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Approximate Controllability of Nonlocal Impulsive Fractional Order Semilinear Time Varying Delay Systems

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Abstract: This paper concerns with approximate (exact) controllability of nonlocal impulsive fractional order semilinear control system with time varying delay. Simple sufficient conditions for the controllability are derived by assuming that the corresponding linear control system is controllable. The results are established under the Lipschitz continuity of nonlinear function. In particular, compactness of the semigroup and uniform boundedness of nonlinear function both are dropped. Finally, some examples are given to illustrate the developed theory.

Keywords: *fractional order semilinear systems; time varying delay; reachable set; approximate controllability.*

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1 Introduction

During the last three decades, various problems on fractional order systems have been investigated. Fractional order semilinear equations arise in the modeling of the problems in engineering, physics, medicine, finance, control and many other fields. Particularly, fractional order equations frequently appear in diffusion process, electrical science, electrochemistry, control science and several more. For more details see [1–6] and the references cited therein.

Controllability is the qualitative property of dynamical systems and is of particular importance in mathematical control theory. In literature various controllability problems for different types of semilinear dynamical systems have been studied [7–19] using several

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methods. Among these methods, the fixed point approach is frequently used to show the controllability of the system, in which the authors converted the controllability problem into a fixed point problem with the assumption that the controllability operator has an induced inverse in a function space [20–24]. In this approach, an inequality condition is always required that involves various system parameters and sometimes this condition is difficult to verify in applications.

A large number of physical dynamic systems and biological processes include time varying delay. The delays in engineering systems such as electric systems are often time-varying and sometimes vary violently with time. It is however not necessary that a system containing either time-invariant or time-varying delays is controllable. Thus the study of various types of controllability is important for application points of view. Tomar and Kumar [25] proved the approximate controllability of first order semilinear system with time varying delay. In [26] Muthukumar et al. showed the approximate controllability of nonlinear stochastic evolution time-varying delay systems. The approximate controllability of semilinear system in which the nonlinear term contains fixed delay in the state has been addressed in [14, 27]. The approximate controllability of semilinear fractional control systems, where the control function depends on multi-delay arguments and the nonlocal condition is fractional, is discussed by Debbouche and Torres [28]. Recently, Ji [29] studied the approximate controllability of fractional order control system without the compactness conditions or Lipschitz conditions for the nonlocal function.

The dynamics of many processes are subject to abrupt changes, such as shocks, harvesting and natural disasters. Short term perturbations from continuous and smooth dynamics are involved in these phenomena and the duration of these perturbations is negligible in comparison with the duration of an entire evolution. Impulsive equations have been developed in important fields of science and technology such as modeling of impulsive problems in physics, population dynamics, ecology, biotechnology, etc. and hence the study of such systems is important. The existence and uniqueness of the mild solution of fractional order impulsive semilinear system is discussed in [30, 31]. Using Krasnoselskii's fixed point theorem Tai and Wang [32] studied the controllability of fractional order impulsive neutral functional integrodifferential systems in Banach space. Sufficient conditions for the controllability of the impulsive fractional evolution integrodifferential equations in Banach spaces are established using Banach's fixed point theorem [33]. Kumar and Sukavanam [34] proved approximate controllability of fractional order semilinear delayed systems under the Lipschitz continuity of nonlinear function and extended the results for impulsive systems also. Using Darbo-Sadovskii's fixed point theorem, sufficient conditions for approximate controllability of impulsive fractional integro-differential systems with nonlocal conditions in Hilbert space are derived by Balasubramaniam et al. [35]. However, it should be stressed here that there is no paper on approximate controllability of impulsive nonlocal fractional order system so far in which the nonlinear term contains time varying delay. This is the motivation of the present paper.

The main objective of this paper is to provide simple sufficient conditions for approximate controllability of semilinear systems (2). In this approach, uniform boundedness of nonlinear function, compactness of C_0-semigroup and inequality condition involving system parameters are not required. Hence the results are more general and applied to a large number of class of semilinear systems. To establish the results a relation between the reachable set of semilinear system and that of the corresponding linear system is shown. Finally, sufficient conditions for the controllability of fractional order impulsive system (1) are obtained. The nonlinear term and nonlocal condition make the paper different from [34].

