (6) in the equivalent form

$$X(t) = X_0 + \mu \int\limits_0^t F(s, X(s))\, ds, \quad X_0 \in D \subset K_c(E),$$

and

$$Y(t) = Y_0 + \mu \int\limits_0^t G(s, Y(s))\, ds, \quad Y_0 \in D^* \subset D,$$

and assume that $D[X_0, Y_0] \ne 0$ for all $X_0$ and $Y_0$ in the domain of their values. It is easy to get the following estimates

$$\begin{aligned} D[X(t),Y(t)] &= D\Big[X_0 + \mu \int\limits_0^t F(s, X(s)) ds, Y_0 + \mu \int\limits_0^t G(s, Y(s)) ds\Big] \\ &= D[X_0, Y_0] + \mu D\Big[\int\limits_0^t F(s, X(s)) ds, \int\limits_0^t G(s, Y(s)) ds\Big] \\ &\le D[X_0, Y_0] + \mu D\Big[\int\limits_0^t F(s, X(s)) ds, \int\limits_0^t G(s, Y(s)) ds\Big] \\ &+ \mu D\Big[\int\limits_0^t G(s, X(s)) ds, \int\limits_0^t G(s, Y(s)) ds\Big] \end{aligned}$$

$$
\begin{aligned}
&\leq D[X_0, Y_0] + \mu \int\limits_0^t D[F(s, X(s)), G(s, Y(s))] ds \\
&+ \mu \int\limits_0^t D[G(s, X(s)), G(s, Y(s))] ds.
\end{aligned} \tag{9}
$$

In view of

$$D[F(t, X), G(t, X)] \leq D[F(t, X), G(t, Y)] + D[G(t, Y), G(t, X)],$$

under hypotheses $H_1$ and $H_2$ we get from estimate (9) that

$$D[X(t), Y(t)] \leq D[X_0, Y_0] + 2 \int\limits_0^t M(s, \mu) D[X(t), Y(t)]\, ds + \int\limits_0^t f(s, \mu) D^\alpha [X(s), Y(s)]\, ds, \tag{10}$$

where $0 < \mu < \mu_0,\ \mu_0 = \min(\mu_1, \mu_2)$.

We designate $D[X(t), Y(t)] = m(t)$ and represent inequality (10) as

$$m(t) \leq m(t_0) + 2 \int\limits_0^t M(s, \mu) m(s)\, ds + \int\limits_0^t f(s, \mu) m^\alpha (s)\, ds, \tag{11}$$

where $0 < \mu < \mu_0,\ \mu_0 = \min(\mu_1, \mu_2)$, and $t \in J$. Inequality (11) is rewritten in the pseudo-linear form

$$m(t) \leq m(t_0) + \int\limits_0^t (2M(s, \mu) m(s)\, ds + f(s, \mu) m^{\alpha - 1}(s)) m(s)\, ds$$

for all $t \in J$. Applying to this inequality the summand estimation technique from [10, 11] we get

$$m(t) \leq \frac{m(t_0) \exp\left(2 \int\limits_0^t M(s, \mu) ds\right)}{\left(1 - (\alpha - 1) m^{\alpha - 1}(t_0) \int\limits_0^t f(s, \mu) \exp\left(2(\alpha - 1) \int\limits_0^s M(\tau, \mu) d\tau\right) ds\right)^{\frac{1}{\alpha - 1}}} \tag{12}$$

under condition (5) of Theorem 1. Estimate (12) yields the assertion of Theorem 1.

Estimate (8) allows one to establish sufficient conditions for the presence of $(A, \lambda)$-estimate of approximate integration of the family of equations (5) in the sense of the definition below.

**Definition 1** The set of solutions $Y(t)$ of the family of differential equations (6) satisfies $(A, \lambda)$-estimate of approximate integration of the family of equations (5) if, given the values $\lambda$ and $A$ $(0 < \lambda < A)$, the condition $D[X_0, Y_0] < \lambda$ implies that $D[X(t), Y(t)] < A$ for $0 < \mu < \mu_0$ on the common existence interval of solutions $X(t)$ and $Y(t)$.

