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Reduced Order Bilinear Time Invariant System by Means of Error Transfer Function Least Upper Bounds

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Abstract: The order selection problem of the reduced bilinear time invariant systems is considered in this paper. The r-*th* order reduced bilinear time invariant systems are chosen by using the least upper bound of the difference bilinear system in the proposed H_2-norm. The H_2-norm of the difference bilinear system is computed by the H_2-norm of the error transfer function between the full order and the reduced order of a bilinear time invariant system. The reduced bilinear systems are obtained by using the balanced truncation and the singular perturbation methods. The H_2-norm of the difference bilinear systems is a function of controllability gramian or observability gramian of the difference bilinear system. The simulation results in the example confirm the proposed method for obtaining the reduced bilinear system which is similar to the full order bilinear system.

Keywords: *bilinear systems; controllability and observability gramians; H_2-norm; reduced order bilinear systems; balanced truncation; singular perturbation.*

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1 Introduction

In this paper, a criteria for selecting order of a reduced order model of bilinear time invariant systems based on the value alteration of the least upper bounds of a transfer function of difference bilinear systems in the proposed H_2-norm is considered. The order selection based on the value alteration of the singular Hankel values, see [3], is not apparent because the decision is influenced by knowledge of the decision makers. The measurement of the model reduction, which is calculated by using the H_2-norm is able to characterize the virtue of the reduced order model. The definition of the H_2-norm based on transfer function of the bilinear time invariant system which includes the controllability gramian or the observability gramian is then proposed.

The least upper bounds of the error transfer function between the full order and the reduced order model of the bilinear systems in the H_2-norm become a tool for modern controller design. The least upper bounds of the error transfer function between full order and reduced order model for the linear systems in the H_2-norm have been discussed in [10] and [13]. Therefore, the least upper bounds of the bilinear time invariant systems discussed in [23, 24] are important in model order reduction.

The reduced order bilinear systems are obtained by using the balanced truncation [3] and the singular perturbation methods [22]. Two methods are used because they preserve the dominant state of the original bilinear systems which are based on the controllability or observability gramians. These methods result in the reduced bilinear systems which are nearly optimal for a given least upper bound. The comparison of the least upper bounds of the difference bilinear system using two methods is investigated in the paper. Another method, for example, the moment-matching method is very efficient and numerically robust, but the reduced bilinear systems are not guaranteed as an optimal reduced bilinear system.

In the high order of the bilinear systems, the bottleneck of the balanced truncation and singular perturbation methods can occur in the calculation of controllability or observability gramians. The controllability or observability gramians can be approximated in the frequency domain to reduce the computational cost. Therefore, it is suggested to use the Poor man's truncated balanced realization of the bilinear systems. This approach uses frequency-weighted finite summation to approximate the infinite integration. This method approximates the gramian in the frequency domain without solving the Lyapunov equations [20]. The reduced bilinear systems will be accurate when the bilinear systems have finite bandwidth inputs.

A class of nonlinear system which is linear in inputs and linear in states with a nonlinearity in a product of states and inputs is known as bilinear systems [3]. Mathematical modeling and control design of bilinear systems were discussed in [1] and [8]. The identification of time-invariant bilinear system models in the error-in-variables framework has been discussed in [16]. The error-in-variables framework is dedicated to problem of dynamic system identification in the presence of noise corrupting both input and output measurements. The bilinear control systems have been discussed by using the Lie groups approach in [9] and [19], whereas in [14] it has been discussed how to stabilize the homogeneous bilinear system by sliding mode control. The bilinear systems are naturally found in science and technology problems, for example induction motor drives in [1], paper making machines in [1], quantum mechanics in [19], power systems in [3], suspension systems in [26], circuit electricity in [17], and immunity problems in [18].

The control design problem of a bilinear system is to seek a controller that stabilizes and satisfies a given norm of the closed loop of the bilinear system. Many problems in science and technology are usually formulated in terms of a high order bilinear system. In fact, the order of robust control design is always higher than the order of the system so a reduced-order controller is necessary for application in real problems. Hence, model order reduction and reduced order controller are an important part in the high order control system design.

