Nonlinear Dynamics and Systems Theory, **16** (3) (2016) 260–267

# The Jacobi Elliptic Method and Its Applications to the Generalized Form of the Phi-Four Equation

R.B. Djob[1*], E. Tala-Tebue [1], A. Kenfack-Jiotsa[2] and T.C. Kofane[1]

[1] *Laboratory of Mechanics, Department of Physics, Faculty of Science, University of Yaoundé 1, P.O. Box 812, Yaoundé, Cameroon*
[2] *Nonlinear Physics and Complex Systems Group, Department of Physics, The Higher Teachers' Training College, University of Yaoundé I, P.O. Box 47 Yaoundé, Cameroon*

Received: June 30, 2015; Revised: June 9, 2016

**Abstract:** In order to investigate the generalized periodic solutions of the generalized phi-four equation, we use the Jacobi elliptic functions. Many kinds of solutions are obtained. For some parameters, these envelope periodic solutions can degenerate to the envelope shock wave solutions (dark solitons) and the envelope solitary wave solutions (bright solitons). The existence of these solutions is determined by the parameters of the initial equation. The solutions found in this work can be used in many areas of physics such as telecommunications.

**Keywords:** *generalized periodic solutions, generalized phi-four equation; Jacobi elliptic functions; envelope periodic solutions.*

**Mathematics Subject Classification (2010):** 00A69, 35Q51.

## 1 Introduction

Before the discovery of solitons, scientists had taken the nonlinear terms in an equation as perturbations. The history of solitons (the wave of translation), in fact, dates back to 1834, the year in which John Scott Russell observed that a heap of water in a canal propagated undistorted over several kilometers. The results obtained in the linear theory of waves, by ignoring the nonlinear parts, are most frequently too far from reality to be useful. The transition from linear to nonlinear description is justified by the necessity to take into account all the details of the observed phenomena. The wave of translation was regarded as a curiosity until the 1960$s$, when scientists began to use computers to study nonlinear wave propagation. The discovery of mathematical solutions started

* Corresponding author: mailto:rogerdjob@yahoo.fr

with the analysis of nonlinear partial differential equations, such as in the works of Boussinesq and Rayleigh, carried out independently. Recently, a new direction related to the investigation of nonlinear phenomena and processes has been actively developed in various areas, including hydrodynamics, nonlinear optics, plasma physics, and biology [1–8], to mention a few. A remarkable number of evolution equations (sine-Gordon, Korteweg de Vries, Boussinesq, Schrodinger and others) considered by the end of the $19^{th}$ century, radically changed the thinking of scientists about the nature of nonlinearity. It then becomes necessary to solve these nonlinear equations. The exact analytical solutions of nonlinear equations are hardly obtained. In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations in mathematics and physics have been proposed. We can list the generalized iterative methods [9], computational methods [10], travelling wave solutions method [11], the sine-cosine method [12,13], Backlund transform method [14], the sinc-collocation method [15], Darboux transform method [16], Painleve's singularity structure analysis [17], homotopy perturbation method [18], variational iteration method [19], inverse scattering transform method [20], the (G'/G)-expansion method [21], the Hirota's bilinear method [22], exp-function method [23], tanh method [24, 25], extended three-wave method [26]. These methods, however, can only obtain the shock and solitary wave solutions or the periodic solutions with the elementary functions [27–32], but cannot get the generalized periodic solutions of nonlinear equations. The objective of this work is to use the Jacobi elliptic method [33] to obtain the generalized periodic solutions with the phi-four equation.

The standard form of phi-four equation

$$u_{tt} - u_{xx} + u^3 - u = 0 \tag{1}$$

arises in many branches of mathematical physics. Its special solutions are known as kink and antikink solitons. In our investigations, we consider the following form of equation (1):

$$(u^l)_{tt} - a(u^n)_{xx} - bu^m + cu^n = 0, \tag{2}$$

where $a$, $b$ and $c$ are arbitrary nonzero constants and $l$, $m$ and $n$ are integers; $u(x,t)$ is the unknown function depending on the spatial variable $x$ and the temporal variable $t$. The subscripts $x$ and $t$ denote partial derivatives with respect to these variables. The technique that will be used is the most effective direct method to construct generalized wave solutions of nonlinear evolution equations.

