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Extremal Mild Solutions for Nonlocal Semilinear Differential Equations with Finite Delay in an Ordered Banach Space

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Abstract: This paper is concerned with the existence and uniqueness of extremal mild solutions for nonlocal semilinear differential equations with finite delay in an ordered Banach space with the help of the monotone iterative technique based on lower and upper solutions. We use the theory of semigroup and measures of noncompactness to obtain the main results. The existence results are proved by assuming compact or non compact semigroup. An example is provided to illustrate the applicability of the main results.

Keywords: *initial value problem; finite delay; semigroup theory; monotone iterative technique; lower and upper solutions; Kuratowskii measure of noncompactness.*

Mathematics Subject Classification (2010): 34G20, 34K30.

1 Introduction

In this paper, we consider the following nonlocal semilinear differential equations with finite delay in an ordered Banach space:

$$\begin{cases} \frac{d}{dt}x(t) = & Ax(t) + f(t, x_t, Bx(t)), \quad t \in J = [0, b], \\ \quad x(t) = & \phi(t) + g(x)(t), \quad t \in [-a, 0], \end{cases} \tag{1}$$

where the state $x(\cdot)$ takes values in the Banach space X endowed with norm $\|\cdot\|$; $A\colon D(A) \subset X \to X$ is a closed linear densely defined operator and an infinitesimal generator of strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ of bounded linear operator in X;

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the nonlinear function $f\colon [0,b] \times \mathcal{D} \times X \to X$ is continuous, here $\mathcal{D} = C([-a,0],X)$; the term $Bx(t)$ is given by $Bx(t) = \int_0^t K(t,s)x(s)ds$, here $K \in C(\Sigma, \mathbb{R}^+)$ is the set of all positive functions which are continuous on $\Sigma = \{(t,s)|0 \le s \le t \le T\}$; $\phi(\cdot) \in \mathcal{D}$ and $g\colon C([-a,b],X) \to \mathcal{D}$ is a continuous operator. If $x\colon [-a,b] \to X$ is a continuous function, then x_t denotes the function in $\mathcal{D}$ defined as $x_t(\nu) = x(t+\nu)$ for $\nu \in [-a,0]$, here $x_t(\cdot)$ represents the time history of the state from the time $t-a$ up to the present time t.

It is well known that time delays are frequently encountered in various industrial and practical systems, such as chemical processing, bio engineering, fuzzy systems, automatic control, neural networks, circuits, vehicle suspension systems and so on. Hence, in recent years, the researchers have paid more attention to delay differential equations (see [1–7]). Some authors have studied differential equations with nonlocal initial conditions, see for instance, [7–13]. Nonlocal initial condition, in many cases, is more suitable and produces better results in applications of physical problems than the classical initial value of the type $x(0) = x_0$.

The monotone iterative technique based on lower and upper solutions provides an effective way to investigate the existence of solutions for the nonlinear differential equations (fractional or non-fractional ordered), see for instance, [6, 14–18]. It constructs monotone sequences of lower and upper solutions that converge uniformly to the extremal mild solutions between the lower and upper solutions.

This paper is motivated by recent works [6, 7, 16]. We extend a monotone iterative technique for nonlocal semilinear differential equations with finite delay (1) to study the existence and uniqueness of extremal mild solutions in an ordered Banach space. We use the semigroup theory and measures of noncompactness to obtain the results. The existence results are discussed by assuming compact or non compact semigroup. To the best of our knowledge, up to now, no work has been reported on nonlocal semilinear differential equations with finite delay by using the monotone iterative technique.

The rest of the paper is organized as follows: In the next section, we introduce some basic definitions, notations and preliminary results. In Section 3, we prove the existence and uniqueness of extremal mild solutions of the delay system (1) by using monotone iterative technique. Finally, in Section 4, we present an example to show the application of the main result.

2 Preliminaries

Throughout this paper, we assume that X is a Banach space with the norm $\|\cdot\|$ and $P = \{y \in X : y \ge \theta\}$ (θ is a zero element of X) is a positive cone in X which defines a partial ordering in X by $x \le y$ if and only if $y - x \in P$. If $x \le y$ and $x \ne y$, we write $x < y$. The cone P is said to be normal if there exists a positive constant N such that $\theta \le x \le y$ implies $\|x\| \le N\|y\|$. We also assume that $A\colon D(A) \subset X \to X$ is a closed linear densely defined operator that generates a strongly continuous semigroup $\{T(t), t \ge 0\}$. By Pazy [19], there exists a constant $M \ge 1$ such that $\sup_{t\in J} \|T(t)\| \le M$. For the sake of convenience, we write $B^* = \sup_{t\in J} \int_0^t K(t,s)ds$.