Model reduction for linear time invariant (LTI) and linear time varying (LTV) systems has been discussed in [2], whereas the model reduction for bilinear systems has been developed by many researchers in [3–7,12,15,21,22,25,27]. Model order reduction methods for nonlinear model have been discussed in [11]. Balanced truncation [3] and singular perturbation [22] methods are used to obtain the reduced order bilinear time invariant systems. In the balanced truncation method, the original bilinear system is transformed to the balanced system. The characterizations of the original bilinear system and the balanced system are the same. In the singular perturbation method, the original bilinear system is transformed into a balanced system which is then divided into two subsystems, i.e. slow and fast mode systems. After that, the reduced bilinear systems are obtained by defining that the velocity of fast mode is zero.

The paper is organized as follows. Section 2 presents the least upper bounds of the transfer function of the bilinear time invariant systems in the H_2-norm. Section 3 reviews the balanced truncation and singular perturbation methods for bilinear systems. Section 4 gives the main result that is the least upper bounds of the difference bilinear system. In Section 5, the procedure of selecting the reduced order bilinear system is presented. Section 6 shows the simulation results which illustrate the performance of the proposed algorithm and Section 7 gives conclusions.

2 The Least Upper Bounds of Bilinear Systems

Consider a bilinear time invariant system $\mathfrak{B}$ characterized by the following differential equations

$$\mathfrak{B}: \begin{aligned} \dot{x}(t) &= Ax(t) + \sum_{i=1}^{m} N_i u_i(t) x(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1}$$

where $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the control input, $u_i(t)$ is the $i-$th element of $u(t)$, $y \in \Re^q$ is the output system, $A \in \Re^{nxn}, N_i \in \Re^{nxn}, i = 1, 2, \ldots, m, B \in \Re^{nxm}$, $C \in \Re^{qxn}$, and $D \in \Re^{qxm}$. Suppose the bilinear system (1) is locally stable, (A, B) is controllable, and (A, C) is observable. The bilinear system is called locally stable if the real parts of all eigenvalues of A are negative. The relation of inputs and outputs of the bilinear system (1) can be expressed by the following Volterra series [18]

$$y(t) = \sum_{i=1}^{\infty} \int_{i=0}^{t} \int_{i=0}^{t_1} \cdots \int_{i=0}^{t_{k-1}} \sum_{i1,i2,\ldots,i_k=1}^{m} h_k^{(i_1,i_2,\ldots,i_k)}(t_1, t_2, ..., t_k)$$

$$u_{i1}(t - t_k) \ldots u_{ik}(t - \sum_{k=1}^{i} t_k) dt_1 \ldots dt_k.$$

The regular Volterra kernel h_k can be expressed as [18]

$$h_k^{(i_1,i_2,\ldots,i_k)}(t_1, t_2, ..., t_k) = Ce^{At_k} N_{i1} e^{At_{k-1}} \ldots N_{ik-1} e^{At_1} b_{i_k},$$

where b_{i_k} denotes the i_k-th column of B matrix. For the sake of simplicity, $h_k^{(i1,...,ik)}(t_1, t_2, ..., t_k)$ is denoted by h_k. The notation h_k^T denotes transpose of the h_k.

To deal with the least upper bounds problem, the paper treats the controllability and the observability gramians defined in [3] as follows

Definition 2.1 The controllability gramian matrix P is defined by

$$P = \sum_{i=1}^{\infty} \int_0^{\infty} \dots \int_0^{\infty} P_i P_i^T dt_1 \dots dt_i,$$

where $P_1(t_1) = e^{At_1}B$, and $P_i(t_1, \dots, t_i) = e^{At_i} \begin{bmatrix} N_1 P_{i-1} & \dots & N_m P_{i-1} \end{bmatrix}, i = 2, 3, \dots$. Analogously, observability gramian matrix Q is defined by

$$Q = \sum_{i=1}^{\infty} \int_0^{\infty} \dots \int_0^{\infty} Q_i^T Q_i dt_1 \dots dt_i,$$

where $Q_1(t_1) = Ce^{At_1}$, and $Q_i(t_1, \dots, t_i) = \begin{bmatrix} Q_{i-1}N_1 \\ Q_{i-1}N_2 \\ \dots \\ Q_{i-1}N_m \end{bmatrix} e^{At_i}, i = 2, 3, \dots$.