**Corollary 1** Let all conditions of Theorem 1 be satisfied and for given values $\lambda$ and $A$ the inequality

$$\frac{\exp\Big(2\int\limits_0^t M(s,\mu)ds\Big)}{\Big(1-(\alpha-1)\lambda^{\alpha-1}\int\limits_0^t f(s,\mu)\exp\Big(2(\alpha-1)\int\limits_0^s M(\tau,\mu)d\tau\Big)ds\Big)^{\frac{1}{\alpha-1}}} < \frac{A}{\lambda}$$

holds true for all $0<\mu<\mu_0$ and all $t\in J$.

Then for the set of solutions $X(t)$ of the family of equations (5) the $(A,\lambda)$-estimate of approximate integration takes place.

The assertion of Corollary 1 follows immediately from the estimate (8) and Definition 1.

Further we consider the quasilinear equation

$$D_H X = A(t)X + \mu F(t,X), \quad X(0) = X_0 \subset D^*, \tag{13}$$

where $A(t)$ is a bounded operator on $R_+$, $F(t,X)$ is a mapping containing $X$ in power higher than 2.

The solution of problem (13) is the mapping $X(t) = X(t,t_0,X_0)$ satisfying the family of equations (13) almost everywhere on $J$.

Together with the family of equations (13) we consider a family of averaged equations

$$D_H Y = \overline{A}(t)Y + \mu G(t,Y), \quad Y(t_0) = Y_0 \subset D^*, \tag{14}$$

where

$$\overline{A} = \lim_{T\to\infty} \frac{1}{T}\int\limits_0^T A(s)\,ds \tag{15}$$

and $G(t,Y)$ satisfies relation (7). We assume on the family of equations (14) as follows.

$H_3$. There exists an integrable function $b(t)>0$ for all $t\in J$ such that

$$\|A(t)-\overline{A}\| \le b(t).$$

We find the estimate of deviation of the set of solutions to the averaged equation (14) from the solutions to the initial equations (13).

**Theorem 2** *In the domain* $Q=\{(t,X)\colon t\ge 0$ *and* $X\in D\subset K_c(E)\}$ *and let the following conditions be satisfied:*

(1) *there exists a limit* (15) *and the correlation* (7) *holds;*

(2) *the conditions of hypotheses* $H_1$ *and* $H_3$ *are satisfied;*

(3) *for* $Y_0\in D^*$ *the solution of averaged equations* (14) *is defined for all* $t\ge 0$ *and* $0<\mu<\mu_0$*;*

(4) *for all* $t\in J$ *and* $0<\mu<\mu_0$ *the inequality*

$$2(\alpha-1)m^{\alpha-1}(t_0)\int\limits_0^t M(s,\mu)\exp\left((\alpha-1)\int\limits_0^s (b(\tau)+f(\tau,\mu))d\tau\right)ds < 1$$

*is true.*

*Then the deviation between the sets of solutions to equations* (13) *and* (14) *is estimated as*

$$D[X(t),Y(t)] \leq$$
$$\leq \frac{D[X_0,Y_0]\exp\Big(\int\limits_0^t (b(s)+2M(s,\mu))ds\Big)}{\Big[1-2(\alpha-1)m^{\alpha-1}(0)\int\limits_0^t M(s,\mu)\exp\Big((\alpha-1)\int\limits_0^s (b(\tau)+f(\tau,\mu))d\tau\Big)ds\Big]^{\frac{1}{\alpha-1}}} \tag{16}$$

*for all* $t \in J$ *and* $0 < \mu < \mu_0$.

