Nonlinear Dynamics and Systems Theory, **16** (4) (2016) 335–349

Minima of Some Integral Functional: Existence and Regularity

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Received: August 25, 2015; Revised: October 22, 2016

Abstract: We prove the existence and the regularity of minima for functional whose prototype is:

$$J(u) = \int_\Omega \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\, dx - \int_\Omega F.\nabla u\, dx, \qquad u \in W_0^{1,p}(\Omega),$$

where Ω is a bounded domain of $I\!R^N$, $p > 1$ and $\alpha > 0$. The function F belongs to some Lebesgue space.

Keywords: *non-linear elliptic equations; degenerate coercive truncations; calculus of variations.*

Mathematics Subject Classification (2010): 35J60, 35J70, 46E35, 35B45.

1 Introduction and Statement of Results

In this paper, we deal with the study of minima for functional whose prototype is:

$$J(u) = \int_\Omega \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\, dx - \int_\Omega F.\nabla u\, dx, \qquad u \in W_0^{1,p}(\Omega), \tag{1.1}$$

where Ω is a bounded open subset of $I\!R^N$, $N \geq 2, \alpha > 0$, and and $1 < p < N$. The datum F belongs to the space $(L^r(\Omega))^N$ for some $r \geq 1$.

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The search of sufficient condition to secure that the functional $J(u) = \int_\Omega a(x,u,\nabla u)\,dx$ attained an extreme value has a long history (see B. Dacorogna [8]). R. Tahraoui, A. Cellina and S. Perrotta in [6,12] prove that the functional J admits a unique minimum, without any assumptions on a, except for the lower semi-continuity and the growth condition. Landes in [10] has shown that if J is weakly lower semi-continuous at one fixed level set, then this level set is an extreme value of J or the defining a is convex in the gradient.

The functional J (see (1.1)) is defined on $W_0^{1,p}(\Omega)$, when $r \geq p'$, but it may not be coercive on the same space as u becomes large (see Example 3.3 of [3]). Thus even if J is lower semi-continuous on $W_0^{1,p}(\Omega)$ as a consequence of the De Giorgi theorem, the lack of coerciveness implies that J may not attain its minimum on $W_0^{1,p}(\Omega)$ even in the case in which J is bounded from below (see Example 3.2 of [3]). To overcome this difficulty we will reason (as in [3]) by extending the functional J to $W_0^{1,q}(\Omega)$ for some $q < p$ depending on α. Thus functional attains its minimum on this larger space when $r \geq q'$. In the same way we cite the recent works of Boccardo and Orsina [1, 2].

In this paper, we will prove several results of existence and regularity of minima (depending on the summability of the datum F) for functional J.

Let us give the precise assumptions on the problem that we will study. Let Ω be a bounded open subset of $\mathbb{R}^N, N \geq 2$. Let $1 < p < N$, and let $a : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a Caratheodory function (that is, $a(.,t)$ is measurable on Ω for every $t \in \mathbb{R}$, and $a(x,.)$ is continuous on $\mathbb{R}$ for almost every x in Ω), such that the following assumption

$$\frac{\beta_0}{(1+|t|)^{\alpha p}} \leq a(x,t) \leq \beta_1 \tag{1.2}$$

for almost every x in Ω, for every t in $\mathbb{R}$ where α, β_0 and β_1 are positive constants. We furthermore suppose that:

$$0 < \alpha < \frac{1}{p'}. \tag{1.3}$$

The function F is such that:

$$|F| \in \quad L^r(\Omega) \quad \text{for some} \quad r \geq p'. \tag{1.4}$$

Example of the function a that satisfies (1.2) is:

$$a(x,t) = \frac{\beta_0}{(b(x)+|t|)^{\alpha p}},$$

where b is a measurable function on Ω such that:

$$0 < \beta_2 \leq b(x) \leq \beta_3 \quad \text{for almost everywhere in} \quad \Omega, \tag{1.5}$$

where β_2 and β_3 are two positive constants.

Similar problems have been considered in [3], more precisely the authors have studied the existence and the regularity of minima for functional:

$$I(u) = \int_\Omega a(x,u)|\nabla u|^p\,dx - \int_\Omega f.u\,dx, \qquad u \in \ \alpha > 0, \tag{1.6}$$

where f belongs to $L^r(\Omega)$ for some $r \geq p'$. The following regularity was proved in [3] in light of various summability of the source term

$$\begin{array}{ll} r > \dfrac{r}{p} & \Rightarrow\ u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \\ \left(\dfrac{p^*}{1+\alpha p}\right)' \leq r < \dfrac{r}{p} & \Rightarrow\ u \in W_0^{1,p}(\Omega) \cap L^s(\Omega), \\ (p^*(1-\alpha))' \leq r < \left(\dfrac{p^*}{1+\alpha p}\right)' & \Rightarrow\ u \in W_0^{1,\rho}(\Omega) \cap L^s(\Omega), \end{array}$$

where

$$s = \frac{Nr(p(1-\alpha)-1)}{N-rp}, \qquad \rho = \frac{Nr(p(1-\alpha)-1)}{N-r(1+\alpha p)}.$$

Following this way, in this paper, we are interested in the existence and the regularity of minima for functional $J(v)$.