$C([-a,b],X)$ is the Banach space of all continuous X-valued functions on interval $[-a,b]$ with norm $\|\cdot\|_C = \sup_{t\in[-a,b]} \|x(t)\|$. Then $C([-a,b],X)$ is an ordered Banach space whose partial ordering $\le$ is induced by positive cone $P_C = \{x \in C([-a,b],X) \mid x(t) \ge \theta,\ t \in [-a,b]\}$. Similarly D is also an ordered Banach space with norm $\|\cdot\|_D = \sup_{t\in[-a,0]} \|x(t)\|$ and partial ordering $\le$ induced by $P_D = \{x \in$

$C([-a,0],X) \mid x(t) \geq \theta, t \in [-a,0]\}$. If the cone P is normal with a normal constant N, then P_C and P_D are also normal cones with the same normal constant N. For $x, y \in C([-a,b],X)$ with $x \leq y$, denote the ordered interval $[x,y] = \{z \in C([-a,b],X),\ x \leq z \leq y\}$ in $C([-a,b],X)$, and $[x(t),y(t)] = \{u \in X \colon x(t) \leq u \leq y(t)\}$ $(t \in [-a,b])$ in X.

Let us recall some basic definitions and lemmas which are used to prove our main results.

Definition 2.1 A C_0-semigroup $\{T(t)\}_{t\geq 0}$ is called a positive semigroup, if $T(t)x \geq \theta$ for all $x \geq \theta$ and $t \geq 0$.

Lemma 2.1 (see [19]) *If $h \in C^1(J,X)$, then for every $x_0 \in D(A)$ the following initial value problem*

$$\begin{cases} \frac{d}{dt}x(t) = Ax(t) + h(t), \quad t \in J, \\ x(0) = x_0, \end{cases} \tag{2}$$

has a unique solution x on J given by

$$x(t) = T(t)x_0 + \int_0^t T(t-s)h(s))\,ds, \quad t \in J.$$

Definition 2.2 (see [19]) A continuous function $x \colon [-a,b] \to X$ is said to be a mild solution of the system (1) if $x(t) = \phi(t) + g(x)(t)$ on $[-a,0]$ and the following integral equation is satisfied:

$$x(t) = T(t)(\phi(0) + g(x)(0)) + \int_0^t T(t-s)f(s,x_s,Bx(s))\,ds, \quad t \in J.$$

Lemma 2.2 (see [19]) *If $h \in L^1((0,b),X)$, then for every $x_0 \in X$ the initial value problem* (2) *has a unique mild solution.*

Let $C_1([-a,b],X) = \{u \in C([-a,b],X) \colon u'$ exists on J, $u'|_J \in C(J,X)$ and $u(t) \in D(A)$ for $t \geq 0\}$. An abstract function $u \in C_1([-a,b],X)$ is called a solution of (1) if $u(t)$ satisfies the equation (1).

Definition 2.3 (see [16]) The function $x \in C_1([-a,b],X)$ is called a lower solution of the system (1) if it satisfies the following inequalities

$$\begin{cases} \frac{d}{dt}x(t) \leq Ax(t) + f(t,x_t,Bx(t)), \quad t \in J, \\ x(\nu) \leq \phi(\nu) + g(x)(\nu), \quad \nu \in [-a,0]. \end{cases} \tag{3}$$

If all inequalities of (3) are reversed, we call x an upper solution of the system (1).

Now we recall the definition of Kuratowski's measure of noncompactness and its properties.

Definition 2.4 (see [20, 21]) Let X be a Banach space and $\mathcal{B}(X)$ be a family of bounded subset of X. Then $\mu : \mathcal{B}(X) \to \mathbb{R}^+$, defined by

$$\mu(S) = \inf\{\delta > 0 \colon S \text{ admits a finite cover by sets of diameter } \leq \delta\},$$

where $S \in \mathcal{B}(X)$, is called the Kuratowski measure of noncompactness. Clearly $0 \leq \mu(S) < \infty$.

Lemma 2.3 (see [20, 21]) *Let S, S_1 and S_2 be bounded sets of a Banach space X. Then*

(i) $\mu(S) = 0$ *if and only if S is a relatively compact set in X.*

(ii) $\mu(S_1) \leq \mu(S_2)$ *if* $S_1 \subset S_2$.

(iii) $\mu(S_1 + S_2) \leq \mu(S_1) + \mu(S_2)$.

(iv) $\mu(\lambda S) \leq |\lambda| \mu(S)$ *for any* $\lambda \in \mathbb{R}$.

Lemma 2.4 (see [20, 21]) *If $S \subset C([c, d], X)$ is bounded and equicontinuous on $[c, d]$, then $\mu(S(t))$ is continuous for $t \in [c, d]$ and*

$$\mu(S) = \sup\{\mu(S(t)),\ t \in [c, d]\}, \quad \textit{where } S(t) = \{x(t) \colon x \in S\} \subseteq X.$$

Remark 2.1 (see [20,21]) If S is a bounded set in $C([c, d], X)$, then $S(t)$ is bounded in X, and $\mu(S(t)) \leq \mu(S)$.

