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On Antagonistic Game With a Constant Initial Condition. Marginal Functionals and Probability Distributions

J.H. Dshalalow [1*], W. Huang [1], H.-J. Ke [1], and A. Treerattrakoon [2]

[1] *Department of Mathematics, Florida Institute of Technology*
[2] *Department of Industrial Engineering Kasetsart University, Thailand*

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Abstract: This paper continues dealing with a class of antagonistic games with two players initiated in Dshalalow et al. [1]. Here we validate our claim of analytic tractability in the results obtained in [1] under various transforms.

Keywords: *noncooperative stochastic games; fluctuation theory; marked point processes; Poisson process; ruin time; exit time; first passage time; modified Bessel functions.*

Mathematics Subject Classification (2010): 82B41, 60G51, 60G55, 60G57, 91A10, 91A05, 91A60, 60K05.

1 Introduction

In this paper we continue our studies of a stochastic game of two players of a fully antagonistic nature initiated in [1] by the same authors. The game evolves as a mutual conflict involving two players A and B hitting each other at random and continued until one of the players is "exhausted." In short, the players attack each other in accordance with two independent marked point processes

$$\mathcal{A} := \sum_{j\geq 1} w_j \varepsilon_{s_j}, \text{ and } \mathcal{B} := \sum_{k\geq 1} z_k \varepsilon_{t_k}, s_1, t_1 > 0,$$

representing respective attacks to players A and B. Here ε_a is the Dirac point mass at point $a \in \mathbb{R}$, $\sum_{j\geq 1} \varepsilon_{s_j}$, and $\sum_{k\geq 1} \varepsilon_{t_k}$ are underlying point random measures of the times of attacks, while the marks w_j's and z_k's represent respective damages to players A and

* Corresponding author: mailto:eugene@fit.edu

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B. Players A and B can sustain the attacks until their respective cumulative casualties cross thresholds M and N (positive real numbers). At a time when it takes place (at the first passage time), i.e., when one of the players loses the game, the game should formally stop. However, the game was assumed to be tracked by a third party observer upon random epochs of time $\tau_1, \tau_2, \ldots$ and consequently, the outcome of the game is unknown in real time. The first passage time is then shifted to epoch τ_ρ (called the first observed passage time) that takes place upon one of the observation epochs. Thus, the narrative of the game is delayed allowing the players to continue fighting even after one of the players lost the game thereby letting the game to proceeed in a more realistic scenario.

We further assumed in [1] that $\mathcal{A}$ and $\mathcal{B}$ are marked Poisson random measures and $\tau := \sum_{i\geq 1} \varepsilon_{\tau_i}$, $\tau_0 > 0$ was a renewal process with interrenewal times being exponentially distributed. If X_i and Y_i are increments of the casualties to players A and B on $(\tau_{i-1}, \tau_i]$ observed at time τ_i, then

$$A_k = X_0 + X_1 + \ldots + X_k,\ B_k = Y_0 + Y_1 + \ldots + Y_k$$

form the cumulative damages to players A and B by time τ_k. With the exit indices

$$\mu := inf\{j \geq 0 : A_j = X_0 + X_1 + \ \ldots + X_j > M\}$$

and

$$\nu := inf\{k \geq 0 : B_k = Y_0 + Y_1 + \ \ldots + Y_k > N\},$$

A_μ and B_ν are the respective cumulative damages to players A and B at their respective observed or virtual ruin times. In [1], the functional of interest was

$$\varPhi_{\mu\nu} = \varPhi_{\mu\nu}(\alpha, \beta, \theta) = Ee^{-\alpha A_\mu - \beta B_\mu - \theta\tau_\mu} \mathbf{1}_{\{\mu<\nu\}}$$

giving the joint transform of the first observed passage time τ_μ (the ruin time of player A), along with the status of the respective casualties to players A and B at $\tau_\mu = \tau_\rho$ on the confine σ-algebra $\mathcal{F}(\Omega) \cap \{\mu < \nu\}$. This functional was obtained in terms of the double Laplace-Carson and Laplace-Stieltjes transforms under the claim that it was analytically invertible. We succeeded in doing this. The inverse formulas contained various special functions but seemed to be cumbersome. We go on the further claim that the results are numerically tame.