Notations :

In the sequel we will use the following functions of a real variable depending on a parameter $k > 0$:

$$T_k(s) = \max(-k, \min(k, s)), \qquad G_k(s) = s - T_k(s). \tag{1.7}$$

Furthermore, we will denote by c or $c_1, c_2, ...$, various constants which may depend on the data of the problem, whose value may vary from line to line.

If $1 < \sigma < N$, we denote by $\sigma^* = \frac{N\sigma}{N-\sigma}$ the Sobolev embedding exponent for the space $W_0^{1,\sigma}(\Omega)$.

If $u\ :\ \Omega \to I\!\!R$ is a Lebesgue measurable function, we define, for all $k \geq 0$

$$A_k = \{x \in \Omega\ \ :\ \ |u(x)| \geq k\}\,, \quad B_k = \{x \in \Omega\ \ :\ \ k \leq |u(x)| \leq k+1\}\,. \tag{1.8}$$

If E is a Lebesgue measurable subset of $I\!\!R^N$, we denote by $|E|$ its N-dimensional Lebesgue measure.

We extend the definition of J to a larger space, namely $W_0^{1,q}(\Omega)$, with $q = \frac{Np(1-\alpha)}{N-\alpha p} < p$, in the following way:

$$I(v) = \begin{cases} J(v), & \text{if } \displaystyle\int_\Omega a(x,v)|\nabla v|^p\, dx < +\infty, \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.9}$$

For the sake of simplicity, in the following we suppose that:

$$a(x,t) = \frac{1}{(1+|t|)^{\alpha p}}. \tag{1.10}$$

Our results are the following:

Theorem 1.1 *Let* $q = \dfrac{Np(1-\alpha)}{N-\alpha p}$*, and let* F *be a function such that* $|F| \in L^r(\Omega)$ *with* $r \geq q'$*. Then there exists a minimum* u *of* I *on* $W_0^{1,q}(\Omega)$*.*

The second result considers the case where $|F|$ has a high summability.

Theorem 1.2 *Let F be such that $|F| \in L^r(\Omega)$ with $r > \frac{N}{p-1}$. Then any minimum u of I on $W_0^{1,q}(\Omega)$ belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$; thus J attains its minimum on $W_0^{1,p}(\Omega)$.*

Remark 1.1 Note that the condition $0 < \alpha < \dfrac{1}{p'}$ implies that $\dfrac{N}{p-1} > q'$.

Remark 1.2 Observe that the condition on r does not depend on α, and the result also does not depend on α. The main tool of the proof will be an $L^\infty(\Omega)$ estimate, which then implies the $W_0^{1,p}(\Omega)$ estimate.

Theorem 1.3 *Let F be such that $|F| \in L^r(\Omega)$ with*

$$\frac{Np'}{N - \alpha p'(N-p)} \leq r < \frac{N}{p-1}.$$

Then any minimum u of I on $W_0^{1,q}(\Omega)$ belongs to $W_0^{1,p}(\Omega) \cap L^s(\Omega)$; thus J attains its minimum on $W_0^{1,p}(\Omega)$, where

$$s = \frac{Nr\,(p(1-\alpha)-1)}{N - r(p-1)}.$$

Remark 1.3 Since $0 < \alpha < \dfrac{1}{p'}$ we have:

$$\frac{Np'}{N - \alpha p'(N-p)} < \frac{N}{p-1}.$$

Remark 1.4 Observe that if the minima are not bounded, we still have that they belong to $W_0^{1,p}(\Omega)$. The $W_0^{1,p}(\Omega)$ regularity result will be proveded combining the information that u belongs to $L^s(\Omega)$ with the fact that u is minimum.

Remark 1.5 As a consequence of the previous theorem, if $r = \dfrac{N}{p-1}$, we have that any minimum u belongs to $W_0^{1,p}(\Omega)$ and to $L^s(\Omega)$, for every $s < +\infty$.

If we decrease the summability of F, we find minima of I which do not in general belong any more to $W_0^{1,p}(\Omega)$.

Theorem 1.4 *Let F be such that $|F| \in L^r(\Omega)$ with*

$$q' \leq r < \frac{Np'}{N - \alpha p'(N-p)}.$$

Then any minimum u of I on $W_0^{1,q}(\Omega)$ belongs to $W_0^{1,\rho}(\Omega) \cap L^s(\Omega)$; thus J attains its minimum on $W_0^{1,\rho}(\Omega)$, where

$$\rho = \frac{Nr\,(p(1-\alpha)-1)}{N - \alpha pr}.$$

Remark 1.6 Note that the condition $0 < \alpha < \dfrac{1}{p'}$ implies that:

$$q' < \frac{Np'}{N - \alpha p'(N-p)}.$$

Remark 1.7 If α tends to $\dfrac{1}{p'}$ both $\dfrac{Np'}{N-\alpha p'(N-p)}$ and q' converge to $\dfrac{N}{p-1}$, so that Theorems 1.3 and 1.4 cannot be applied if $\alpha = \dfrac{1}{p'}$.