The existence and properties of the controllability gramian P and the observability gramian Q which satisfy the generalized Lyapunov equations are presented in [27]. The generalized Lyapunov equations are given by the following equations

$$AP + PA^T + \sum_{i=1}^{m} N_i P N_i^T + BB^T = 0, \tag{2}$$

$$A^T Q + QA + \sum_{i=1}^{m} N_i^T Q N_i + C^T C = 0. \tag{3}$$

If the equation (2) is taken *vec* on two sides then

$$\left(A \otimes I + I \otimes A + \sum_{i=1}^{m} N_i \otimes N_i \right) vec(P) = -vec(BB^T).$$

Therefore, if $A \otimes I + I \otimes A + \sum_{i=1}^{m} N_i \otimes N_i$ is a nonsingular matrix, then a single solution P will be found. If P is a nonnegative matrix then P is called the controllability gramian. The observability gramian Q is obtained by using the similar manner and properties to the equation (3) [27].

Let us introduce a definition of H_2-norm of the bilinear system $\mathfrak{B}$ in [23, 24].

Definition 2.2 Consider the bilinear system (1). The H_2-norm of the bilinear system $\mathfrak{B}$ is defined by

$$\|\mathfrak{B}\|_2 = \sqrt{\lambda_{max} \left(\sum_{k=1}^{\infty} \int_0^{\infty} \dots \int_0^{\infty} \sum_{i1,\dots,ik=1}^{m} h_k h_k^T dt_1 ... dt_k \right)},$$

where $\lambda_{max}(.)$ denotes the maximum of (.) eigenvalues and h_k is the regular Voltera kernel.

Definition 2.2 is an extended form of the Euclidian-induced norm of matrix M which is equivalent to the square root of the maximum eigenvalue of M^TM over a time interval of integration from $t = 0$ to $t = \infty$. It is clear that $h_kh_k^T$ is a symmetry and a semi definite positive matrix because h_k is $k-$variate impulse response. The following lemma is obtained from Definition 2.2.

Lemma 2.1 *[23, 24] Suppose the bilinear system (1) is locally stable. If there exists the controllability gramian P of bilinear system (1) then $\|\mathfrak{B}\|_2 = \sqrt{\lambda_{max}(CPC^T)}$. If there exists the observability gramian Q of bilinear system (1) then $\|\mathfrak{B}\|_2 = \sqrt{\lambda_{max}(B^TQB)}$.*

Proof. Suppose that

$$J_k^2 = \int_0^\infty \int_0^\infty \dots \int_0^\infty \sum_{i1,\dots,ik=1}^{m} h_kh_k^T dt_1 \dots dt_k.$$

When $k = 1$ then $J_1^2 = \int_0^\infty \sum_{i1=1}^m Ce^{At_1}b_{i1}b_{i1}^Te^{A^Tt_1}C^Tdt_1 = C\int_0^\infty P_1P_1^Tdt_1C^T$. When $k = 2$ then $J_2^2 = \int_0^\infty \int_0^\infty \sum_{i1=1}^m \phi\phi^T dt_1dt_2 = C\int_0^\infty \int_0^\infty P_2P_2^Tdt_1dt_2C^T$, where $\phi = Ce^{At_2}N_1e^{At_1}b_{i1}$, b_{i1} denotes the $i1$-th column of the matrix B, and generally $J_k^2 = C\int_0^\infty \dots \int_0^\infty P_kP_k^Tdt_1 \dots dt_kC^T, i = 2, 3, \dots$. Therefore, the following result will be obtained by taking the sum from $k = 1$ to infinite

$$\sum_{k=1}^\infty J_k^2 = C\sum_{k=1}^\infty \int_0^\infty \dots \int_0^\infty P_kP_k^Tdt_1 \dots dt_kC^T = CPC^T.$$

Hence, the H_2 norm can also be computed by using

$$\|\mathfrak{B}\|_2 = \sqrt{\lambda_{max}\left(\sum_{k=1}^\infty J_k^2\right)} = \sqrt{\lambda_{max}(CPC^T)},$$

where P is the controllability gramian of bilinear system (1). Similar reasoning holds for the second case. □

The least upper bounds of H_2-norm of the transfer function of the bilinear system are determined as a function of the controllability gramian (the observability gramian) of the bilinear system.