## 2 Jacobi Elliptic sn Function

By means of the Jacobi elliptic function, $u(x,t)$ can be expressed as follows:

$$u(x,t) = Asn^p\xi, \quad \xi = q(x - v_0 t), \tag{3}$$

where $p > 0$ is a constant which will be determined later. $A$ represents the amplitude of the wave, while $v_0$ is the velocity of the wave; $q$ can represent the inverse width of the wave. From equation (3), we have:

$$\begin{cases} u^m = A^m sn^{pm}\xi, \\ u^n = A^n sn^{pn}\xi, \\ (u^l)_{tt} = A^l plq^2v_0^2\Big[(pl-1)sn^{pl-2}\xi - pl(1+k^2)sn^{pl}\xi + (pl+1)k^2sn^{pl+2}\xi\Big], \\ (u^n)_{xx} = A^n pnq^2\Big[(pn-1)sn^{pn-2}\xi - pn(1+k^2)sn^{pn}\xi + (pn+1)k^2sn^{pn+2}\xi\Big]. \end{cases} \tag{4}$$

The following relations are taken into account:

$$cn^2(\xi,k) + sn^2(\xi,k) = 1, \quad dn^2(\xi,k) + k^2sn^2(\xi,k) = 1. \tag{5}$$

Substituting the expression (4) into (2) yields

$$\begin{aligned} &A^l plq^2v_0^2\Big[(pl-1)sn^{pl-2}\xi - pl(1+k^2)sn^{pl}\xi + (pl+1)k^2sn^{pl+2}\xi\Big] \\ &-aA^n pnq^2\Big[(pn-1)sn^{pn-2}\xi - pn(1+k^2)sn^{pn}\xi + (pn+1)k^2sn^{pn+2}\xi\Big] \\ &\qquad -bA^m sn^{pm}\xi + cA^n sn^{pn}\xi = 0. \end{aligned} \tag{6}$$

From equation (6), equating the exponents of $sn^{pn+2}\xi$ and $sn^{pm}\xi$ functions we get

$$p = \frac{2}{m-n}. \tag{7}$$

Also from equation (6), equating the exponents of $sn^{pl}\xi$ and $sn^{pn}\xi$ functions we have

$$l = n. \tag{8}$$

If we make the same gymnastic with the exponents of $sn^{pl+2}\xi$ and $sn^{pn+2}\xi$ and for $sn^{pl-2}\xi$ and $sn^{pn-2}\xi$ functions, we also obtain $l = n$. Now, in view of equation (8), the functions $sn^{pl+j}\xi$ with $j = -2, 0, 2$ in (6) are linearly independent. Thus, their respective coefficients must vanish. Setting their coefficients to zero gives the system of algebraic equations:

$$A^n pnq^2(pn-1)(v_0^2 - a) = 0, \tag{9}$$

$$A^n p^2n^2q^2(1+k^2)(a - v_0^2) + cA^n = 0, \tag{10}$$

$$A^n pnq^2k^2(pn+1)(v_0^2 - a) - bA^m = 0. \tag{11}$$

If $v_0^2 - a \neq 0$, then equation (9) gives the relation between the two parameters $p$ and $n$, that is

$$p = \frac{1}{n}, \tag{12}$$

and using relation (7), we have:

$$m = 3n. \tag{13}$$

From equation (10), one obtains

$$q^2 = \frac{c}{(1+k^2)(v_0^2 - a)}. \tag{14}$$

Inserting (14) into (11) yields

$$A = \Big[\frac{2k^2 c}{b(1+k^2)}\Big]^{\frac{1}{2n}}. \tag{15}$$

Thus, the generalized solutions of equation (3) are given by

$$u(x,t) = \Big\{\sqrt{\frac{2k^2 c}{b(1+k^2)}}sn\Big[\sqrt{\frac{c}{(1+k^2)(v_0^2-a)}}(x-v_0t)\Big]\Big\}^{\frac{1}{n}}. \tag{16}$$

We clearly observe that these solutions exist if and only if $c(v_0^2 - a) > 0$ and $bc > 0$. As $k \to 1$, corresponding envelope solitary wave solutions are

$$u(x,t) = \Big\{\sqrt{\frac{c}{b}}\tanh\Big[\sqrt{\frac{c}{2(v_0^2-a)}}(x-v_0t)\Big]\Big\}^{\frac{1}{n}}. \tag{17}$$

Namely dark solitons of equation (17) look like those found by Triki and Wazwaz in [34]. This justifies the fact that the present method is more explicit.