The paper is organized as follows. In the next section we prove the existence of minima for J, in the third section we give the proof of Theorem 1.2 (proof of bounded minima), while the fourth section is devoted to the proof of Theorems 1.3 and 1.4.

2 Existence of Minima

In order to prove that there exists a minimum of I on $W_0^{1,q}(\Omega)$, we are going to prove that I is both coercive and weakly lower semicontinuous on $W_0^{1,q}(\Omega)$.

Theorem 2.1 *Let F be such that: $|F| \in L^r(\Omega)$ with $r \geq q'$. Then I is coercive and weakly lower semi-continuous on $W_0^{1,q}(\Omega)$.*

Proof. The weak lower semi-continuity is a consequence of a theorem by De Giorgi (see [9]). As far as the coerciveness is concerned, it is enough to consider v in $W_0^{1,q}(\Omega)$ such that $I(v)$ is finite.

We have

$$\int_\Omega |\nabla v|^q \, dx = \int_\Omega \frac{|\nabla v|^q}{(1+|v|)^{\alpha q}} (1+|v|)^{\alpha q} \, dx,$$

therefore, by the Hölder inequality we get:

$$\int_\Omega |\nabla v|^q \, dx \leq c \left(\int_\Omega \frac{|\nabla v|^p}{(1+|v|)^{\alpha p}} \, dx \right)^{\frac{q}{p}} \left(\int_\Omega (1+|v|)^{\frac{\alpha p q}{p-q}} \, dx \right)^{1-\frac{q}{p}}.$$

By the fact that $q^* = \frac{\alpha p q}{p-q}$ and Sobolev embedding theorem we obtain:

$$\int_\Omega |\nabla v|^q \, dx \leq c \left(\int_\Omega \frac{|\nabla v|^p}{(1+|v|)^{\alpha p}} \, dx \right)^{\frac{q}{p}} \left(1 + \left(\int_\Omega |\nabla v|^q \, dx \right)^{\frac{q}{q}} \right)^{1-\frac{q}{p}},$$

which implies that if $R = \|v\|_{W_0^{1,q}(\Omega)}$

$$R^p \leq \left(\int_\Omega \frac{|\nabla v|^p}{(1+|v|)^{\alpha p}} \, dx \right)^{\frac{q}{p}} \left(1 + R^{q^*} \right)^{1-\frac{q}{p}}. \tag{2.1}$$

On the other hand we have:

$$\begin{aligned} \left| \int_\Omega F.\nabla v \, dx \right| &\leq c \left(\int_\Omega |F|^{q'} \, dx \right)^{\frac{1}{q'}} \left(\int_\Omega |\nabla v|^q \, dx \right)^{\frac{1}{q}}. \\ &\leq cR. \end{aligned} \tag{2.2}$$

Thus, by (2.1) and (2.2) we obtain:

$$I(v) \geq c \frac{R^p}{(1+R^{q^*})^{\frac{p}{q}-1}} - cR.$$

Using the definition of q, it is easy to check that:

$$p - q^* \left(\frac{p}{q} - 1 \right) > 1,$$

so that

$$\lim_{R \to +\infty} I(v) = +\infty.$$

That is I is coercive on $W_0^{1,q}(\Omega)$.

By standard results, we deduce that there exists the minimum of I on $W_0^{1,q}(\Omega)$ and then Theorem 1.1 is proved.

3 Bounded Minima

By Theorem 2.1 there exists u in $W_0^{1,q}(\Omega)$ such that

$$I(u) = \min \left\{ I(u), \quad v \in W_0^{1,q}(\Omega) \right\},$$

i.e.

$$I(u) \le I(v) \quad \text{for all} \quad v \in W_0^{1,q}(\Omega). \tag{3.1}$$

3.1 Some lemmas

To prove the bounded minima, we need the following lemmas.

Lemma 3.1 *[4] Let w be a function in $W_0^{1,\sigma}(\Omega)$ such that, for k greater than some k_0*

$$\int_{A_k} |\nabla w|^\sigma \, dx \le c k^{\theta \sigma} |A_k|^{\frac{\sigma}{\sigma^*} + \varepsilon},$$

where $\varepsilon > 0$, $0 \le \theta < 1$. Then the norm of w in $L^\infty(\Omega)$ is bounded by a constant which depends on $c, \theta, \sigma, N, \varepsilon, k_0$.

The proof of this lemma can be found in the Appendix of [4], its proof is based on the lemma according to Stampacchia [11].