We ended [1] with obtaining the marginal functional $Ee^{-\alpha A_\mu}\mathbf{1}_{\{\mu<\nu\}}$ in terms of modified Bessel functions and their integrals. The objective of this paper is to continue with other marginal functionals and a subsequent inversion of their Laplace-Stieltjes transforms to arrive at explicit probability distributions and then illustrate the result with computational examples. Note that either the present paper and [1] are abridged and their complete version is available in [2].

2 Further Cases of Marginal Functionals

Our next goal is to get the other marginal transforms. They are to be obtained from $\varPhi_{\mu\nu}(\alpha, \beta, \theta) = Ee^{-\alpha A_\mu - \beta B_\mu - \theta\tau_\mu}\mathbf{1}_{\{\mu<\nu\}}$ in (2.27) and (3.21-3.73) of [1]. In Case 1 [1], we gave $\varPhi_{\mu\nu}(\alpha, 0, 0) = Ee^{-\alpha A_\mu}\mathbf{1}_{\{\mu<\nu\}}$. We continue with the other cases.

Case 2. Setting $\alpha = \theta = 0$ in $\varPhi_{\mu\nu}(\alpha, \beta, \theta)$ leads us to the marginal Laplace-Stieltjes transform of the casualties to player B at the exit from the game to be lost by player A,

$\Phi_{\mu\nu}(0,\beta,0) := Ee^{-\beta B_\mu}\mathbf{1}_{\{\mu<\nu\}}$. After setting $\alpha = \theta = 0$ in (3.70-3.71) [1], we arrive at the following.

(i) Case $\boldsymbol{\delta \neq \lambda_A}$. Proceeding as in Case 1 (see more details in [2]) we have

$$\begin{aligned}
\Phi^{(1)}_{\mu\nu}(0,\beta,0) = \Bigg\{ & \frac{\lambda_A\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_B)} \cdot e^{-N\beta} e^{-(\frac{\lambda_B g}{\lambda_A+\lambda_B})(M-X_0)} e^{-(\frac{\lambda_A h}{\lambda_A+\lambda_B})(N-Y_0)} \\
& \times I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)(N-Y_0)}{(\lambda_A+\lambda_B)^2}}\big) \\
& + \int_{z=0}^{N-Y_0} \Big[\Big(\frac{\lambda_A\delta\beta}{(\lambda_A+\lambda_B)(\delta+\lambda_B)} + \frac{\lambda_A h(\delta^2+2\lambda_B\delta)}{(\lambda_A+\lambda_B)(\delta+\lambda_B)^2} + \frac{\lambda_A\lambda_B^2 h^2\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_B)^3} \\
& \times \frac{1}{\beta+\frac{h\delta}{\delta+\lambda_B}}\Big) e^{-(Y_0+z)\beta} e^{-(\frac{\lambda_B g}{\lambda_A+\lambda_B})(M-X_0)} e^{-(\frac{\lambda_A h}{\lambda_A+\lambda_B})z} \\
& \times I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{(\lambda_A+\lambda_B)^2}}\big)\Big] dz + \frac{-\lambda_A\lambda_B^2 h^2\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_B)^3} \cdot \frac{1}{\beta+\frac{h\delta}{\delta+\lambda_B}} \\
& \times e^{-N\beta} e^{-(\frac{\lambda_B g}{\lambda_A+\lambda_B})(M-X_0)} e^{-(\frac{h\delta}{\delta+\lambda_B})(N-Y_0)} \\
& \times \int_{z=0}^{N-Y_0} e^{(\frac{\lambda_B h(\delta-\lambda_A)}{(\lambda_A+\lambda_B)(\delta+\lambda_B)})z} I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{(\lambda_A+\lambda_B)^2}}\big) dz \Bigg\} \\
& \times \mathbf{1}_{(X_0,\infty)}(M)\mathbf{1}_{(Y_0,\infty)}(N). \qquad (2.1)
\end{aligned}$$