Lemma 2.5 (see [20, 21]) *Let $S = \{u_n\} \subset C([c, d], X)(n = 1, 2, \dots)$ be a bounded and countable set. Then $\mu(S(t))$ is Lebesgue integrable on $[c, d]$, and*

$$\mu\left(\left\{\int_c^d u_n(t)\, dt \mid n = 1, 2, \dots\right\}\right) \leq 2 \int_c^d \mu(S(t))\, dt. \tag{4}$$

3 Main Result

In this section, we prove the existence and the uniqueness of extremal mild solutions of the system (1).

Theorem 3.1 *Let X be an ordered Banach space, whose positive cone P is normal with a normal constant N. Also assume that A is the infinitesimal generator of a positive and compact C_0-semigroup $\{T(t)\}_{t \geq 0}$ on X. If the system (1) has a lower solution $x^{(0)} \in C([-a, b], X)$ and an upper solution $y^{(0)} \in C([-a, b], X)$ with $x^{(0)} \leq y^{(0)}$ and satisfies the following assumptions:*

(H1) The function $f \colon J \times \mathcal{D} \times X \to X$ satisfies that $f(t, \cdot, \cdot) \colon \mathcal{D} \times X \to X$ is continuous for $t \in J$, and $f(\cdot, \varphi, x)$ is strongly measurable for all $(\varphi, x) \in \mathcal{D} \times X$.

(H2) For any $t \in J$, the function $f(t, \cdot, \cdot) \colon \mathcal{D} \times X \to X$ satisfies the following

$$f(t, \varphi_1, u_1) \leq f(t, \varphi_2, u_2),$$

where $u_1, u_2 \in X$ with $Bx^{(0)}(t) \leq u_1 \leq u_2 \leq By^{(0)}(t)$ and $\varphi_1, \varphi_2 \in \mathcal{D}$ with $x_t^{(0)} \leq \varphi_1 \leq \varphi_2 \leq y_t^{(0)}$.

(H3) The function $g \colon C([-a, b], X) \to \mathcal{D}$ is increasing, continuous and compact.

Then the delay system (1) has minimal and maximal mild solutions between $x^{(0)}$ and $y^{(0)}$.

Proof. Let $B = [x^{(0)}, y^{(0)}] = \{x \in C([-a,b],X) \mid x^{(0)} \leq x \leq y^{(0)}\}$. Define $Q\colon B \to C([-a,b],X)$ by

$$Qx(t) = \begin{cases} T(t)(\phi(0) + g(x)(0)) + \int_0^t T(t-s)f(s,x_s,Bx(s))\,ds, & t \in [0,b], \\ \phi(t) + g(x)(t), & t \in [-a,0]. \end{cases} \tag{5}$$

For any $x \in B$ and in view of (H2), we have

$$\begin{aligned} f(t,x_t^{(0)},Bx^{(0)}(t)) &\leq f(t,x_t,Bx(t)) \\ &\leq f(t,y_t^{(0)},By^{(0)}(t)). \end{aligned}$$

By the normality of the positive cone P, there exists a constant $k > 0$ such that

$$\|f(t,x_t,Bx(t))\| \leq k, \quad x \in B. \tag{6}$$

Firstly we prove that Q is a continuous and monotonically increasing operator from B to B. Let $x, y \in B$ with $x \leq y$, then $x(t) \leq y(t),\ t \in [-a,b]$. Therefore $x_t \leq y_t$ in $\mathcal{D}$ for all $t \in [0,b]$. By the positivity of the semigroup $T(t)$ and the assumptions (H2) and (H3), we get

$$Qx \leq Qy. \tag{7}$$

Let $\frac{d}{dt}x^{(0)}(t) = Ax^{(0)}(t) + h(t),\ t \in J$. In view of Lemma 2.2 and Definition 2.3, we get

$$\begin{aligned} x^{(0)}(t) =& T(t)x^{(0)}(0) + \int_0^t T(t-s)h(s)ds \\ \leq& T(t)(\phi(0) + g(x^{(0)})(0)) + \int_0^t T(t-s)f(s,x_s^{(0)},Bx^{(0)}(s))ds \\ =& Qx^{(0)}(t), \quad t \in J. \end{aligned}$$

Also $x^{(0)}(t) \leq \phi(t) + g(x^{(0)})(t) = Qx^{(0)}(t),\ t \in [-a,0]$. Thus $x^{(0)}(t) \leq Qx^{(0)}(t),\ t \in [-a,b]$. Similarly we can show that $Qy^{(0)}(t) \leq y^{(0)}(t),\ t \in [-a,b]$. Now let $\{x^{(n)}\} \subset B$ with $x^{(n)} \to x \in B$ as $n \to \infty$. By (6), (H1) and (H3) for any $t \in J$, we have

(i) $f(t,x_t^{(n)},Bx^{(n)}(t)) \to f(t,x_t,Bx(t))$.

(ii) $g(x^{(n)}) \to g(x)$.

(iii) $\|f(t,x_t^{(n)},Bx^{(n)}(t)) - f(t,x_t,Bx(t))\| \leq 2k$.