Lemma 3.2 *Let u be the minima of I in $W_0^{1,q}(\Omega)$, then*

$$\int_{A_k} \frac{|\nabla u|^p}{(1 + |u|)^{\alpha p}} \, dx \le \int_{A_k} F.\nabla G_k(u) \, dx, \quad \forall \ k > 0, \tag{3.2}$$

where A_k is as in (1.8) *and G_k is the function defined in* (1.7)*.*

Proof. We have, $I(u) \le I(0) = 0$, then

$$\int_\Omega \frac{|\nabla u|^p}{(1 + |u|)^{\alpha p}} \, dx \le \int_\Omega F.\nabla u \, dx < +\infty.$$

On the other hand, we have for all $k > 0$

$$\int_\Omega \frac{|\nabla T_k(u)|^p}{(1 + |T_k(u)|)^{\alpha p}} \, dx = \int_{\{|u| \le k\}} \frac{|\nabla u|^p}{(1 + |u|)^{\alpha p}} \, dx \le \int_\Omega \frac{|\nabla u|^p}{(1 + |u|)^{\alpha p}} \, dx < +\infty.$$

We take $v = T_k(u)$ as a test function in (3.1) to obtain:

$$\int_{A_k} \frac{|\nabla u|^p}{(1 + |u|)^{\alpha p}} \, dx \le \int_{A_k} F.\nabla G_k(u) \, dx, \quad \forall \ k > 0.$$

3.2 Proof of Theorem 1.2

Let σ be such that $r' < \sigma < q < p$, this implies that $\dfrac{1}{r}+\dfrac{1}{\sigma} < 1$, then by Hölder inequality, we have:

$$\begin{aligned}\int_{A_k} |F.\nabla G_k(u)|\,dx &\leq \|F\|_{L^r}\left[\int_{A_k} |\nabla G_k(u)|^{\sigma}\,dx\right]^{\frac{1}{\sigma}}.|A_k|^{1-\frac{1}{\sigma}-\frac{1}{r}}\\ &\leq c\left[\int_{A_k} |\nabla G_k(u)|^{\sigma}\,dx\right]^{\frac{1}{\sigma}}.|A_k|^{1-\frac{1}{\sigma}-\frac{1}{r}}\end{aligned}$$

and by Lemma 3.2, we deduce that:

$$\int_{A_k} \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\,dx \leq c\left[\int_{A_k} |\nabla G_k(u)|^{\sigma}\,dx\right]^{\frac{1}{\sigma}}.|A_k|^{1-\frac{1}{\sigma}-\frac{1}{r}}. \tag{3.3}$$

Moreover, by the Hölder inequality, we obtain:

$$\begin{aligned}\int_{A_k} |\nabla u|^{\sigma}\,dx &= \int_{A_k} \frac{|\nabla u|^{\sigma}}{(1+|u|)^{\alpha\sigma}}(1+|u|)^{\alpha\sigma}\,dx\\ &\leq \left[\int_{A_k} \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\,dx\right]^{\frac{\sigma}{p}}\left[\int_{A_k}(1+|u|)^{\frac{\alpha\sigma p}{p-\sigma}}\,dx\right]^{1-\frac{\sigma}{p}},\end{aligned}$$

therefore, by (3.3), we have:

$$\int_{A_k} |\nabla u|^{\sigma}\,dx \leq c|A_k|^{\left(1-\frac{1}{\sigma}-\frac{1}{r}\right)\frac{\sigma}{p-1}}\left[\int_{A_k}(1+|u|)^{\frac{\alpha\sigma p}{p-\sigma}}\,dx\right]^{\frac{p-\sigma}{p-1}}. \tag{3.4}$$

Since if $k \geq 1$, one has on A_k that $1+|u| \leq 2(k+|G_k(u)|)$, we can write:

$$\begin{aligned}\int_{A_k} |\nabla u|^{\sigma}\,dx \leq\ & c\Big\{k^{\frac{\alpha\sigma p}{p-1}}|A_k|^{\left(1-\frac{1}{\sigma}-\frac{1}{r}\right)\frac{\sigma}{p-1}+\frac{p-\sigma}{p-1}}\\ &+|A_k|^{\left(1-\frac{1}{\sigma}-\frac{1}{r}\right)\frac{\sigma}{p-1}}\left[\int_{A_k}|G_k(u)|^{\frac{\alpha\sigma p}{p-\sigma}}\,dx\right]^{\frac{p-\sigma}{p-1}}\Big\}.\end{aligned}$$

Now, we choose σ such that $\dfrac{\alpha\sigma p}{p-\sigma} < \sigma^*$, and therefore, using Hölder's and Sobolev's inequalities one obtains:

$$\begin{aligned}\int_{A_k} |\nabla u|^{\sigma}\,dx \leq\ & c\Big\{k^{\frac{\alpha\sigma p}{p-1}}|A_k|^{\left(1-\frac{1}{\sigma}-\frac{1}{r}\right)\frac{\sigma}{p-1}+\frac{p-\sigma}{p-1}}\\ &+|A_k|^{\left(1-\frac{1}{\sigma}-\frac{1}{r}\right)\frac{\sigma}{p-1}-\frac{\alpha p}{p-1}\cdot\frac{\sigma}{\sigma^*}}\left[\int_{A_k}|\nabla u|^{\sigma}\,dx\right]^{\frac{\alpha p}{p-1}}\Big\}.\end{aligned}$$