(ii) Case $\boldsymbol{\delta = \lambda_A}$. Furthermore,

$$\begin{aligned}
\Phi^{(2)}_{\mu\nu}(0,\beta,0) = \Bigg\{ & \frac{\lambda_A^2}{(\lambda_A+\lambda_B)^2} \cdot e^{-N\beta} e^{-(\frac{\lambda_B g}{\lambda_A+\lambda_B})(M-X_0)} e^{-(\frac{\lambda_A h}{\lambda_A+\lambda_B})(N-Y_0)} \\
& \times I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)(N-Y_0)}{(\lambda_A+\lambda_B)^2}}\big) \\
& + \int_{z=0}^{N-Y_0} \Big[\Big(\frac{\lambda_A^2\beta}{(\lambda_A+\lambda_B)^2} + \frac{\lambda_A h(\lambda_A^2+2\lambda_A\lambda_B)}{(\lambda_A+\lambda_B)^3} + \frac{\lambda_A^2\lambda_B^2 h^2}{(\lambda_A+\lambda_B)^4} \cdot \frac{1}{\beta+\frac{\lambda_A h}{\lambda_A+\lambda_B}}\Big) \\
& \times e^{-(Y_0+z)\beta} e^{-(\frac{\lambda_B g}{\lambda_A+\lambda_B})(M-X_0)} e^{-(\frac{\lambda_A h}{\lambda_A+\lambda_B})z} I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{(\lambda_A+\lambda_B)^2}}\big)\Big] dz \\
& + \frac{-\lambda_A^2\lambda_B^2 h^2}{(\lambda_A+\lambda_B)^3} \sqrt{\frac{N-Y_0}{\lambda_A\lambda_B hg(M-X_0)}} \cdot \frac{1}{\beta+\frac{\lambda_A h}{\lambda_A+\lambda_B}} \cdot e^{-N\beta} e^{-(\frac{\lambda_B g}{\lambda_A+\lambda_B})(M-X_0)} \\
& \times e^{-(\frac{\lambda_A h}{\lambda_A+\lambda_B})(N-Y_0)} I_1\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)(N-Y_0)}{(\lambda_A+\lambda_B)^2}}\big) \Bigg\} \\
& \times \mathbf{1}_{(X_0,\infty)}(M)\mathbf{1}_{(Y_0,\infty)}(N). \qquad (2.2)
\end{aligned}$$

Case 3. With $\alpha = \beta = 0$ we obtain the Laplace-Stieltjes transform of the exit time of the game to be lost by player A, $\Phi_{\mu\nu}(0,0,\theta) := Ee^{-\theta\tau_\mu}\mathbf{1}_{\{\mu<\nu\}}$.

(i) Case $\boldsymbol{\delta \neq \lambda_A}$.

$$\Phi^{(1)}_{\mu\nu}(0,0,\theta) = \Bigg\{ \frac{\lambda_A \delta}{\Lambda(\delta+\theta+\lambda_B)} \cdot e^{-(g-\frac{\lambda_A g}{\Lambda})(M-X_0)} e^{-(h-\frac{\lambda_B h}{\Lambda})(N-Y_0)}$$
$$\times I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)(N-Y_0)}{\Lambda^2}}\big) + \frac{\lambda_A h\delta}{\Lambda(\delta+\theta)} \cdot e^{-(g-\frac{\lambda_A g}{\Lambda})(M-X_0)}$$
$$\times \int_{z=0}^{N-Y_0} e^{-(h-\frac{\lambda_B h}{\Lambda})z} I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{\Lambda^2}}\big)dz$$
$$+ \frac{-\lambda_A\lambda_B^2 h\delta}{\Lambda(\delta+\theta)(\delta+\theta+\lambda_B)^2} \cdot e^{-(g-\frac{\lambda_A g}{\Lambda})(M-X_0)} \cdot e^{-(h-\frac{\lambda_B h}{\delta+\theta+\lambda_B})(N-Y_0)}$$
$$\times \int_{z=0}^{N-Y_0} e^{(\frac{\lambda_B h}{\Lambda}-\frac{\lambda_B h}{\delta+\theta+\lambda_B})z} I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{\Lambda^2}}\big)dz \Bigg\}$$
$$\times \mathbf{1}_{(X_0,\infty)}(M)\mathbf{1}_{(Y_0,\infty)}(N). \tag{2.3}$$

(ii) Case $\boldsymbol{\delta = \lambda_A}$.