These, together with Lebesgue's dominated convergence theorem, imply that

$$\begin{aligned} \|Qx^{(n)}(t) - Qx(t)\| \leq& M\|g(x^{(n)})(0) - g(x)(0)\| + M\int_0^t \|f(s,x_s^{(n)},Bx(s)) \\ &- f(s,x_s,Bx(s))\|\,ds \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

In view of (H3), for any $t \in [-a,0]$, we have $\|Qx^{(n)}(t) - Qx(t)\| = \|g(x^{(n)})(t) - g(x)(t)\| \to 0$ as $n \to 0$. Therefore $Q : B \to B$ is a monotonically increasing and continuous operator.

Next we show that $Q(B)$ is equicontinuous on $[-a, b]$. Since semigroup $T(t)$ is compact for $t > 0$, $T(t)$ is continuous in uniform operator topology for $t > 0$. For any $x \in B$ and $t_1, t_2 \in J$ with $t_1 < t_2$, we have that

$$\begin{aligned}
\|Qx(t_2) - Qx(t_1)\| \leq & \|T(t_2)(\phi(0) + g(x)(0)) - T(t_1)(\phi(0) + g(x)(0))\| \\
& + \Big\| \int_0^{t_1} [T(t_2 - s) - T(t_1 - s)]\, f(s, x_s, Bx(s))\, ds \Big\| \\
& + \Big\| \int_{t_1}^{t_2} T(t_2 - s) f(s, x_s, Bx(s))\, ds \Big\| \\
\leq & \|T(t_2)(\phi(0) + g(x)(0)) - T(t_1)(\phi(0) + g(x)(0))\| \\
& + k \int_0^{t_1 - \epsilon} \|T(t_2 - s) - T(t_1 - s)\|\, ds \\
& + k \int_{t_1 - \epsilon}^{t_1} \|T(t_2 - s) - T(t_1 - s)\|\, ds + Mk(t_2 - t_1) \\
\leq & \|T(t_2)(\phi(0) + g(x)(0)) - T(t_1)(\phi(0) + g(x)(0))\| \\
& + k(t_1 - \epsilon) \sup_{s \in [0, t_1 - \epsilon]} \|T(t_2 - s) - T(t_1 - s)\| \\
& + 2Mk\epsilon + Mk(t_2 - t_1),
\end{aligned}$$

where $\epsilon \in (0, t_1)$ is arbitrary. Therefore $\|Qx(t_2) - Qx(t_1)\| \to 0$ as $t_1 \to t_2$ and $\epsilon \to 0$ independently of $x \in B$. Thus $Q(B)$ is equicontinuous on J. Since $g : C([-a, b], X) \to \mathcal{D}$ is continuously compact operator and $\phi \in \mathcal{D}$, $Q(B)$ is equicontinuous on $[-a, 0]$. Hence $Q(B)$ is equicontinuous on $[-a, b]$.

Further we show that for each $t \in [-a, b]$, the set $G(t) = \{Qx(t) : \ x \in B\}$ is relatively compact in X. Let $t \in (0, b]$ be a fixed real number and κ be a given real number satisfying $0 < \kappa < t$. For $x \in B$, we define

$$\begin{aligned}
Q^\kappa x(t) = & T(t)\big(\phi(0) + g(x)(0)\big) + \int_0^{t-\kappa} T((t - s) f(s, x_s, Bx(s))\, ds \\
= & T(\kappa) \left[T(t - \kappa)\big(\phi(0) + g(x)(0)\big) + \int_0^{t-\kappa} T(t - \kappa - s) f(s, x_s, Bx(s))\, ds \right].
\end{aligned}$$

By (6), (H3) and the compactness of $T(\kappa)$, the set $\{Q^\kappa x(t) : \ x \in B\}$ is relatively compact in X for each $t \in (0, b]$. Also

$$\begin{aligned}
\|Qx(t) - Q^\kappa x(t)\| \leq & \Big\| \int_{t-\kappa}^{t} T(t - s) f(s, x_s, Bx(s))\, ds \Big\| \\
\leq & Mk\kappa \to 0 \text{ as } \kappa \to 0^+.
\end{aligned}$$

Thus there are relatively compact sets $\{(Q^\kappa x)(t) \colon x \in B\}$ arbitrary close to the set $G(t)$ for each $t \in (0, b]$. Also $G(t),\ t \in [-a, 0]$, is relatively compact in X as $g \colon C([-a, b], X) \to \mathcal{D}$ is a continuously compact operator and $\phi(\cdot) \in \mathcal{D}$. Hence the set $G(t)$ is relatively compact in X for all $t \in [-a, b]$.