Using the Young's inequality with exponents $\frac{1}{\alpha p'}$ and $\frac{1}{1-\alpha p'}$, on the second term on the right side, we get:

$$\begin{aligned}&|A_k|^{\left(1-\frac{1}{\sigma}-\frac{1}{r}\right)\frac{\sigma}{p-1}-\frac{\alpha p}{p-1}\cdot\frac{\sigma}{\sigma^*}}\left[\int_{A_k}|\nabla u|^{\sigma}\,dx\right]^{\frac{\alpha p}{p-1}}\\ &\qquad\leq \frac{1}{2}\int_{A_k}|\nabla u|^{\sigma}\,dx + c|A_k|^{\left(p-1-\frac{\sigma}{r}-\alpha p\frac{\sigma}{\sigma^*}\right)\frac{1}{(p-1)(1-\alpha p')}}\end{aligned}$$

so that we have:

$$\int_{A_k} |\nabla u|^\sigma \, dx \leq c\Big\{k^{\frac{\alpha\sigma p}{p-1}}|A_k|^{1-\frac{\sigma}{r(p-1)}} + |A_k|^{\left(p-1-\frac{\sigma}{r}-\alpha p\frac{\sigma}{\sigma^*}\right)\frac{1}{(p-1)(1-\alpha p')}}\Big\}. \tag{3.5}$$

As can be seen by means of straightforward calculations, the assumptions on r and α, imply that:

$$1-\frac{\sigma}{r(p-1)} < \left(p-1-\frac{\sigma}{r}-\alpha p\frac{\sigma}{\sigma^*}\right)\frac{1}{(p-1)(1-\alpha p')}.$$

Moreover, since u belongs to $W_0^{1,q}(\Omega)$, we have that $|A_k|$ tends to zero as k tends to infinity, thus there exists k_0 such that if $k \geq k_0$, we have:

$$|A_k|^{\left(p-1-\frac{\sigma}{r}-\alpha p\frac{\sigma}{\sigma^*}\right)\frac{1}{(p-1)(1-\alpha p')}} < |A_k|^{1-\frac{\sigma}{r(p-1)}}$$

and so (3.5) implies that:

$$\int_{A_k} |\nabla u|^\sigma \, dx \leq ck^{\frac{\alpha\sigma p}{p-1}}|A_k|^{1-\frac{\sigma}{r(p-1)}} \quad \forall\, k \geq k_0.$$

It is easy to see that $1-\frac{\sigma}{r(p-1)}-\frac{\sigma}{\sigma^*} > 0$ since $r > \frac{N}{p-1}$ and $\frac{\alpha p}{p-1}$ belongs to $(0,1)$ since $0 < \alpha < \frac{1}{p'}$.

Thus, by Lemma 3.1 u belongs to $L^\infty(\Omega)$. On the other hand,

$$\int_\Omega \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\, dx \leq \int_\Omega F.\nabla u \, dx < +\infty.$$

The $L^\infty(\Omega)$ estimate implies that:

$$\frac{1}{(1+\|u\|_{L^\infty(\Omega)})^{\alpha p}}\int_\Omega |\nabla u|^p \, dx \leq \int_\Omega \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\, dx \leq c$$

and so u belongs to $W_0^{1,p}(\Omega)$.

Theorem 1.3 is proved.

Remark 3.1 Observe that the condition $\frac{\alpha\sigma p}{p-\sigma} < \sigma^*$ is equivalent to $\sigma < q$.

4 Summability of Unbounded Minima

This section will be devoted to the proof of Theorems 1.3 and 1.4. We begin with technical results, which will be used later.

4.1 Preliminary lemmas

Lemma 4.1. *Let u be the minima of I in $W_0^{1,q}(\Omega)$, then for all $k \in \mathbb{N}$, we have:*

$$\int_{B_k} \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\, dx \leq \frac{c}{1+k}\int_{A_k} F.\nabla u \, dx + \int_{B_k} F.\nabla u \, dx, \tag{4.1}$$

where A_k and B_k are as in (1.8).

Proof.

– If $k = 0$, the result is trivial since u is minimum of I.

– Let $k > 0$, we take $v = u - T_1(u - T_k(u))$ as test function in (3.1), we obtain:

$$\int_\Omega \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\,dx - \int_\Omega F.\nabla u\,dx \le \int_\Omega \frac{|\nabla v|^p}{(1+|v|)^{\alpha p}}\,dx - \int_\Omega F.\nabla v\,dx$$

which implies that:

$$\int_\Omega \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\,dx - \int_\Omega \frac{|\nabla v|^p}{(1+|v|)^{\alpha p}}\,dx \le \int_{B_k} F.\nabla u\,dx$$

and by definition of v, we deduce that:

$$\int_{B_k} \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\,dx + \int_{A_{k+1}} \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\,dx - \int_{A_{k+1}} \frac{|\nabla u|^p}{(1+|v|)^{\alpha p}}\,dx \le \int_{B_k} F.\nabla u\,dx$$

and then

$$\begin{aligned}\int_{B_k} \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\,dx &\le \int_{A_{k+1}} |\nabla u|^p \left\{\frac{1}{(1+|v|)^{\alpha p}} - \frac{1}{(1+|u|)^{\alpha p}}\right\} dx \\ &\quad + \int_{B_k} F.\nabla u\,dx \\ &\le \int_{A_{k+1}} |\nabla u|^p \frac{(1+|u|)^{\alpha p} - (1+|v|)^{\alpha p}}{(1+|v|)^{\alpha p}(1+|u|)^{\alpha p}}\,dx \\ &\quad + \int_{B_k} F.\nabla u\,dx.\end{aligned} \tag{4.2}$$