The paper is organized as follows: in Section 2, the problem formulation is presented. We give some basic definitions and lemma in Section 3. Sufficient conditions for approximate controllability are obtained in Section 4. To illustrate the theory some examples are provided in Section 5.

2 Problem Formulation

Let V, $\hat{V}$ be Banach spaces and $Z = L_2([0,\tau];V)$, $Y = L_2([0,\tau];\hat{V})$ be the corresponding function spaces. Further, let $\mathcal{C}_t := C([-r,t];V)$, $r > 0$, $0 \le t \le \tau < \infty$ be a Banach space of all continuous functions from $[-r,t]$ into V and the norm on $\mathcal{C}_t$ be defined by

$$\|\varphi\|_{\mathcal{C}_t} = \sup_{-r\le\eta\le t} \|\varphi(\eta)\|_V .$$

Let $0 < t_1 < t_2 < ... < t_m < \tau$. Consider the following fractional order nonlocal impulsive system with time varying delay

$$\left.\begin{aligned} {}^cD_t^\alpha x(t) &= Ax(t) + Bu(t) + f(t, x(\sigma(t))), && t \in]0,\tau]; \\ h(x) &= \varphi, && \text{on } [-r,0]; \\ \Delta x|_{t=t_k} &= I_k(x(t_k)), && k = 1,2,...,m, \end{aligned}\right\} \tag{1}$$

where ${}^cD_t^\alpha$ is the Caputo fractional derivative of order α; $1/2 < \alpha < 1$. The state $x(\cdot)$ takes values in Banach space V; the control function $u(\cdot)$ takes values in Y; $A : D(A) \subseteq V \to V$ is a linear operator with dense domain $D(A)$ generating a C_0-semigroup $T(t)$; B is a bounded linear operator from $\hat{V}$ to V; the function $f : [0,\tau]\times V \to V$ is nonlinear; $\sigma : [0,\tau] \to [-r,\tau]$ is a nondecreasing, non-expensive map such that it satisfies delay property i.e. $\sigma(t) \le t$, $\forall\ t \in [0,\tau]$; $h : \mathcal{C}_0 \to \mathcal{C}_0$ and there exists a function $\chi \in \mathcal{C}_0$ such that $h(\chi) = \varphi$. For some examples of h one can see [36]. Here I_k, $k = 1,2,...,m$ are appropriate functions and $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. Let $PC([-r,\tau],V) = \{x : [-r,\tau] \to V : x(t)$ be continuous everywhere except for some t_k at which $x(t_k^-)$ and $x(t_k^+)$ exist and $x(t_k^-) = x(t_k)\}$. It is easy to see that $PC([-r,\tau],V)$ is a Banach space with the norm

$$\|x\|_{PC} = \sup\{\|x(t)\| : \ t \in [0,\tau]\}.$$

To establish sufficient conditions for controllability of system (1), we first discuss controllability of the following nonlocal fractional order semilinear control system with time varying delay

$$\left.\begin{aligned} {}^cD_t^\alpha x(t) &= Ax(t) + Bu(t) + f(t, x(\sigma(t))), && t \in]0,\tau]; \\ h(x) &= \varphi, && \text{on } [-r,0]. \end{aligned}\right\} \tag{2}$$

3 Preliminaries

In this section some basic definitions and lemma, which are useful for further developments, are given.

Definition 3.1 A real function $f(t)$ is said to be in the space C_α, $\alpha \in \mathbb{R}$ if there exists a real number $p > \alpha$, such that $f(t) = t^p g(t)$, where $g \in C[0, \infty[$ and it is said to be in the space C_α^m iff $f^{(m)} \in C_\alpha$, $m \in N$.

Definition 3.2 The Riemann-Liouville fractional integral operator of order $\beta > 0$ of function $f \in C_\alpha$, $\alpha \geq -1$ is defined as

$$I^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) ds,$$

where Γ is the Euler gamma function.