In view of Ascoli-Arzela theorem, we conclude that $Q(B)$ is relatively compact. Now we define the sequences as

$$x^{(n)} = Qx^{(n-1)} \text{ and } y^{(n)} = Qy^{(n-1)}, \quad n = 1, 2, \ldots, \tag{8}$$

and from (7), we have

$$x^{(0)} \leq x^{(1)} \leq \ldots x^{(n)} \leq \ldots \leq y^{(n)} \leq \ldots \leq y^{(1)} \leq y^{(0)}. \tag{9}$$

Since $Q(B)$ is relatively compact, the sequence $\{x^{(n)}\}$ has a convergent subsequence $\{x^{(n_j)}\}$. Let x^* be its limit. Then for each $\varepsilon > 0$ there exists an n_j (depending upon ε) such that

$$\|x^{(n_j)} - x^*\|_C < \frac{\varepsilon}{1+N}.$$

To show that the sequence $\{x^{(n)}\}$ converges to x^*, take any $n \geq n_j$ and in view of (9), we have

$$x^{(n_j)} \leq x^{(n)} \leq x^*,$$

that is

$$0 \leq x^{(n)} - x^{(n_j)} \leq x^* - x^{(n_j)}.$$

By normality of cone P of X, we have

$$\|x^{(n)} - x^{(n_j)}\|_C \leq N\|x^* - x^{(n_j)}\|_C.$$

This implies

$$\begin{aligned}\|x^{(n)} - x^*\|_C \leq & \|x^{(n)} - x^{(n_j)}\|_C + N\|x^{(n_j)} - x^*\|_C \\ \leq & (N+1)\|x^{(n_j)} - x^*\|_C \\ \leq & \varepsilon.\end{aligned}$$

Hence the sequence $\{x^{(n)}\}$ converges to x^*. By (5) and (8), we have that

$$x^{(n)}(t) = \begin{cases} T(t)(\phi(0) + g(x^{(n-1)})(0)) \\ \quad + \int_0^t T(t-s)f(s, x_s^{(n-1)}, Bx^{(n-1)}(s))\, ds, & t \in [0,b], \\ \phi(t) + g(x^{(n-1)})(t), \quad t \in [-a,0]. \end{cases}$$

In view of Lebesgue's dominated convergence theorem and taking $n \to \infty$, we get

$$x^*(t) = \begin{cases} T(t)(\phi(0) + g(x^*)(0)) + \int_0^t T(t-s)f(s, x_s^*, Bx^*(s))\, ds, & t \in [0,b], \\ \phi(t) + g(x^*)(t), \quad t \in [-a,0]. \end{cases}$$

Thus $x^* \in C([-a,b],X)$ and $x^* = Qx^*$. It means that x^* is a mild solution of (1). Similarly we can prove that there exists $y^* \in C([-a,b],X)$ such that $y^{(n)} \to y^*$ as $n \to \infty$ and $y^* = Qy^*$. Let $x \in B$ be any fixed point of Q, then by (7), $x^{(1)} = Qx^{(0)} \leq Qx = x \leq Qy^{(0)} = y^{(1)}$. By induction, $x^{(n)} \leq x \leq y^{(n)}$. Using (9) and taking the limit as $n \to \infty$, we conclude that $x^{(0)} \leq x^* \leq x \leq y^* \leq y^{(0)}$. Hence x^*, y^* are the minimal and maximal mild solutions of the nonlocal semilinear differential equations with finite delay (1) on $[x^{(0)}, y^{(0)}]$ respectively.

In the next theorem, we again discuss the existence of extremal mild solution of (1) with the help of the measure of noncompactness and the monotone iterative procedure. In this result, semigroup $\{T(t)\}_{t\geq 0}$ does not have to be compact.

Theorem 3.2 *Let X be an ordered Banach space whose positive cone P is normal with a normal constant N and A be the infinitesimal generator of a positive C_0-semigroup $\{T(t)\}_{t\geq 0}$ on X. Also suppose that the delay system* (1) *has a lower solution $x^{(0)} \in C([-a,b],X)$ and an upper solution $y^{(0)} \in C([-a,b],X)$ with $x^{(0)} \leq y^{(0)}$ and the assumptions (H1)-(H3) hold. If the following hypotheses are satisfied*

(H4) The operator $T(t)$ is continuous in the sense of uniform operator topology for $t > 0$.

(H5) There exists a constant $L \geq 0$ such that

$$\mu(f(t,E,S)) \leq L\Big[\sup_{-a\leq\nu\leq 0}\mu(E(\nu)) + \mu(S)\Big],$$

for $t \in J$ and $E \subset \mathcal{D}$, $S \subset X$, where $E(\nu) = \{\varphi(\nu)\colon \varphi \in E\}$,

and $2MLb(1+2B^*) < 1$, then the delay system (1) *has minimal and maximal mild solutions between $x^{(0)}$ and $y^{(0)}$.*

Proof. Let $B = [x^{(0)}, y^{(0)}] = \{x \in C([-a,b],X) \mid x^{(0)} \leq x \leq y^{(0)}\}$. We define a map $Q : B \to C([-a,b],X)$ as defined in Theorem 3.1. Proceeding as in the proof of Theorem 3.1 and in view of (H4), we get that the operator $Q : B \to B$ is monotonically increasing and continuous, and $Q(B)$ is equicontinuous on $[-a,b]$. Also we define the sequences $x^{(n)}$ and $y^{(n)}$ as defined by (8) in Theorem 3.1. Since $x^{(0)} \leq Qx^{(0)}$, $Qy^{(0)} \leq y^{(0)}$ and the map Q is increasing, the equation (9) holds.