Since $|v| = |u| - 1$ on A_{k+1}, we easily obtain that there exists a positive constant c such that

$$(1+|u|)^{\alpha p} - (1+|v|)^{\alpha p} \le c(1+|v|)^{\alpha p-1}.$$

Thus (4.2) becomes

$$\int_{B_k} \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\,dx \le c\int_{A_{k+1}} \frac{|\nabla u|^p}{(1+|v|)(1+|u|)^{\alpha p}}\,dx + \int_{B_k} F.\nabla u\,dx.$$

Since $|v| \ge k$ on A_{k+1}, we have:

$$\int_{B_k} \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\,dx \le \frac{c}{1+k}\int_{A_{k+1}} \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\,dx + \int_{B_k} F.\nabla u\,dx.$$

Using (3.2) we thus obtain (4.1).

Lemma 4.2 *Let u be the minima of I in $W_0^{1,q}(\Omega)$, then for all $\gamma \ge 1$, we have:*

$$\int_\Omega |\nabla u|^p |u|^{p(\gamma-1)}\,dx \le c_1 + c_2\int_\Omega |F|^{p'} |u|^{p(\alpha p'+\gamma-1)}\,dx, \tag{4.3}$$

where c_1 and c_2 are two positive constants.

Proof. Let $\gamma \geq 1$, we have:

$$\begin{aligned}\int_\Omega |\nabla u|^p |u|^{p(\gamma-1)}\,dx &= \sum_{k=0}^{+\infty}\int_{B_k} |\nabla u|^p |u|^{p(\gamma-1)}\,dx \\ &\leq \sum_{k=0}^{+\infty}\int_{B_k} |\nabla u|^p (1+k)^{p(\gamma-1)}\,dx \\ &\leq c\sum_{k=0}^{+\infty}\int_{B_k} \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}(1+k)^{p(\gamma-1)+\alpha p}\,dx.\end{aligned} \tag{4.4}$$

Thus, by (4.1) we obtain:

$$\begin{aligned}\int_\Omega |\nabla u|^p |u|^{p(\gamma-1)}\,dx \leq\; & c\sum_{k=0}^{+\infty}(1+k)^{p(\gamma-1)+\alpha p-1}\int_{A_k} |F|.|\nabla u|\,dx \\ & +c\sum_{k=0}^{+\infty}(1+k)^{p(\gamma-1)+\alpha p}\int_{B_k} |F|.|\nabla u|\,dx.\end{aligned} \tag{4.5}$$

Observe that, for $k \in I\!N$, we have:

$$\int_{A_k} |F|.|\nabla u|\,dx = \sum_{h=k}^{+\infty}\int_{B_h} |F|.|\nabla u|\,dx. \tag{4.6}$$

Hence,

$$\begin{aligned}&\sum_{k=0}^{+\infty}(1+k)^{p(\gamma-1)+\alpha p-1}\int_{A_k} |F|.|\nabla u|\,dx \\ &\qquad = \sum_{k=0}^{+\infty}(1+k)^{p(\gamma-1)+\alpha p-1}\sum_{h=k}^{+\infty}\int_{B_h} |F|.|\nabla u|\,dx.\end{aligned} \tag{4.7}$$

Therefore, changing the order of summation, and recalling that:

$$\sum_{k=0}^{h} k^l \leq c(1+h)^{l+1} \tag{4.8}$$

with $c = c(l)$, we have:

$$\begin{aligned}&\sum_{k=0}^{+\infty}(1+k)^{p(\gamma-1)+\alpha p-1}\int_{A_k} |F|.|\nabla u|\,dx \\ &\qquad = \sum_{h=0}^{+\infty}\sum_{k=0}^{h}(1+k)^{p(\gamma-1)+\alpha p-1}\int_{B_h} |F|.|\nabla u|\,dx \\ &\qquad = \sum_{h=0}^{+\infty}(1+h)^{p(\gamma-1)+\alpha p}\int_{B_h} |F|.|\nabla u|\,dx.\end{aligned} \tag{4.9}$$

We obtain from (4.5) that:

$$\begin{aligned}\int_\Omega |\nabla u|^p |u|^{p(\gamma-1)}\,dx &\leq c\sum_{k=0}^{+\infty}(1+k)^{p(\gamma-1)+\alpha p}\int_{B_k}|F|.|\nabla u|\,dx\\ &\leq c\sum_{k=0}^{+\infty}\int_{B_k}|F|.|\nabla u|(1+|u|)^{p(\gamma-1)+\alpha p}\,dx\\ &\leq c\int_\Omega |F|.|\nabla u|\,dx + c\int_\Omega |F|.|\nabla u||u|^{p(\gamma-1)+\alpha p}\,dx.\end{aligned}$$

By Young's inequality and the fact that $\int_\Omega |F|.|\nabla u|\,dx < +\infty$, we deduce (4.3).