Definition 3.3 If the function $f \in C_{-1}^m$ and m is a positive integer then we can define the fractional derivative of $f(t)$ in the Caputo sense as

$$^c D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^m(s) ds, \text{ where } m-1 \leq \alpha < m.$$

Definition 3.4 [37] A function $x \in \mathcal{C}_\tau$ is said to be the mild solution of (2) if it satisfies

$$\begin{aligned} x(t) &= S_\alpha(t)\chi(0) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[Bu(s) + f(s, x(\sigma(s)))] ds, \ t \in [0, \tau]; \\ x(t) &= \chi(t), \quad t \in [-r, 0], \end{aligned}$$

where

$$\begin{aligned} S_\alpha(t)x &= \int_0^\infty \phi_\alpha(\theta) T(t^\alpha \theta) x d\theta, \\ T_\alpha(t)x &= \alpha \int_0^\infty \theta \phi_\alpha(\theta) T(t^\alpha \theta) x d\theta. \end{aligned}$$

Here $\phi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-1/\alpha}\psi_\alpha(\theta^{-1/\alpha})$ is the probability density function defined on $(0, \infty)$, that is $\phi_\alpha(\theta) \geq 0$, and $\int_0^\infty \phi_\alpha(\theta) d\theta = 1$. We define $\psi_\alpha(\theta)$ as $\psi_\alpha(\theta) = \frac{1}{\pi}\Sigma_{n=1}^\infty (-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha)$, $\theta \in (0, \infty)$.

Definition 3.5 Let $x(\tau)$ be the state value of system (2) at time τ corresponding to the control u. The system (2) is said to be approximately controllable in time interval $[0, \tau]$, if for every desired final state ξ and $\epsilon > 0$ there exists a control function $u \in Y$ such that the solution of (2) satisfies

$$\|x(\tau) - \xi\| \leq \epsilon.$$

The above definition gives exact controllability of system (2) iff $\epsilon = 0$.

The set $K_\tau(f) = \{x(\tau) \in V : x(\cdot)$, is the mild solution of (2)$\}$ and is called the reachable set of the system (2). If $f \equiv 0$, then the system (2) is known as the corresponding linear system and denoted by $(2)^*$. In this case, $K_\tau(0)$ denotes the reachable set of the linear system $(2)^*$.

Definition 3.6 The system (2) is said to be approximately (exactly) controllable on $[0, \tau]$ if $\overline{K_\tau(f)} = V$ $(K_\tau(f) = V)$, where $\overline{K_\tau(f)}$ denotes the closure of $K_\tau(f)$. Clearly, the corresponding linear system $(2)^*$ is approximately controllable if $\overline{K_\tau(0)} = V$.

Lemma 3.1 *For any fixed $t \geq 0$, $S_\alpha(t)$ and $T_\alpha(t)$ are linear and bounded operators, that is, for any $x \in V$, $\|S_\alpha(t)x\| \leq M\|x\|$ and $\|T_\alpha(t)x\| \leq \frac{M\alpha}{\Gamma(\alpha+1)}\|x\|$, where M is a constant such that $\|T(t)\| \leq M$, for all $t \in [0,\tau]$ (see Lemma 3.2 [37]).*

We now define the operator $F : Z \to Z$ as

$$[Fx](t) = f(t, x(\sigma(t))); \; x \in Z.$$

The following conditions are required to establish the results:

$[H1]$ The nonlinear function satisfies the Lipschitz continuity, i.e. there exists some positive constant l such that

$$\|f(t,x) - f(t,y)\|_V \leq l\|x - y\|_{\mathcal{C}_\tau}, \text{ for all } x, y \in V.$$

Remark 3.1 Under assumption $[H1]$ one can easily verify that the mild solution of system (2) exists and is unique.

$[H2]$ Range of function F is a subset of closure of range of B, i. e.

$$R(F) \subseteq \overline{R(B)}.$$

Remark 3.2 To support this condition an example is given. Also if $B = I$ the range condition is trivially true. In several real life problems the above condition is also satisfied [38].

$[H3]$ The linear system $(2)^*$ is approximately controllable.

4 Main Results

4.1 Controllability of semilinear system

Theorem 4.1 *Under the assumptions $[H1]$-$[H3]$ the fractional order semilinear control system (2) is approximately controllable.*

Proof. To prove the result, we will show that $K_\tau(0) \subset K_\tau(f)$. For this, we assume that $x(\cdot)$ is the mild solution of $(2)^*$ corresponding to a control $u \in Y$ which is given by

$$\left.\begin{array}{ll} x(t) = S_\alpha(t)\chi(0) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)Bu(s)ds, & t \in [0,\tau]; \\ x(t) = \chi(t), & t \in [-r, 0]. \end{array}\right\} \tag{3}$$