Let $S = \{x^{(n)}\}_{n=1}^{\infty}$. By (9) and the normality of positive cone P_C , the set S is bounded. As g is a continuously compact operator, we get

$$\begin{aligned}\mu(\{S(t)\}) &= \mu(\{\phi(t) + g(x^{(n-1)})(t)\}_{n=1}^{\infty})\\ &\leq \mu(\{\phi(t)\}) + \mu(\{g(x^{(n-1)})(t)\}_{n=1}^{\infty}) = 0 \text{ for } t \in [-a,0].\end{aligned}$$

Since $S(t) = \{x^{(1)}(t)\} \cup \{Q(S)(t)\}$ for any $t \in J$, $\mu(S(t)) = \mu(Q(S)(t))$, $t \in J$. From (H3), (H5), (5) and (8), we get for $t \in J$ that

$$\begin{aligned}\mu(S(t)) =&\mu\Big(\Big\{T(t)[\phi(0) + g(x^{(n)})(0)] + \int_0^t T(t-s)f(s,x_s^{(n)},Bx^{(n)}(s))\,ds\Big\}\Big)\\ \leq&2M\int_0^t \mu\Big(\Big\{f(s,x_s^{(n)},Bx^{(n)}(s))\,ds\Big\}\Big)\\ \leq&2ML\int_0^t \left[\sup_{-a\leq\nu\leq 0}\mu\left(\left\{x^{(n)}(s+\nu)\right\}\right) + \mu\left(\left\{\int_0^s K(s,r)x^{(n)}(r)\,dr\right\}\right)\right]ds\\ \leq&2ML\int_0^t \left[\sup_{0\leq r\leq s}\mu\left(\left\{x^{(n)}(r)\right\}\right) + 2\int_0^s K(s,r)\mu\left(\left\{x^{(n)}(r)\right\}\right)\,dr\right]ds\\ \leq&2ML(1+2B^*)\int_0^t \sup_{0\leq r\leq s}\mu\left(\left\{x^{(n)}(r)\right\}\right)\,ds\\ \leq&2MLb(1+2B^*)\sup_{-a\leq r\leq b}\mu\left(\{S(r)\}\right).\end{aligned}$$

Since $\left\{Qx^{(n)}\right\}_{n=0}^{\infty}$, i.e. $\left\{x^{(n)}\right\}_{n=1}^{\infty}$, is equicontinuous on $[-a,b]$ and by Lemma 2.4, we get

$$\mu(S) \leq 2MLb(1+2B^*)\mu(S).$$

Since $2MLb(1+2B^*)<1$, this implies that $\mu(S)=0$, i.e. $\mu(\{x^{(n)}\}_{n=1}^{\infty})=0$. Therefore the set $\{x^{(n)}: n\geq 1\}$ is relatively compact in B. So the sequence $\{x^{(n)}\}$ has a convergent subsequence in B. By the proof of Theorem 3.1, the sequence $\{x^{(n)}\}$ is itself convergent sequence. So there exists $x^*\in B$ such that $x^{(n)}\to x^*$ as $n\to\infty$. Similarly there exists $y^*\in B$ such that $y^{(n)}\to y^*$ as $n\to\infty$. Again by Theorem 3.1, x^* and y^* become the minimal and maximal mild solutions of the nonlocal semilinear differential equations with finite delay (1) in B respectively.

In the next theorem, we shall prove the uniqueness of the solution of the system (1) by using monotone iterative procedure. For this purpose, we make the following assumptions:

(H6) The function $f\colon J\times\mathcal{D}\times X\to X$ is continuous and there exists a constant $\eta\geq 0$ such that for some $\nu\in[-a,0]$,

$$f(t,\varphi_2,u_2)-f(t,\varphi_1,u_1)\leq\eta[(\varphi_2(\nu)-\varphi_1(\nu))+(u_2-u_1)],$$

for any $t\in J$, $u_1,u_2\in X$ with $Bx^{(0)}(t)\leq u_1\leq u_2\leq By^{(0)}(t)$ and $\varphi_1,\varphi_2\in\mathcal{D}$ with $x_t^{(0)}\leq\varphi_1\leq\varphi_2\leq y_t^{(0)}$.