Lemma 4.3 *Let $\lambda > 0$ and let $u \in W_0^{1,q}(\Omega)$ be the minimum of I, then we have:*

$$\int_\Omega \frac{|\nabla u|^p}{(1+|u|)^\lambda}\,dx \leq c\int_\Omega |F|^{p'}(1+|u|)^{\alpha p p'-\lambda}\,dx. \tag{4.10}$$

Proof. Let $\lambda > 0$ and let $u \in W_0^{1,q}(\Omega)$ be the minimum of I, we have:

$$\begin{aligned}\int_\Omega \frac{|\nabla u|^p}{(1+|u|)^\lambda}\,dx &= \int_\Omega \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}(1+|u|)^{\alpha p-\lambda}\,dx\,dx\\ &\leq c\sum_{k=0}^{+\infty}(1+k)^{\alpha p-\lambda}\int_{B_k}\frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\,dx\end{aligned} \tag{4.11}$$

and this implies, by (4.1) that:

$$\begin{aligned}\int_\Omega \frac{|\nabla u|^p}{(1+|u|)^\lambda}\,dx &\leq c\sum_{k=0}^{+\infty}(1+k)^{\alpha p-\lambda-1}\int_{A_k}|F|.|\nabla u|\,dx\\ &\quad +c\sum_{k=0}^{+\infty}(1+k)^{\alpha p-\lambda}\int_{B_k}|F|.|\nabla u|\,dx.\end{aligned} \tag{4.12}$$

Using (4.6) one has

$$\begin{aligned}&\sum_{k=0}^{+\infty}(1+k)^{\alpha p-\lambda-1}\int_{A_k}|F|.|\nabla u|\,dx\\ &\qquad = \sum_{k=0}^{+\infty}(1+k)^{\alpha p-\lambda-1}\sum_{h=k}^{+\infty}\int_{B_h}|F|.|\nabla u|\,dx.\end{aligned}$$

Changing the order of summation and using (4.8), we have:

$$\begin{aligned}&\sum_{k=0}^{+\infty}(1+k)^{\alpha p-\lambda-1}\int_{A_k}|F|.|\nabla u|\,dx\\ &\qquad \leq \sum_{k=0}^{+\infty}(1+k)^{\alpha p-\lambda}\int_{B_k}|F|.|\nabla u|\,dx.\end{aligned} \tag{4.13}$$

Combining (4.12) and (4.13), we get:

$$\begin{aligned}\int_\Omega \frac{|\nabla u|^p}{(1+|u|)^\lambda}\,dx &\le c\sum_{k=0}^{+\infty}(1+k)^{\alpha p-\lambda}\int_{B_k}|F|.|\nabla u|\,dx\\ &\le c\sum_{k=0}^{+\infty}\int_{B_k}|F|.|\nabla u|(1+|u|)^{\alpha p-\lambda}\,dx\\ &\le c\int_\Omega |F|.|\nabla u|(1+|u|)^{\alpha p-\lambda}\,dx.\end{aligned}$$

Now, the Young's inequality implies that:

$$\int_\Omega \frac{|\nabla u|^p}{(1+|u|)^\lambda}\,dx \le c\int_\Omega |F|^{p'}(1+|u|)^{\alpha pp'-\lambda}\,dx.$$

4.2 Proof of Theorem 1.3

We begin with the following technical lemma.

Lemma 4.4 *Let* $\gamma = \dfrac{(1-\alpha p')(r(p-1))^*}{p^*}$, *we have*

i) $s = \gamma p^* = \dfrac{pr(\alpha p'+\gamma-1)}{r-p'}$

ii) $\gamma \ge 1$ *if and only if* $r \ge \dfrac{Np'}{N-\alpha p'(N-p)}$

iii) $1-\dfrac{p'}{r} < \dfrac{p}{p^*}$ *if and only if* $r < \dfrac{N}{p-1}$.

Theorem 4.1 *Under the hypotheses of Theorem* 1.3, *we have the following estimations*

i) $\displaystyle\int_\Omega |u|^s\,dx \le c_3$,

ii) $\displaystyle\int_\Omega |\nabla u|^p\,dx \le c_4$,

where c_3 *and* c_4 *are two positive constants.*

Proof.

i) We have, by Lemmas 4.2, 4.4 and Sobolev embedding

$$\begin{aligned}\left(\int_\Omega |u|^s\,dx\right)^{\frac{p}{p^*}} &= \left(\int_\Omega |u|^{\gamma p^*}\,dx\right)^{\frac{p}{p^*}} \le \int_\Omega |\nabla u|^p|u|^{p(\gamma-1)}\,dx\\ &\le c+\int_\Omega |F|^{p'}|u|^{p(\alpha p'+\gamma-1)}\,dx.\end{aligned} \tag{4.14}$$