## 3 Jacobi Elliptic cn Function

In this section, $u(x,t)$ is expressed as follows:

$$u(x,t) = Acn^p\xi, \quad \xi = q(x - v_0t). \tag{18}$$

In this equation, $p > 0$. From equation (18), we get:

$$\begin{cases} u^m = A^m cn^{pm}\xi, \\ u^n = A^n cn^{pn}\xi, \\ (u^l)_{tt} = A^l plq^2v_0^2\Big[(pl-1)(1-k^2)cn^{pl-2}\xi + pl(2k^2-1)cn^{pl}\xi - (pl+1)k^2cn^{pl+2}\xi\Big], \\ (u^n)_{xx} = A^n pnq^2\Big[(pn-1)(1-k^2)cn^{pn-2}\xi + pn(2k^2-1)cn^{pn}\xi - (pn+1)k^2cn^{pn+2}\xi\Big]. \end{cases} \tag{19}$$

Inserting (19) into (2), one obtains:

$$\begin{aligned} A^l plq^2v_0^2\Big[(pl-1)(1-k^2)cn^{pl-2}\xi + pl(2k^2-1)cn^{pl}\xi - (pl+1)k^2cn^{pl+2}\xi\Big] \\ -aA^n pnq^2\Big[(pn-1)(1-k^2)cn^{pn-2}\xi + pn(2k^2-1)cn^{pn}\xi - (pn+1)k^2cn^{pn+2}\xi\Big] \\ -bA^m cn^{pm}\xi + cA^n cn^{pn}\xi = 0. \end{aligned} \tag{20}$$

In equation (20), equating the exponents of $cn^{pn+2}\xi$ and $cn^{pm}\xi$ functions gives

$$p = \frac{2}{m-n}. \tag{21}$$

Also from equation (20), equating the exponents of $cn^{pl}\xi$ and $cn^{pn}\xi$ functions we get

$$l = n. \tag{22}$$

The same work can be done with the exponents of $cn^{pl+2}\xi$ and $cn^{pn+2}\xi$ and for $cn^{pl-2}\xi$ and $cn^{pn-2}\xi$ functions; we also obtain $l = n$. Now, the functions $cn^{pl+j}\xi$ with $j = -2, 0, 2$

in (20) are linearly independent. Thus, their respective coefficients must vanish. Setting their coefficients to zero gives the system of algebraic equations:

$$A^n pnq^2(pn-1)(1-k^2)(v_0^2-a)=0, \tag{23}$$

$$A^n p^2n^2q^2(2k^2-1)(v_0^2-a)+cA^n=0, \tag{24}$$

$$A^n pnq^2k^2(pn+1)(a-v_0^2)-bA^m=0. \tag{25}$$

If $v_0^2-a\neq 0$, then equation (23) gives the following two relations, that is

$$\begin{cases} p=\frac{1}{n}, \\ k^2=1. \end{cases} \tag{26}$$

Equation (24) gives

$$q^2=\frac{c}{p^2n^2(2k^2-1)(a-v_0^2)}, \tag{27}$$

and (25) yields

$$A=\Big[\frac{(pn+1)k^2c}{bpn(2k^2-1)}\Big]^{\frac{1}{m-n}}. \tag{28}$$

Thus, the generalized solutions of equation(3) are given by:

**Case 1:** $p=\frac{1}{n}$, *i.e.* $m=3n$; $q^2=\frac{c}{(2k^2-1)(a-v_0^2)}$, $A=\Big[\frac{2k^2c}{b(2k^2-1)}\Big]^{\frac{1}{2n}}$ and

$$u(x,t)=\Big\{\sqrt{\frac{2k^2c}{b(2k^2-1)}}cn\Big[\sqrt{\frac{c}{(2k^2-1)(a-v_0^2)}}(x-v_0t)\Big]\Big\}^{\frac{1}{n}}. \tag{29}$$