(H7) For any $t\in[-a,0]$ and $x,y\in B$ with $x\leq y$, there exists a constant $\gamma(0\leq\gamma<\frac{1}{N})$ such that

$$g(y)(t)-g(x)(t)\leq\gamma(y(t)-x(t)).$$

Theorem 3.3 *Let X be an ordered Banach space whose positive cone P is normal with a normal constant N and A be the infinitesimal generator of a positive C_0-semigroup $\{T(t)\}_{t\geq 0}$ on X. Also suppose that the system (1) has a lower solution $x^{(0)}\in C([-a,b],X)$ and an upper solution $y^{(0)}\in C([-a,b],X)$ with $x^{(0)}\leq y^{(0)}$. If the assumptions (H2), (H3), (H4), (H6) and (H7) hold, and $2MLb(1+2B^*)<1$, where $L=N\eta$, then the delay system (1) has a unique mild solution between $x^{(0)}$ and $y^{(0)}$.*

Proof. Let $\{\varphi_n\}\subset\mathcal{D}$ and $\{u_n\}\subset X$ be two monotone increasing sequences. Take any $m,n=1,2,\ldots,$ with $m>n$. By (H2), (H3) and (H6), we get for some $\nu\in[-a,0]$ that

$$\theta\leq f(t,\varphi_m,u_m)-f(t,\varphi_n,u_n)\leq\eta\Big[(\varphi_m(\nu)-\varphi_n(\nu))+(u_m-u_n)\Big].$$

Using the normality of the positive cone P, we get

$$\|f(t,\varphi_m,u_m)-f(t,\varphi_n,u_n)\|\leq N\eta\Big[\|\varphi_m(\nu)-\varphi_n(\nu)\|+\|u_m-u_n\|\Big]. \tag{10}$$

By the definition of measure of noncompactness, we get

$$\begin{aligned}\mu\left(\{f\left(s,\varphi_n\right)\}\right)\leq& L\left[\mu\left(\{\varphi_n(\nu)\}\right)+\mu\left(\{u_n\}\right)\right]\\ \leq& L\left[\sup_{-a\leq\nu\leq 0}\mu\left(\{\varphi_n(\nu)\}\right)+\mu\left(\{u_n\}\right)\right],\end{aligned}$$

where $L=N\eta$. Clearly the assumption (H5) is satisfied. The assumption (H1) is satisfied by the inequality (10). Thus the assumptions (H1)-(H5) hold and $2MLb(1+2B^*)<1$. So by Theorem 3.2, the delay system (1) has minimal and maximal mild solutions between $x^{(0)}$ and $y^{(0)}$.

Let $x^*(t)$ and $y^*(t)$ be the minimal and maximal solutions of the delay system (1) respectively on the ordered interval $B = [x^{(0)}, y^{(0)}]$. By (5) and H(7) for any $t \in [-a, 0]$, we have

$$\begin{aligned} \theta \le y^*(t) - x^*(t) &= Qy^*(t) - Qx^*(t) \\ &= g(y^*)(t) - g(x^*)(t) \\ &\le \gamma(y^*(t) - x^*(t)) \end{aligned}$$

By using the normality of positive cone P, we get $\|y^*(t) - x^*(t)\| \le N\gamma\|y^*(t) - x^*(t)\|$ for all $t \in [-a, 0]$. This implies that $y^*(t) = x^*(t)$ for all $t \in [-a, 0]$ as $N\gamma < 1$. Let $t \in [0, b]$. In view of (5) and (H6), we have

$$\begin{aligned} \theta \le & y^*(t) - x^*(t) = Qy^*(t) - Qx^*(t) \\ = & \int_0^t T(t-s)\left[f(s, y_s^*, By^*(s)) - f(s, x_s^*, Bx^*(s))\right] ds \\ \le & \eta \int_0^t T(t-s)\left[(y_s^*(\nu) - x_s^*(\nu)) + \int_0^s K(s,r)(y^*(r) - x^*(r))\, dr\right] ds \end{aligned}$$

where $\nu \in [-a, 0]$. By applying the normality of the positive cone P, we get

$$\begin{aligned} \|y^*(t) - x^*(t)\| &\le N\eta \Big\| \int_0^t T(t-s)\Big[(y_s^*(\nu) - x_s^*(\nu)) \\ &\quad + \int_0^s K(s,r)(y^*(r) - x^*(r))\, dr\Big]\, ds \Big\| \\ &\le MN\eta \int_0^t \Big[\|y^*(s+\nu) - x^*(s+\nu)\| \\ &\quad + \int_0^s K(s,r)\|y^*(r) - x^*(r)\|\, dr\Big]\, ds \\ &\le MN\eta b(1 + B^*)\|y^* - x^*\|_C. \end{aligned} \tag{11}$$

Since $y^*(t) = x^*(t)$ for $t \in [-a, 0]$ and due to the inequality (11), we get that $\|y^* - x^*\|_C \le MN\eta b(1 + B^*)\|y^* - x^*\|_C$. But $MLb(1 + 2B^*) < \frac{1}{2}$, so $\|y^* - x^*\|_C = 0$, i.e., $y^*(t) = x^*(t),\ t \in [-a, b]$. Hence $y^* = x^*$ is the unique mild solution of the delay system (1) between $x^{(0)}$ and $y^{(0)}$.