***Proof.*** The relations

$$\begin{aligned} X(t) &= X_0 + \int\limits_0^t (A(s)X(s) + \mu F(s,X(s)))\,ds, \\ Y(t) &= Y_0 + \int\limits_0^t (\overline{A}(s)Y(s) + \mu G(s,Y(s)))\,ds \end{aligned} \tag{17}$$

for $D[X_0,Y_0] \neq 0$ yield

$$\begin{aligned} &D[X(t),Y(t)] \\ &\quad = D\Bigg[X_0 + \int\limits_0^t (A(s)X(s)+\mu F(s,X(s)))ds,\ Y_0 + \int\limits_0^t (\overline{A}(s)Y(s)+\mu G(s,Y(s)))ds\Bigg] \\ &\quad \leq D[X_0,Y_0] + D\Bigg[\int\limits_0^t A(s)X(s)ds, \int\limits_0^t \overline{A}(s)Y(s)ds\Bigg] \\ &\quad + \mu D\Bigg[\int\limits_0^t F(s,X(s))ds, \int\limits_0^t G(s,Y(s))ds\Bigg] \\ &\quad \leq D[X_0,Y_0] + \int\limits_0^t (A(s)-\overline{A})D[X(s),Y(s)]ds + \mu\int\limits_0^t D[F(s,X(s)),G(s,X(s))]ds \\ &\quad + \mu\int\limits_0^t D[F(s,X(s)),G(s,Y(s))]ds. \end{aligned}$$

Hence, according to hypotheses $H_1$ and $H_3$ it follows that

$$\begin{aligned} D[X(t),Y(t)] \leq D[X_0,Y_0] &+ \int\limits_0^t (b(s)+2M(s,\mu))D[X(s),Y(s)]\,ds \\ &+ \int\limits_0^t f(s,\mu)D^\alpha[X(s),Y(s)]\,ds \end{aligned} \tag{18}$$

for all $t > t_0$ and $0 < \mu < \mu_0$. Inequality (18) is rewritten as

$$m(t) \leq m(t_0) + \int\limits_0^t [(b(s) + 2M(s,\mu))m(s) + f(s,\mu)m^\alpha(s)]\, ds$$

and further

$$m(t) \leq m(t_0) + \int\limits_0^t [(b(s) + 2M(s,\mu)) + f(s,\mu)m^{\alpha-1}(s)]m(s)\, ds. \tag{19}$$

As in the analysis of inequality (11), we get from estimate (19) the inequality

$$m(t) \leq \frac{m(t_0)\exp\Big(\int\limits_0^t (b(s) + 2M(s,\mu))ds\Big)}{\Big[1 - (\alpha-1)(0)\int\limits_0^t f(s,\mu)\exp\Big((\alpha-1)\int\limits_0^s (b(\tau) + 2M(\tau,\mu))d\tau\Big)ds\Big]^{\frac{1}{\alpha-1}}} \tag{20}$$

provided that condition (4) of Theorem 2 is satisfied for all $t \in J$ and $0 < \mu < \mu_0$. In view of the designation $m(t) = D[X(t), Y(t)]$, for all $t \in J$ estimate (20) completes the proof of Theorem 2.

**Corollary 2** Let all conditions of Theorem 2 be satisfied and for the given estimates of the values $\lambda$ and $A$ the inequality

$$\frac{\exp\Big(\int\limits_0^t (b(s) + 2M(s,\mu))ds\Big)}{1 - 2(\alpha-1)m^{\alpha-1}(0)\int\limits_0^t M(s,\mu)\exp\Big((\alpha-1)\int\limits_0^s (b(\tau) + f(\tau,\mu))d\tau\Big)ds} < \frac{A}{\lambda}$$

holds for all $t > t_0$ and $0 < \mu < \mu_0$. Then for the set of solutions $X(t)$ of the family of equations (13) the $(A, \lambda)$-estimate of approximate integration takes place.

The assertion of Corollary 2 follows from estimate (16) and Definition 1.

## 4 Conditions of $(\lambda, A, J)$-stability of Averaged Equation

Further we shall consider a family of averaged equations (13).

Assume that the following conditions are satisfied.