**Case 2:** $k^2=1$; $q^2=\frac{c(m-n)^2}{4n^2(a-v_0^2)}$, $A=\Big[\frac{(m+n)c}{2nb}\Big]^{\frac{1}{m-n}}$ and

$$u(x,t)=\Big\{\frac{(m+n)c}{2nb}sech^2\Big[\sqrt{\frac{c(m-n)^2}{4n^2(a-v_0^2)}}(x-v_0t)\Big]\Big\}^{\frac{1}{m-n}}. \tag{30}$$

It is evident that these solutions have a physical sense if and only if $c(a-v_0^2)>0$ and $bc>0$. $k$ must be different from zero in (29). Equation (30) is an exact bright soliton solution of (3).

## 4 Jacobi Elliptic dn Function

In this section, $u(x,t)$ is expressed as follows:

$$u(x,t)=Adn^p\xi, \quad \xi=q(x-v_0t). \tag{31}$$

Here again, $p$ has to be positive. From (31), we get:

$$\begin{cases} u^m=A^m dn^{pm}\xi, \\ u^n=A^n dn^{pn}\xi, \\ (u^l)_{tt}=A^l plq^2v_0^2\Big[(pl-1)(k^2-1)dn^{pl-2}\xi+pl(2-k^2)dn^{pl}\xi-(pl+1)dn^{pl+2}\xi\Big], \\ (u^n)_{xx}=A^n pnq^2\Big[(pn-1)(k^2-1)dn^{pn-2}\xi+pn(2-k^2)dn^{pn}\xi-(pn+1)dn^{pn+2}\xi\Big]. \end{cases} \tag{32}$$

Inserting (32) into (3), one obtains:

$$A^l plq^2 v_0^2 \Big[(pl-1)(k^2-1)dn^{pl-2}\xi + pl(2-k^2)dn^{pl}\xi - (pl+1)dn^{pl+2}\xi\Big]$$
$$-aA^n pnq^2 \Big[(pn-1)(k^2-1)dn^{pn-2}\xi + pn(2-k^2)dn^{pn}\xi - (pn+1)dn^{pn+2}\xi\Big]$$
$$-bA^m dn^{pm}\xi + cA^n dn^{pn}\xi = 0. \quad (33)$$

In (33), equating the exponents of $dn^{pn+2}\xi$ and $dn^{pm}\xi$ functions gives

$$p = \frac{2}{m-n}. \quad (34)$$

Also from (33), equating the exponents of $dn^{pl}\xi$ and $dn^{pn}\xi$ functions we have

$$l = n \quad (35)$$

which is also obtained by equating the exponents' pairs $pl+2$ and $pn+2$, $pl-2$ and $pn-2$. Setting the coefficients of the linearly independent functions $dn^{pl+j}\xi$, where $j = -2, 0, 2$, to zero gives the system of algebraic equations:

$$A^n pnq^2(pn-1)(k^2-1)(v_0^2-a) = 0, \quad (36)$$
$$A^n p^2n^2q^2(2-k^2)(v_0^2-a) + cA^n = 0, \quad (37)$$
$$A^n pnq^2(pn+1)(a-v_0^2) - bA^m = 0. \quad (38)$$

If $v_0^2 - a \neq 0$, then equation (36) gives the following two relations

$$\begin{cases} p = \frac{1}{n}, \\ k^2 = 1. \end{cases} \quad (39)$$

Equation (37) gives

$$q^2 = \frac{c}{p^2n^2(2-k^2)(a-v_0^2)}, \quad (40)$$

and (38) yields

$$A = \left[\frac{(pn+1)c}{bpn(2-k^2)}\right]^{\frac{1}{m-n}}. \quad (41)$$

The generalized solutions of equation (3) are given by:

**Case 1:** $p = \frac{1}{n}$, *i.e.* $m = 3n$; $q^2 = \frac{c}{(2-k^2)(a-v_0^2)}$, $A = \left[\frac{2c}{b(2-k^2)}\right]^{\frac{1}{2n}}$ and

$$u(x,t) = \left\{\sqrt{\frac{2c}{b(2-k^2)}}dn\Big[\sqrt{\frac{c}{(2-k^2)(a-v_0^2)}}(x-v_0t)\Big]\right\}^{\frac{1}{n}}. \quad (42)$$

**Case 2:** $k^2 = 1$; $q^2 = \frac{c(m-n)^2}{4n^2(a-v_0^2)}$, $A = \left[\frac{(m+n)c}{2nb}\right]^{\frac{1}{m-n}}$ and

$$u(x,t) = \left\{\frac{(m+n)c}{2nb}sech^2\Big[\sqrt{\frac{c(m-n)^2}{4n^2(a-v_0^2)}}(x-v_0t)\Big]\right\}^{\frac{1}{m-n}}, \quad (43)$$

with $c(a-v_0^2) > 0$ and $bc > 0$.

These soliton solutions can be controlled well by adjusting the parameters of the system. From one ansatz, we carry out many types of solutions, and we conclude that the present method is straightforward and concise.

## 5 Conclusion

In this work, we have considered a generalized phi-four equation with arbitrary constant coefficients and general values of the exponents in the dissipation and nonlinear terms. With the aid of Jacobi elliptic functions, the generalized periodic solutions are obtained. We have noted that the existence of these solutions depends on whether $c(v_0^2 - a) > 0$ or $c(a - v_0^2) > 0$ and $bc > 0$. We have also pointed out that for some parameters, these envelope periodic solutions can degenerate to the non-topological and topological solitons.

## References

[1] Tabiryan, N.V., Sukhov, A.V. and Zeldovich, B.Y. The orientational optical nonlinearity of liquid crystals. *Mol Cryst Liq Cryst.* **136** (1986) 1–140.

[2] Khoo, I.C. *Progress in Optics*. Elsevier, Amsterdam, 1988.

[3] Khoo, I.C. and Wu, N.T. *Optics and Nonlinear Optics of Liquid Crystals.* World Scientific, Singapore, 1993.

[4] Braun, E., Faucheux, L.P., Libchaber, A., McLaughlin, D.W., Muraki, D.J. and Shelley, M.J. Filamentation and Undulation of Self-Focused Laser Beams in Liquid Crystals. *Europhys. Lett.* **23** (1993) 239–244.

[5] Braun, E., Faucheux, L.P. and Libchaber, A. Strong self-focusing in nematic liquid crystals. *Phys. Rev. A* **48** (1993) 611–622.

[6] McLaughlin, D.W., Muraki, D.J., Shelley, M.J. and Wang, X. A paraxial model for optical self focussing in a nematic liquid crystal. *Physica D* **88** (1995) 55–81.

[7] McLaughlin, D.W., Muraki, D.J. and Shelley, M.J. Self-focussed optical structures in a nematic liquid crystal. *Physica D* **97** (1996) 471–497.

[8] Janossy, I. Molecular Interpretation of the Absorption-Induced Optical Reorientation of. Nematic Liquid Crystals. *Phys. Rev. E* **49** (1994) 2957–2963.

[9] Sowmya, M. and Vatsala, A.S. Generalized Iterative Methods for Caputo Fractional Differential Equations via Coupled Lower and Upper Solutions with Superlinear Convergence. *Nonlinear Dynamics and Systems Theory* **15** (2) (2015) 198–208.

[10] Jafari, H. and Azad, A. A Computational Method for Solving a System of Volterra Integro-differential Equations. *Nonlinear Dynamics and Systems Theory* **12** (4) (2012) 389–396.

[11] Danylenko, V.A. and Skurativskyi, S.I. Travelling Wave Solutions of Nonlocal Models for Media with Oscillating Inclusions. *Nonlinear Dynamics and Systems Theory* **12** (4) (2012) 365–374.

[12] Bekir, A. New solitons and periodic wave solutions for some nonlinear physical models by using the sine-cosine method. *Phys. Scr.* **77** (2008) 501–504.

[13] Gambo, B., Bouetou, T.B., Kuetche, V.K. and Kofane, T.C. Explicit Series Solutions to Nonlinear Evolution Equations: The Sine-Cosine Method. *Appl. Math. Comput.* **215** (2010) 4239–4247.