$$\Phi^{(2)}_{\mu\nu}(0,0,\theta) = \Bigg\{ \frac{\lambda_A^2}{\Lambda^2} \cdot e^{-(g-\frac{\lambda_A g}{\Lambda})(M-X_0)} e^{-(h-\frac{\lambda_B h}{\Lambda})(N-Y_0)}$$
$$\times I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)(N-Y_0)}{\Lambda^2}}\big) + \frac{\lambda_A^2 h}{\Lambda(\theta+\lambda_A)} \cdot e^{-(g-\frac{\lambda_A g}{\Lambda})(M-X_0)}$$
$$\times \int_{z=0}^{N-Y_0} e^{-(h-\frac{\lambda_B h}{\Lambda})z} I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{\Lambda^2}}\big)dz$$
$$+ \frac{-\lambda_A^2\lambda_B^2 h}{\Lambda^2(\theta+\lambda_A)}\sqrt{\frac{N-Y_0}{\lambda_A\lambda_B hg(M-X_0)}} \cdot e^{-(g-\frac{\lambda_A g}{\Lambda})(M-X_0)} e^{-(h-\frac{\lambda_B h}{\Lambda})(N-Y_0)}$$
$$\times I_1\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)(N-Y_0)}{\Lambda^2}}\big)\Bigg\}\mathbf{1}_{(X_0,\infty)}(M)\mathbf{1}_{(Y_0,\infty)}(N). \tag{2.4}$$

Here I_j's are modified Bessel functions.

3 The Probability Distribution of the Casualties Values to Players A and B

Here we will find the probability distribution function F_{A} of the exit value of casualties to player A (special case 1) by taking the inverse Laplace transform with respect to variable α. The Laplace inverse formula that we use, along with (3.64-3.67) [1], is:

$$\mathcal{L}_y^{-1}(e^{-\alpha y} \cdot \frac{1}{(y+b)^2})(q) = (q-\alpha)e^{-b(q-\alpha)}\mathbf{1}_{(\alpha,\infty)}(q). \tag{3.1}$$

The above formula can be found in references [3,4] as well. After that, we apply the Laplace inverse to $\Phi_{\mu\nu}(\alpha,0,0) = Ee^{-\alpha A_\mu}\mathbf{1}_{\{\mu<\nu\}}$, arriving at

$$F_{\mathrm{A}}(t) = \mathcal{L}_{\alpha}^{-1}\Big\{\Phi_{\mu\nu}(\alpha,0,0)\Big\}(t) = \Big\{\frac{\lambda_A g\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)}\cdot e^{-\left(\frac{(\delta+\lambda_B)g}{\delta+\lambda_A+\lambda_B}\right)(t-M)}$$
$$\times e^{-\left(\frac{\lambda_B g}{\lambda_A+\lambda_B}\right)(M-X_0)} e^{-\left(\frac{\lambda_A h}{\lambda_A+\lambda_B}\right)(N-Y_0)} I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)(N-Y_0)}{(\lambda_A+\lambda_B)^2}}\big)$$
$$+\frac{\lambda_A hg\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)} e^{-\left(\frac{g\delta}{\delta+\lambda_A}\right)(t-M)} e^{-(\frac{\lambda_B g}{\lambda_A+\lambda_B})(M-X_0)} \int_{z=0}^{N-Y_0} e^{-(\frac{\lambda_A h}{\lambda_A+\lambda_B})z}$$
$$\times I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{(\lambda_A+\lambda_B)^2}}\big)dz + \int_{z=0}^{N-Y_0}\Big[\frac{-\lambda_A\lambda_B^2 hg\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)(\delta+\lambda_A+\lambda_B)^2}$$
$$\times e^{-(\frac{(\delta+\lambda_B)g}{\delta+\lambda_A+\lambda_B})(t-M)} I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(N-Y_0-z)(t-M)}{(\delta+\lambda_A+\lambda_B)^2}}\big)$$
$$+\frac{-\lambda_A^2\lambda_B hg^2\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)^2(\delta+\lambda_A+\lambda_B)} e^{-(\frac{g\delta}{\delta+\lambda_A})(t-M)} \int_{w=0}^{t-M} e^{-(\frac{\lambda_A\lambda_B g}{(\delta+\lambda_A)(\delta+\lambda_A+\lambda_B)})w}$$
$$\times I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(N-Y_0-z)w}{(\delta+\lambda_A+\lambda_B)^2}}\big)dw + \frac{\lambda_A^2\lambda_B hg^2\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)^2}$$
$$\times\sqrt{\frac{t-M}{\lambda_A\lambda_B hg(N-Y_0-z)}}\cdot e^{-(\frac{(\delta+\lambda_B)g}{\delta+\lambda_A+\lambda_B})(t-M)} I_1\big(2\sqrt{\frac{\lambda_A\lambda_B hg(N-Y_0-z)(t-M)}{(\delta+\lambda_A+\lambda_B)^2}}\big)\Big]$$
$$\times e^{-(\frac{\lambda_B g}{\lambda_A+\lambda_B})(M-X_0)} e^{-(\frac{(\delta+\lambda_A)h}{\delta+\lambda_A+\lambda_B})(N-Y_0)} e^{(\frac{\lambda_B h\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)})z}$$
$$\times I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{(\lambda_A+\lambda_B)^2}}\big)dz\Big\}\mathbf{1}_{(X_0,\infty)}(M)\mathbf{1}_{(Y_0,\infty)}(N)\mathbf{1}_{(M,\infty)}(t). \tag{3.2}$$