$H_4$. There exists a constant $a > 0$ such that

$$\|\overline{A}\| < a, \quad a = \text{const} > 0.$$

$H_5$. There exists a function $N(*, t) > 0$, which is integrable on J, such that

$$\mu D[G(t, Y), \Theta_0] \leq N(\mu, t) D^\alpha[Y, \Theta_0]$$

for all $t \in J$ and $0 < \mu < \mu_0$ in the domain of values $Y \subset D^*$.

We shall show that the following result is valid.

**Theorem 3** *In the domain* $Q = \{(t, Y) \colon t \geq 0, Y \in D^* \subset K_c(E)\}$ *let the following conditions be satisfied.*

(1) *there exists a solution* $Y(t) = Y(t, t_0, Y_0)$ *of the averaged equation* (14) *for all* $t \geq 0$ *and* $Y^* \in D^*$*;*

(2) *the conditions of hypotheses* $H_4$ *and* $H_5$ *be satisfied.*

*Then the deviation of the set of solutions* $Y(t)$ *from the equilibrium state is estimated as*

$$D[Y(t), \Theta_0] \leq \frac{n(0) \exp \int\limits_0^t \|\overline{A}\| ds}{\Big[1 - (\alpha - 1) n^{\alpha-1}(0) \int\limits_0^t N(\mu, s) \exp \Big((\alpha - 1) \int\limits_0^s \|\overline{A}\| d\tau\Big) ds\Big]^{\frac{1}{\alpha-1}}} \tag{21}$$

*for all* $t \in J$ *and* $0 < \mu < \mu_0$ *provided that*

$$(\alpha - 1) n^{\alpha-1}(t_0) \int\limits_0^t N(\mu, s) \exp \Big((\alpha - 1) \int\limits_0^s \|\overline{A}\| d\tau\Big) ds < 1$$

*for all* $t \in J$ *and* $0 < \mu < \mu_0$.

***Proof.*** For correlation (17) we have

$$\begin{aligned} D[Y(t), \Theta_0] &\leq D[Y_0, \Theta_0] + \int\limits_0^t \overline{A} D[Y(s), \Theta_0] \, ds + \mu \int\limits_0^t D[G(s, Y(s)), \Theta_0] \, ds \\ &\leq D[Y_0, \Theta_0] + \int\limits_0^t \|\overline{A}\| D[Y(s), \Theta_0] \, ds + \int\limits_0^t N(\mu, s) D^\alpha[Y(s), \Theta_0] \, ds. \end{aligned}$$

Hence, for the function $n(t) = D[Y(t), 0]$ estimating the deviation of the set of solutions to the averaged equations from zero in $K_c(E)$, we have the inequality

$$n(t) \leq n(t_0) + \int\limits_0^t \|\overline{A}\| n(s) \, ds + \int\limits_0^t N(\mu, s) n^\alpha(s) \, ds$$

for all $t \in J$ and $0 < \mu < \mu_0$. Applying to this inequality the technique used for the proof of Theorem 1 we arrive at estimate (21). This proves Theorem 3.

Estimate (21) allows one to establish conditions for $(\lambda, A, J)$-stability on finite interval of the set of solutions to equations (14).

**Definition 2** For given estimates of the values $\lambda, A, J$ the set of solutions to the averaged equations (14) is $(\lambda, A, J)$-stable if $D[Y(t), \Theta_0] < A$ for all $t \in J$, whenever $D[Y_0, \Theta_0] < \lambda$ and $0 < \mu < \mu_0$.

**Corollary 3** Let all conditions of Theorem 2 be satisfied and for given estimates of the values $\lambda, A$ and $J$ the inequality

$$\frac{\exp\Big(\int\limits_0^t \|\overline{A}\|ds\Big)}{\Big[1-(\alpha-1)\lambda^{\alpha-1}\int\limits_0^t N(\mu,s)\exp\Big((\alpha-1)\int\limits_0^s\|\overline{A}\|d\tau\Big)ds\Big]^{\frac{1}{\alpha-1}}} < \frac{A}{\lambda}$$

holds true for all $t\in J$ and $0<\mu<\mu_0$. Then the set of solutions $Y(t)$ of the family of equations (14) is $(\lambda, A, J)$-stable.