[14] Tsigaridas, G., Fragos, A., Polyzos, I., Fakis, M., Ioannou, A., Giannetas, V. and Persephonis, P. Evolution of near-soliton initial profiles in nonlinear wave equations through their Backlund transforms. *Chaos Solitons Fract.* **23** (2005) 1841–1854.

[15] Yeganeh, S., Saadatmandi, A., Soltanian, F., Dehghan, M. The numerical solution of differential-algebraic equations by sinc-collocation method. *Comput. and Appl. Math.* **32** (2013) 343–354.

[16] Suzo, A.A. Intertwining technique for the matrix Schrodinger equation. *Phys. Lett. A* **335** (2005) 88–102.

[17] Banerjee, R.S. Painlev analysis of the K(m,n) equations which yield compactons. *Phys. Scr.* **57** (1998) 598–600.

[18] Mehdi, D., Fatemeh, S. Solution of a partial differential equation subject to temperature overspecification by He's homotopy perturbation method. *Phys. Scr.* **75** (2007) 778–787.

[19] He, J.H. The variational iteration method for eighth-order initial-boundary value problems. *Phys. Scr.* **76** (2007) 680–682.

[20] Vakhnenko, V.O., Parkes, E.J. and Morrison, A.J. A Backlund transformation and the inverse scattering transform method for the generalised Vakhnenko equation. *Chaos Solitons Fract.* **17** (2003) 683–692.

[21] Wang, M., Li, X. and Zhang, J. The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. *Phys. Lett. A* **372** (2008) 417–423.

[22] Hirota, R. Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons, *Phys Rev Lett* **27** (1971) 1192–1194.

[23] Zayed, E.M., Abdelaziz, M.A. Exact solutions for the nonlinear Schrdinger equation with variable coefficients using the generalized extended tanh-function, the sine-cosine and the exp-function methods. *Appl. Math. Comput.* **218** (2011) 2259–2268.

[24] Wazwaz, A.M. The tanh method: solitons and periodic solutions for the Dodd-Bullough-Mikhailov and the Tziteica-Dodd-Bullough equations. *Chaos Solitons Fract.* **25** (2005) 55–63.

[25] Zayed, E.M. and Abdelaziz, M.A. The tanh-function method using a generalized wave transformation for nonlinear equations. *Int. J. Nonlinear Sci. Numer. Simula.* **11** (2010) 595–601.

[26] Li, Z., Dai, Z., Liu, J. Exact three-wave solutions for the(3+1)-dimensional Jimbo-Miwa equation. *Comput. Math. Appl.* **61** (2011) 2062–2066.

[27] Wang, M.L. Solitary wave solution for variant Boussinesq equations. *Phys. Lett. A* **199** (1995) 169–172.

[28] Wang, M.L., Zhou, Y.B. and Li, Z.B. Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics. *Phys. Lett. A* **216** (1996) 67–75.

[29] Yang, L., Zhu, Z. and Wang, Y. Exact Solutions of Nonlinear Equations. *Phys. Lett. A* **260** (1999) 55–59.

[30] Hirota, R. Exact N-solutions of the wave equation of long waves in shallow water and in nonlinear lattices. *J. Math. Phys.* **14** (1973) 810–814.

[31] Kudryashov, N.A. Exact solutions of the generalized Kuramoto Sivashinsky equation. *Phys. Lett. A* **147** (1990) 287–291.

[32] Otwinowski, M., Paul, R. and Laidlaw, W.G. Exact Traveling Wave Solutions of a Class of Nonlinear Diffusion Equations by Reduction to Quadrature. *Phys. Lett. A* **128** (1988) 483–487.

[33] Huang, W.H., Liu, Y.L. Jacobi elliptic function solutions of the Ablowitz-Ladik discrete nonlinear Schrdinger system. *Chaos Solitons Fract.* **40** (2009) 786–792.

[34] Triki, H., Wazwaz, A.M. Bright and dark soliton solutions for a $K(m, n)$ equation with $t$-dependent coefficients. *Phys. Lett. A* 373 (2009) 2162–2165.