Since $Fx \in \overline{R(B)}$ (by $[H2]$), for a given $\epsilon > 0$ there exists a control function $w \in Y$ such that

$$\|Fx - Bw\|_Z \leq \epsilon. \tag{4}$$

We now assume that $y(t)$ is the mild solution of (2) corresponding to the control $(u - w)$ in Y then

$$\left.\begin{array}{ll} y(t) = S_\alpha(t)\chi(0) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\{B(u-w) + [Fy]\}(s)ds, & t \in [0,\tau]; \\ y(t) = \chi(t), & t \in [-r, 0]. \end{array}\right\} \tag{5}$$

If $t \in [0, \tau]$ then from (3) and (5), we have

$$\begin{aligned} x(t) - y(t) &= \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[Bw - Fy](s)ds \\ &= \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[Bw - Fx](s)ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[Fx - Fy](s)ds. \end{aligned}$$

Taking norm on both sides and using (4), we get

$$\begin{aligned} \|x(t) - y(t)\|_V &\leq \int_0^t (t-s)^{\alpha-1} \|T_\alpha(t-s)\| \|Bw(s) - Fx(s)\|_V ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} \|T_\alpha(t-s)\| \|[Fx](s) - [Fy](s)\| ds \\ &\leq \frac{M\alpha}{\Gamma(\alpha+1)} \Big(\int_0^t (t-s)^{2\alpha-2} ds \Big)^{1/2} \times \\ &\quad \Big(\int_0^t \|Bw(s) - Fx(s)\|^2 ds \Big)^{1/2} \\ &\quad + \frac{M\alpha}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1} \|[Fx](s) - [Fy](s)\|_V ds \\ &\leq \frac{M\alpha}{\Gamma(\alpha+1)} \Big(\int_0^t (t-s)^{2\alpha-2} ds \Big)^{1/2} \Big(\|Fx - Bw\|_Z \Big) \\ &\quad + \frac{M\alpha}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1} \|[Fx](s) - [Fy](s)\|_V ds \\ &\leq \frac{M\alpha\epsilon}{\Gamma(\alpha+1)} \Big(\int_0^t (t-s)^{2\alpha-2} ds \Big)^{1/2} + \frac{M\alpha}{\Gamma(\alpha+1)} \times \\ &\quad \int_0^t (t-s)^{\alpha-1} \|f(s, x(\sigma(s))) - f(s, y(\sigma(s)))\|_V ds \\ &\leq \frac{M\alpha\epsilon}{\Gamma(\alpha+1)} \sqrt{\frac{\tau^{2\alpha-1}}{2\alpha-1}} \\ &\quad + \frac{Ml\alpha}{\Gamma(\alpha+1)} \int_0^\tau (\tau-s)^{\alpha-1} \|x - y\|_{\mathcal{C}_\tau} ds. \end{aligned}$$

For all values of $t \in [-r, \tau]$, we have

$$\|x(t) - y(t)\|_V \leq \frac{M\alpha\epsilon}{\Gamma(\alpha+1)} \sqrt{\frac{\tau^{2\alpha-1}}{2\alpha-1}} + \frac{Ml\alpha}{\Gamma(\alpha+1)} \int_0^\tau (\tau-s)^{\alpha-1} \|x - y\|_{\mathcal{C}_\tau} ds.$$

Using Gronwall's inequality, we get

$$\begin{aligned} \|\|x - y\|_{\mathcal{C}_\tau} &\leq \frac{M\alpha\epsilon}{\Gamma(\alpha+1)} \sqrt{\frac{\tau^{2\alpha-1}}{2\alpha-1}} \exp\left(\frac{Ml\alpha}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1} ds \right) \\ &\leq \frac{M\alpha\epsilon}{\Gamma(\alpha+1)} \sqrt{\frac{\tau^{2\alpha-1}}{2\alpha-1}} \exp\left(\frac{Ml\tau^\alpha}{\Gamma(\alpha+1)} \right). \end{aligned}$$

Since the right hand side of above inequality depends on $\epsilon > 0$ and ϵ is arbitrary, it is clear that $\|x - y\|_{\mathcal{C}_\tau}$ can be made arbitrary small by choosing suitable value of control function w. It now follows that the reachable set of system (2) is dense in the reachable set of system $(2)^*$, which is dense in V due to condition $[H3]$. Hence the approximate controllability of $(2)^*$ implies that of the semilinear control system (2). This completes the proof.