The assertion of Corollary 3 follows from estimate (21) and Definition 2.

## 5 Boundedness of Trajectories of Standard Affine Systems

Consider a family of affine systems of the form

$$D_H X(t) = \mu(f(t,X) + g(t,X)U(t)), \tag{22}$$

$$X(t_0) = X_0 \in D \subset K_c(\mathbb{R}^n), \tag{23}$$

where $f(t,X)\colon \mathbb{R}_+\times D \to K_c(E)$, $g(t,X)$ is an $n\times n$-matrix, $g(t,X)\colon \mathbb{R}_+\times D^* \to K_c(E)$, $U(t)\in W \subset K_c(E)$ is a control.

Together with the family of equations (22) we consider the averaged equations

$$\begin{gathered} D_H Y(t) = \mu(\overline{f}(t,Y) + \overline{g}(t,Y)V(t)),\\ Y(t_0) = Y_0 \in D \subset K_c(\mathbb{R}^n), \end{gathered} \tag{24}$$

where

$$\lim_{T\to\infty}\frac{1}{T}D\left[\int\limits_0^T f(s,X(s))ds, \int\limits_0^T \overline{f}(s,Y(s))ds\right] = 0; \tag{25}$$

$$\lim_{T\to\infty}\frac{1}{T}D\left[\int\limits_0^T g(s,X(s))U(s)ds, \int\limits_0^T \overline{g}(s,Y(s))V(s)ds\right] = 0; \tag{26}$$

for all $(U,V)\in W \subset K_c(E)$.

We shall estimate the deviation of solutions to the family of equations (24) from the state $\Theta_0 \in K_c(E)$. Assume as follows:

$H_6$. There exists a function $f_1(t,\mu)>0$, integrable on $J$ and such that

$$D[\overline{f}(t,Y),\Theta_0] \le f_1(t,\mu)D[Y,\Theta_0]$$

for all $Y\in D$ and $0<\mu<\mu_1$.

$H_7$. There exists a function $f_2(t,\mu)>0$, integrable on $J$ and such that

$$D[\overline{g}(t,Y)V(t),\Theta_0] \le f_2(t,\mu)D^2[Y,\Theta_0]$$

for all $Y\in D^*$, $V(t)\in W$ and $0<\mu<\mu_1$.

**Theorem 4** *In the domain* $Q = \{(t, Y) \colon t \geq 0, Y \in D^* \subset K_c(E)\}$ *for equations* (22) *and* (24) *let*

(1) *there exist limits* (25) *and* (26)*;*

(2) *conditions of hypotheses* $H_6$ *and* $H_7$ *be satisfied;*

(3) *for all* $t \geq 0$ *and* $0 < \mu < \mu_0$ *the inequality*

$$1 - D[Y_0, \Theta_0] \int\limits_0^t f_2(s, \mu) \exp\left( \int\limits_0^t f_1(\tau, \mu) d\tau \right) ds > 0 \tag{27}$$

*hold true.*

*Then the deviation of the set of solutions to the family of equations* (24) *from zero is estimated as*

$$D[Y(t), \Theta_0] \leq \frac{D[Y_0, \Theta_0] \exp\left( \int\limits_0^t f_1(s, \mu) ds \right)}{1 - D[Y_0, \Theta_0] \int\limits_0^t f_2(s, \mu) \exp\left( \int\limits_0^t f_1(\tau, \mu) d\tau \right) ds} \tag{28}$$

*for all* $t \geq 0$, $Y \in D^*$ *and* $0 < \mu < \mu_0$.