4 The Loss Probability

Another special case is the probability that player A loses to player B. This can be directly obtained from $\Phi_{\mu\nu}(\alpha,\beta,\theta) = Ee^{-\alpha A_\mu-\beta B_\mu-\theta\tau_\mu}\mathbf{1}_{\{\mu<\nu\}}$ by setting $\alpha=\beta=\theta=0$:

$$\Phi_{\mu\nu}(0,0,0) := E\mathbf{1}_{\{\mu<\nu\}} = P\{\mu<\nu\} = P\{\tau_\mu<\tau_\nu\}. \tag{4.1}$$

With $\alpha=\beta=\theta=0$ in (3.70-3.73) [1], we have

(i) Case $\boldsymbol{\delta \neq \lambda_A}$,

$$\Phi_{\mu\nu}^{(1)}(0,0,0) = \Big\{\frac{\lambda_A\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_B)}\cdot e^{-\frac{\lambda_B g(M-X_0)}{\lambda_A+\lambda_B}} e^{-\frac{\lambda_A h(N-Y_0)}{\lambda_A+\lambda_B}}$$
$$\times I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)(N-Y_0)}{(\lambda_A+\lambda_B)^2}}\big) + \frac{\lambda_A h}{\lambda_A+\lambda_B}\cdot e^{-\frac{\lambda_B g(M-X_0)}{\lambda_A+\lambda_B}}$$
$$\times\int_{z=0}^{N-Y_0} e^{-\frac{\lambda_A h}{\lambda_A+\lambda_B}z} I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{(\lambda_A+\lambda_B)^2}}\big)dz \frac{-\lambda_A\lambda_B^2 h}{(\lambda_A+\lambda_B)(\delta+\lambda_B)^2}$$
$$\times e^{-\frac{\lambda_B g(M-X_0)}{\lambda_A+\lambda_B}} e^{-\frac{h\delta(N-Y_0)}{\delta+\lambda_B}} \int_{z=0}^{N-Y_0} e^{(\frac{-\lambda_B h(\lambda_A-\delta)}{(\lambda_A+\lambda_B)(\delta+\lambda_B)})z} I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{(\lambda_A+\lambda_B)^2}}\big)dz\Big\}$$
$$\times\mathbf{1}_{(X_0,\infty)}(M)\mathbf{1}_{(Y_0,\infty)}(N). \tag{4.2}$$

(ii) Case $\boldsymbol{\delta = \lambda_A}$,

$$\Phi^{(2)}_{\mu\nu}(0,0,0) = \Bigg\{ \frac{\lambda_A^2}{(\lambda_A+\lambda_B)^2} \cdot e^{-\frac{\lambda_B g(M-X_0)}{\lambda_A+\lambda_B}} e^{-\frac{\lambda_A h(N-Y_0)}{\lambda_A+\lambda_B}}$$

$$\times I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)(N-Y_0)}{(\lambda_A+\lambda_B)^2}}\big) + \frac{\lambda_A h}{\lambda_A+\lambda_B} \cdot e^{-\frac{\lambda_B g(M-X_0)}{\lambda_A+\lambda_B}}$$