4.2 Controllability of Impulsive System

We now prove the approximate controllability of the system (1).

Definition 4.1 [30, 31] The mild solution of the system (1) is a function $x \in PC([-r,\tau];V)$ such that it satisfy the following integral equation

$$x(t) = \begin{cases} S_\alpha(t)\chi(0) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[Bu(s) + f(s, x(\sigma(s)))]ds, & t \in]0, t_1]; \\ S_\alpha(t-t_1)[x(t_1^-) + I(x(t_1^-))] + \int_{t_1}^t (t-s)^{\alpha-1}T_\alpha(t-s)[Bu(s) & \\ +f(s, x(\sigma(s)))]ds, & t \in]t_1, t_2]; \\ \cdots & \\ S_\alpha(t-t_m)[x(t_m^-) + I(x(t_m^-))] + \int_{t_m}^t (t-s)^{\alpha-1}T_\alpha(t-s)[Bu(s) & \\ +f(s, x(\sigma(s)))]ds, & t \in]t_m, \tau]; \\ \chi(t), & t \in [-r, 0]. \end{cases}$$

To establish the result we need one more hypothesis on the impulsive function as follows: $[H4]$ The functions I_k, $k = 1, 2, \cdots, m$ are continuous and uniformly bounded.

Theorem 4.2 *Under the assumptions [H1]–[H4] the fractional order semilinear control system (1) is approximately controllable.*

Proof. Let $y(t)$ be the mild solution of (1) corresponding to the control $(u - w)$ then

$$y(t) = \begin{cases} S_\alpha(t)\chi(0) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[B(u-w)(s) \\ +f(s, y(\sigma(s)))]ds, \quad t \in]0, t_1]; \\ S_\alpha(t-t_1)[y(t_1^-) + I(y(t_1^-))] + \int_{t_1}^t (t-s)^{\alpha-1}T_\alpha(t-s)[B(u-w)(s) \\ +f(s, y(\sigma(s)))]ds, \quad t \in]t_1, t_2]; \\ \cdots \\ S_\alpha(t-t_m)[y(t_m^-) + I(y(t_m^-))] + \int_{t_m}^t (t-s)^{\alpha-1}T_\alpha(t-s)[B(u-w)(s) \\ +f(s, y(\sigma(s)))]ds, \quad t \in]t_m, \tau]; \\ \chi(t), \quad t \in [-r, 0]. \end{cases}$$

The mild solution $x(t)$ of $(2)^*$ corresponding to a control u is given by

$$\begin{aligned} x(t) &= S_\alpha(t)\chi(0) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)Bu(s)ds, \quad t \in]0, \tau]; \\ x(t) &= \chi(t), \quad t \in [-r, 0]. \end{aligned}$$

To show the approximate controllability of semilinear system (1), we divide the interval $[-r, \tau]$ into subintervals $[-r, 0]$, $]0, t_1]$, $]t_1, t_2]$, $\cdots$, $]t_m, \tau]$. Now if $t \in]-r, t_1]$ the approximate controllability of the system follows from Theorem 4.1. If $t \in]t_1, t_2]$, since both $y(t_1^-)$ and $I(y(t_1^-))$ are bounded, we are able to prove the approximate controllability in the interval $t \in]t_1, t_2]$ as shown in Theorem 4.1. Similarly, we can show the approximate controllability in subsequent intervals. This completes the proof of the theorem.

5 Examples

In this section, we give examples to show the effectiveness of the developed theory.

Example 5.1 Let $V = L_2(0, \pi)$ and $A \equiv \frac{d^2}{dx^2}$ with $D(A)$ consisting of all $y \in V$ with $\frac{d^2y}{dx^2}$ and $y(0) = 0 = y(\pi)$. Put

$$\phi_n(x) = (\frac{2}{\pi})^{1/2} \sin(nx); 0 \le x \le \pi,\ n = 1, 2, \cdots,$$

then $\{\phi_n\}$ is an orthonormal base for V and ϕ_n is the eigenfunction corresponding to the eigenvalue $\lambda_n = -n^2$ of the operator A. Then the C_0-semigroup $T(t)$ generated by A has $\exp(\lambda_n t)$ as the eigenvalues and ϕ_n as their corresponding eigenfunctions, see [39].