***Proof.*** Let the limiting relations (25) and (26) be satisfied. From equation (24) we have

$$Y(t) = Y_0 + \mu \int\limits_0^t \left( \overline{f}(s, Y(s)) + \overline{g}(s, Y(s)) V(s) \right) ds$$

and further

$$D[Y(t), Y_0] \leq D[Y_0, \Theta_0] + \mu \int\limits_0^t D[\overline{f}(s, Y(s)), \Theta_0]\, ds + \mu \int\limits_0^t D[\overline{g}(s, Y(s)) V(s), \Theta_0]\, ds. \tag{29}$$

Under conditions of hypotheses $H_6$ and $H_7$ we find from inequality (29) that

$$D[Y(t), \Theta_0] \leq D[Y_0, \Theta_0] + \mu \int\limits_0^t f(s, Y(s)) D[Y(s), \Theta_0] + f_2(s, \mu) D^2[Y(s), \Theta_0]\, ds. \tag{30}$$

Designate $n(t) = D[Y(t), \Theta_0]$ and from (30) we get

$$n(t) \leq n(t_0) + \mu \int\limits_0^t (f_1(s, \mu) + f_2(s, \mu) n(s)) n(s)\, ds. \tag{31}$$

Applying Gronwall-Bellman lemma to inequality (31) we arrive at

$$n(t) \leq n(t_0) \exp\left( \int\limits_0^t (f_1(s, \mu) + f_2(s, \mu)) n(s)\, ds \right).$$

Hence

$$-n(t)\exp\int_0^t(-f_2(s,\mu))n(s))\,ds \geq n(t_0)\exp\left(\int_0^t f_1(s,\mu)\,ds\right).$$

Multiplying both sides of this inequality by $f_2(t,\mu)>0$ we obtain

$$\frac{d}{dt}\left(\exp\left(-\int_0^t f_2(s,\mu)n(s)ds\right)\right) \geq -n(t_0)f_2(t,\mu)\exp\left(\int_0^t f_1(s,\mu)ds\right). \tag{32}$$

Integrating this inequality between $t_0$ and $t$ we get

$$(n(t))^{-1}\exp\left(\int_0^t f_1(s,\mu)ds\right) \geq 1-n(t_0)\int_0^t f_2(s,\mu)\exp\left(\int_0^t f_1(\tau,\mu)d\tau\right)ds. \tag{33}$$

Hence follows the estimate of deviation of the family of solutions to equation (24) from the equilibrium state in the (28) form under condition (27). Theorem 4 is proved.

Estimate (28) allows one to establish boundedness conditions for the set of solutions to the averaged affine system (24).

**Definition 3** The set of solutions to equations (24) is bounded if for any $\varepsilon>0$ and $t_0\in\mathbb{R}_+$ there exists $\delta(t_0,\varepsilon)>0$ such that

$$D[Y(t),\Theta_0]<\varepsilon \quad \text{for all} \quad t\geq t_0$$

and $0<\mu<\mu_0$, whenever $D[Y_0,\Theta_0]<\delta$.

If $\delta$ does not depend on $t_0$, the boundedness of the set of solutions $Y(t)$ is uniform with respect to $t_0$.

**Corollary 4** Let all conditions of Theorem 4 be satisfied and for any $\varepsilon>0$ there exist $\delta(\varepsilon)>0$ such that

$$\frac{\exp\left(\int_0^t f_1(s,\mu)ds\right)}{1-\delta(\varepsilon)\int_0^t f_2(s,\mu)\exp\left(\int_0^t f_1(\tau,\mu)d\tau\right)ds}<\frac{\varepsilon}{\delta(\varepsilon)}$$

for all $t\geq 0$ and $0<\mu<\mu_0$. Then the set of solutions of equations (24) is uniformly bounded.

The assertion of Corollary 4 follows from estimate (28) and Definition 3.

## 6 Conclusion

A key element of the approach is the use of nonlinear integral inequalities in the problems of qualitative analysis of the set of trajectories of generalized standard systems. The resulting estimates of evasion of the set of trajectories of the equilibrium state, and the estimate of the distance between the sets of initial and averaged solutions to systems of equations are applicable in many problems of mechanics and applied mathematics in which processes models are the system of equations (5), (13) and (22).

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