4 Example

Consider the following nonlocal semilinear partial differential equations with finite delay of the form:

$$\begin{cases} \frac{\partial z(t,\xi)}{\partial t} = & \frac{\partial^2}{\partial \xi^2} z(t,\xi) + \int_{-a}^0 (a+\nu)^{\frac{-1}{2}}(-\nu)^{\frac{-1}{2}} z(t+\nu, \xi)\, d\nu \\ & \quad + \int_0^t z(s,\xi)\, ds, \quad \xi \in [0,\pi],\ t \in [0,b], \\ z(t,0) = & z(t,\pi) = 0, \quad t \in [0,b], \\ z(\nu,\xi) = & \phi(\nu,\xi) + \int_0^b \rho(s,\nu)\log(1 + |z(s,\xi)|)ds, \quad -a \le \nu \le 0, \end{cases} \tag{12}$$

where $\phi \in \mathcal{D} = C([-a,0] \times [0,\pi] \colon \mathbb{R}^+)$, the operator $\rho(s,\nu) \colon [0,b] \times [-a,0] \to \mathbb{R}^+$ is continuous.

Let $X = L^2([0,\pi],\mathbb{R})$ and $P = \{v \in X \colon v(\xi) \geq 0,\ \xi \in [0,\pi]\}$. Then P is a normal cone in Banach space X. We define an operator $A : X \to X$ by $Av = v''$ with domain

$$D(A) = \{v \in X \colon v, v' \text{ is absolutely continuous } v'' \in X, v(0) = v(\pi) = 0\}.$$

It is well known that A is an infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$ of uniformly bounded linear operators in X. Now we define $z(t)(\xi) = z(t,\xi)$, $z_t(\nu,\xi) = z(t+\nu,\xi)$, $\phi(\nu)(\xi) = \phi(\nu,\xi)$, $Bz(t)(\xi) = \int_0^t z(s,\xi)\, ds$, $f(t,\varphi,u)(\xi) = \int_{-a}^0 (a+\nu)^{\frac{-1}{2}}(-\nu)^{\frac{-1}{2}}\varphi(\nu,\xi)\, d\nu + u(\xi)$ and $g(z)(\nu)(\xi) = g(z(\nu,\xi)) = \int_0^b \rho(s,\nu)\log(1 + |z(s,\xi)|)ds$. Therefore, the above nonlocal semilinear partial differential equations with finite delay (12) can be written as the abstract form (1).

Since $T(t)$ is continuous in the sense of uniform operator topology for $t > 0$, the assumption (H4) is satisfied. We can also easily see that function f satisfies the assumptions (H1) and (H2). For $t \in [0,b]$, $\varphi_1, \varphi_2 \in C([-a,0],X)$ with $0 \leq \varphi_1 \leq \varphi_2$ and $u_1, u_2 \in X$ with $0 \leq u_1 \leq u_2$, then

$$\begin{aligned} 0 \leq & f(t,\varphi_2,u_2)(\xi) - f(t,\varphi_1,u_1)(\xi) \\ \leq & \int_{-a}^0 (a+\nu)^{\frac{-1}{2}}(-\nu)^{\frac{-1}{2}}[\varphi_2(\nu)(\xi) - \varphi_1(\nu)(\xi)]\, d\nu + [u_2(\xi) - u_1(\xi)]. \end{aligned}$$

By normality of cone P, we have

$$\|f(t,\varphi_2,u_2) - f(t,\varphi_1,u_1)\| \leq \int_{-a}^0 (a+\nu)^{\frac{-1}{2}}(-\nu)^{\frac{-1}{2}}\|\varphi_2(\nu) - \varphi_1(\nu)\|\, d\nu + \|u_2 - u_1\|.$$

Hence, for any bounded set $E \subset C([-a,0],X)$ and $S \subset X$, we have

$$\mu(f(t,E,S)) \leq \left[\pi \sup_{-a \leq \nu \leq 0} \mu(E(\nu)) + \mu(S)\right].$$

Thus f satisfies the assumption H(5). Clearly the function $g : PC([0,b],X) \to X$ is increasing, continuous and compact. Thus g satisfies the assumption (H3).

Let $v(t,\xi) = 0$, $(t,\xi) \in [-a,b] \times [0,\pi]$. Then $f(t, v_t, Bv(t)) = 0$ for $t \in [0,b]$ and $v(\nu,\xi) \leq \phi(\nu,\xi) + g(v(\nu,\xi))$ for $\nu \in [-a,0]$. Now we assume that there is a function $w(t,\xi) \geq 0$ such that $w(t,0) = w(t,\pi) = 0$,

$$\frac{\partial w(t,\xi)}{\partial t} \geq \frac{\partial^2}{\partial y^2} w(t,\xi) + f(t, w_t, Bw(t)),$$

and $w(\nu,\xi) \geq \phi(\nu,\xi) + g(w(\nu,\xi))$ for $\nu \in [-a,0]$. Thus v, w become lower and upper solutions of the system (12) respectively and $v \leq w$. If $2Mb(\pi + 2b) < 1$, then all the conditions of Theorem 3.2 are satisfied. Hence, by Theorem 3.2, the system (12) has the minimal and maximal mild solutions lying between the lower solution 0 and the upper solution w.

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