Define an infinite-dimensional space $\hat{V}$ by

$$\hat{V} = \left\{ u \mid u = \sum_{n=2}^{\infty} u_n \phi_n,\ \text{with}\ \sum_{n=2}^{\infty} u_n^2 < \infty \right\}.$$

The norm in $\hat{V}$ is defined by

$$\|u\|_{\hat{V}} = \left(\sum_{n=2}^{\infty} u_n^2 \right)^{1/2}.$$

Define a continuous linear map B from $\hat{V}$ to V as

$$Bu = 2u_2\phi_1 + \sum_{n=2}^{\infty} u_n \phi_n \text{ for } u = \sum_{n=2}^{\infty} u_n \phi_n \in \hat{V}.$$

Let us consider the following fractional order semilinear control system of the form

$$\begin{aligned} {}^cD_t^{\alpha} y(t,x) &= \frac{\partial^2 y(t,x)}{\partial x^2} + Bu(t,x) + f(t, y(\sigma(t)));\ t \in [0,\tau], 0 < x < \pi \\ y(t,0) &= y(t,\pi) = 0;\ t > 0 \\ y_0(x) &= \frac{1}{r}\int_{-r}^{0} \exp(2s) y(s,x) ds. \end{aligned} \tag{6}$$

Let $\sigma(t) = \frac{t^2}{t^2+1} - r$ be time varying aftereffect such that $\sigma(t) \le t$ for all $t \in [0, \tau]$. If we take $h(y)(t) = g(y)$ for $y \in \mathcal{C}_0$, $t \in [-r, 0]$; $\varphi = y_0$, where $g : \mathcal{C}_0 \to V$ is such that $g(y)(x) = \frac{1}{r}\int_{-r}^{0} \exp(2s) y(s,x) ds$. Thus we are able to define a function $\chi \in \mathcal{C}_0$ such that $\chi(t) = \frac{1}{k} y_0$ on $[-r, 0]$ with $k = \frac{1}{r}\int_{-r}^{0} \exp(2s) ds \neq 0$ and

$$h(\chi)(t) = \frac{1}{r}\int_{-r}^{0} \exp(2s) \left(\frac{1}{k} y_0 \right) ds = y_0 = \varphi(t).$$

Thus the system (6) can be written in the abstract form given by (2). If the conditions $[H1]$–$[H3]$ are satisfied, then the approximate controllability of system (6) follows from Theorem 4.1. For example, if we consider the function f as

$$f(t,z) = l\|z\|(\phi_3(z) + \phi_4(z)),$$

where l is a positive constant. Here it is clear that f satisfies $[H3]$ with Lipschitz constant l and $R(f) \subset R(B)$. However, it should be noted that the nonlinear term is not uniformly bounded.

Example 5.2 Let us consider the following fractional order impulsive system with finite delay

$$\begin{aligned} {}^cD_t^\alpha y(t,x) &= \frac{\partial^2 y(t,x)}{\partial x^2} + Bu(t,x) + f(t,y(t-r,x));\ t\in[0,\tau], 0<x<\pi, \\ y(t,0) &= y(t,\pi)=0;\ t>0, \\ y(t,x) &= \varphi;\ t\in[-r,0], \\ y(t_k^+,x) &- y(t_k^-,x) = I_k(y(t_k^-,x));\ k=0,1,2,\cdots, \end{aligned} \tag{7}$$

where $I_k > 0,\ k = 1,2,\cdots,m$ and $\varphi \in \mathcal{D} = \{\nu : [-r,\tau] \to V : \nu(t)$ is continuous everywhere except for some t_k at which $\nu(t_k^-)$ and $\nu(t_k^+)$ exist and $\nu(t_k^-) = \nu(t_k)\}$.

The system (7) can be reformulated in the abstract form given by (1). The approximate controllability of the system (7) follows from Theorem 4.2 if the conditions [H1]–[H4] are satisfied.

Conclusion

The approximate controllability of nonlocal impulsive fractional order semilinear time varying delay systems is proved. In literature, fixed-point theory has been used to establish the approximate controllability of semilinear control systems. This approach needs certain inequality conditions involving various system parameters which are sometimes difficult to be verified. Here, the approximate controllability of nonlocal impulsive fractional order semilinear control system has been proved for a certain class of nonlinear functions under simple sufficient conditions.

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