$$\times \int_{z=0}^{N-Y_0} e^{-\frac{\lambda_A h}{\lambda_A+\lambda_B}z} I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{(\lambda_A+\lambda_B)^2}}\big)dz$$

$$+ \frac{-\lambda_A\lambda_B^2 h}{(\lambda_A+\lambda_B)^2}\sqrt{\frac{N-Y_0}{\lambda_A\lambda_B hg(M-X_0)}} \cdot e^{-\frac{\lambda_B g(M-X_0)}{\lambda_A+\lambda_B}} e^{-\frac{\lambda_A h(N-Y_0)}{\lambda_A+\lambda_B}}$$

$$\times I_1\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)(N-Y_0)}{(\lambda_A+\lambda_B)^2}}\big)\Bigg\} \mathbf{1}_{(X_0,\infty)}(M)\mathbf{1}_{(Y_0,\infty)}(N). \quad (4.3)$$

5 Numerical Results

Even though the above formulas are totally explicit, they may look quite bulky. We would like to illustrate their tameness by means of simple computations. They also show how changing input parameters alters the trend of the game. For a full version of these computations including a MATLAB routine, see [2]. The program utilizes (4.2) and (4.3) with the results placed in the tables below.

λ_A	45	45	45	45	45
λ_B	45	45	45	45	45
g	18	18	18	18	18
h	18	18	18	18	18
M	35	34	33	32	31
N	33	33	33	33	33
X_0	13	13	13	13	13
Y_0	13	13	13	13	13
δ	45	45	45	45	45
Probability of A losing	0.1708	0.3106	0.4895	0.6749	0.8279

λ_A	45	45	45	45	45
λ_B	45	45	45	45	45
g	18	18	18	18	18
h	18	18	18	18	18
M	33	33	33	33	33
N	33	33	33	33	33
X_0	10	11.5	13	14.5	16
Y_0	13	13	13	13	13
δ	45	45	45	45	45
Probability of A losing	0.0811	0.2345	0.4895	0.7574	0.9268

λ_A	18	18	18	18	18
λ_B	20	20	20	20	20
g	14	14	14	14	14
h	16	12	11	10	6
M	20	20	20	20	20
N	24	24	24	24	24
X_0	7	7	7	7	7
Y_0	5	5	5	5	5
δ	100	100	100	100	100
Probability of A losing	0.9991	0.8014	0.5875	0.3324	0.0003

λ_A	18	18	18	18	18
λ_B	20	20	20	20	20
g	14	14	14	14	14
h	16	16	16	16	16
M	32	28	26	24	20
N	24	24	24	24	24
X_0	7	7	7	7	7
Y_0	5	5	5	5	5
δ	100	100	100	100	100
Probability of A losing	0.0129	0.2650	0.5910	0.8717	0.9991

λ_A	18	18	18	18	18
λ_B	20	20	20	20	20
g	14	14	14	14	14
h	16	16	16	16	16
M	20	20	20	20	20
N	24	24	24	24	24
X_0	0.0001	0.01	1	2	7
Y_0	5	5	5	5	5
δ	100	100	100	100	100
Probability of A losing	0.4191	0.4207	0.5910	0.7505	0.9991

λ_A	8	8	8	8	8
λ_B	10	10	10	10	10
g	28	28	28	28	28
h	24	32	35	38	46
M	10	10	10	10	10
N	12	12	12	12	12
X_0	2	2	2	2	2
Y_0	4	4	4	4	4
δ	50	50	50	50	50
Probability of A losing	0.0033	0.2419	0.4963	0.7431	0.9893

λ_A	8	8	8	8	8
λ_B	10	10	10	10	10
g	28	28	28	28	28
h	24	24	24	24	24
M	10	10	10	10	10
N	12	12	12	12	12
X_0	7	5	4.5	4	2
Y_0	4	4	4	4	4
δ	50	50	50	50	50
Probability of A losing	0.9996	0.7190	0.4888	0.2712	0.0033

where

$\lambda_A, \lambda_B =$ rates of strikes to player A by player B and player B to player A;
$g^{-1}, h^{-1} =$ mean magnitudes of strikes to A by B and B to A;
$M, N =$ thresholds of players A and B;
$X_0, Y_0 =$ initial casualties to players A and B;
$\delta^{-1} =$ observations frequency.

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