Applying the Holder inequality, we obtain:

$$\left(\int_\Omega |u|^s\,dx\right)^{\frac{p}{p^*}} \le \left(\int_\Omega |F|^r\,dx\right)^{\frac{p'}{r}}\left(\int_\Omega |u|^{\frac{pr(\alpha p'+\gamma-1)}{r-p'}}\,dx\right)^{1-\frac{p'}{r}}.$$

Then by $i), iii)$ of Lemma 4.4 and Young's inequality, we deduce

$$\int_\Omega |u|^s \, dx \le c_3.$$

ii) We have:

$$\begin{aligned}\int_\Omega |\nabla u|^p \, dx &= \int_{\{|u|\le 1\}} |\nabla u|^p \, dx + \int_{\{|u|\ge 1\}} |\nabla u|^p \, dx \\ &\le c\int_\Omega \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}} \, dx + \int_\Omega |\nabla u|^p |u|^{p(\gamma-1)} \, dx \\ &\le c\int_\Omega |F||\nabla u| \, dx + \int_\Omega |\nabla u|^p |u|^{p(\gamma-1)} \, dx\end{aligned}$$

and from (4.14), we get:

$$\int_\Omega |\nabla u|^p |u|^{p(\gamma-1)} \, dx \le c_4,$$

which implies that:

$$\int_\Omega |\nabla u|^p \, dx \le c_5.$$

4.3 Proof of Theorem 1.4

We begin with the following technical lemma.

Lemma 4.5 *Let* $\lambda = \dfrac{pN - r(p-1)(N - \alpha p'(N-p))}{N - r(p-1)}$*, we have the following properties :*

i) $s = \dfrac{\lambda\rho}{p-\rho} = \dfrac{r(\alpha p p' - \lambda)}{r - p'}$,

ii) $\lambda > 0$ *if and only if* $r < \dfrac{Np'}{N - \alpha p'(N-p)}$,

iii) $(1 - \dfrac{p'}{r})\dfrac{\rho}{p} + 1 - \dfrac{\rho}{p} < \dfrac{\rho}{s}$.

Theorem 4.2 *Under the hypotheses of Theorem* 1.4*, we have the following estimations:*

i) $\displaystyle\int_\Omega |u|^s \, dx \le c_6$,

ii) $\displaystyle\int_\Omega |\nabla u|^\rho \, dx \le c_7$,

where c_6 *and* c_7 *are two positive constants.*

Proof. Since $\rho^* = s$, we have by Sobolev embedding:

$$\begin{aligned} \left(\int_\Omega |u|^s \, dx\right)^{\frac{\rho}{s}} &= \left(\int_\Omega |u|^{\rho^*} \, dx\right)^{\frac{\rho}{\rho^*}} \le c \int_\Omega |\nabla u|^\rho \, dx \\ &= c \int_\Omega \frac{|\nabla u|^\rho}{(1+|u|)^{\frac{\lambda\rho}{p}}} (1+|u|)^{\frac{\lambda\rho}{p}} \, dx. \end{aligned} \tag{4.15}$$

Applying Hölder inequality, we have:

$$\left(\int_\Omega |u|^s \, dx\right)^{\frac{\rho}{s}} \le c \left[\int_\Omega \frac{|\nabla u|^p}{(1+|u|)^\lambda} \, dx\right]^{\frac{\rho}{p}} \left[\int_\Omega (1+|u|)^{\frac{\lambda\rho}{p-\rho}} \, dx\right]^{1-\frac{\rho}{p}}. \tag{4.16}$$

On the other hand by Lemma 4.2 and Hölder inequality, we deduce that:

$$\int_\Omega \frac{|\nabla u|^p}{(1+|u|)^\lambda} \, dx \le \left[\int_\Omega (1+|u|)^{\frac{r(\alpha p p' - \lambda)}{r-p'}} \, dx\right]^{1-\frac{p'}{r}}. \tag{4.17}$$

From (4.15), (4.16) and (4.17), we get:

$$\begin{aligned} \left(\int_\Omega |u|^s \, dx\right)^{\frac{\rho}{s}} &\le c \int_\Omega |\nabla u|^\rho \, dx \\ &\le \left[\int_\Omega (1+|u|)^{\frac{r(\alpha p p' - \lambda)}{r-p'}} \, dx\right]^{(1-\frac{p'}{r})\frac{\rho}{p}} \\ &\quad \times \left[\int_\Omega (1+|u|)^{\frac{\lambda\rho}{p-\rho}} \, dx\right]^{1-\frac{\rho}{p}} \end{aligned} \tag{4.18}$$

which implies, by using Lemma 4.5

$$\left(\int_\Omega |u|^s \, dx\right)^{\frac{\rho}{s}} \le \left[\int_\Omega (1+|u|)^s \, dx\right]^{(1-\frac{p'}{r})\frac{\rho}{p}+1-\frac{\rho}{p}}.$$

Finally, by the lemma 4.5 and Hölder inequality, we deduce that:

$$\int_\Omega |u|^s \, dx \le c_6.$$

Therefore by (4.18), we also have :

$$\int_\Omega |\nabla u|^\rho \, dx \le c_7.$$

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