Lemma 2.2 *[23, 24] Suppose the bilinear system (1) is locally stable. If there exists the controllability gramian P of bilinear system (1) then $\|\mathfrak{B}\|_2 < \sqrt{\lambda_{max}(P)}\sqrt{\lambda_{max}(C^TC)}$. If there exists the observability gramian Q of bilinear system (1) then $\|\mathfrak{B}\|_2 \leq \sqrt{\lambda_{max}(Q)}\sqrt{\lambda_{max}(BB^T)}$.*

Proof. We shall furnish the proof for the controllability gramian P, having the same arguments for the observability gramian Q. As the controllability gramian P exists, then P is a positive definite matrix. Furthermore, C^TC is a positive semidefinite matrix. According to Lemma 2.1 and properties of the eigenvalues of positive semidefinite matrix, it holds that $\|\mathfrak{B}\|_2 = \sqrt{\lambda_{max}\left(CPC^T\right)} \leq \sqrt{\lambda_{max}\left(P\right)}\sqrt{\lambda_{max}\left(C^TC\right)}$. □

3 Balanced Truncation and Singular Perturbation Methods

According to [3], balanced realization of the bilinear system (1) can be obtained by applying the state space balancing transformation $x_b(t) = T^{-1}x(t)$ to (1). Hence, the new presentation will be obtained as follows

$$\mathfrak{B}_b : \begin{array}{l} \dot{x}_b(t) = A_b x_b(t) + \sum_{i=1}^{m} N_{bi} u_i(t) x_b(t) + B_b u(t), \\ y(t) = C_b x_b(t), \end{array} \tag{4}$$

where $A_b = T^{-1}AT, N_{bi} = T^{-1}N_iT, B_b = T^{-1}B, C_b = CT$. The controllability and the observability gramians of the balanced system are $P_b = T^{-1}PT^{-T}$ and $Q_b = T^TQT$. Furthermore, the system (4) is denoted by $(A_b, B_b, N_{bi}, C_b, D_d)$, $i = 1, ..., m$.

Definition 3.1 *The system $(A_b, B_b, N_{bi}, C_b, D_b), i = 1, ..., m$ is called the balanced realization of the bilinear system (1) if*

$$P_b = Q_b = \Sigma = diag(\sigma_1, \sigma_2, ..., \sigma_n), \sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n \geq 0,$$

where P_b and Q_b are the controllability gramian and the observability gramian, respectively. Furthermore, $\sigma_k = \sqrt{\lambda_k(P_bQ_b)}, k = 1, ..., n$ is called Hankel singular value of the balanced system, where $\lambda_k(P_bQ_b)$ denotes the k-th eigenvalue of the matrix P_bQ_b.

The balanced system (4) can be partitioned as follows

$$\begin{bmatrix} \dot{x}_{b_1} \\ \dot{x}_{b_2} \end{bmatrix} = \begin{bmatrix} A_{b_{11}} & A_{b_{12}} \\ A_{b_{21}} & A_{b_{22}} \end{bmatrix} \begin{bmatrix} x_{b_1} \\ x_{b_2} \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} N_{b_{11_i}} & N_{b_{12_i}} \\ N_{b_{21_i}} & N_{b_{22_i}} \end{bmatrix} \begin{bmatrix} x_{b_1} \\ x_{b_2} \end{bmatrix} u_i + \begin{bmatrix} B_{b_1} \\ B_{b_2} \end{bmatrix} u,$$

$$y = \begin{bmatrix} C_{b_1} & C_{b_2} \end{bmatrix} \begin{bmatrix} x_{b_1} \\ x_{b_2} \end{bmatrix},$$

where $\dot{x}_{b_1}$ is the velocity of slow mode and $\dot{x}_{b_2}$ is the velocity of fast mode. In the balanced truncation method, the system of the slow mode is selected as the reduced bilinear system. The system which is obtained by the balanced truncation method can preserve the stability, but this method gives high error at low frequencies. Let Σ be partitioned as $\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$, where $\Sigma_1 = diag[\sigma_1, \sigma_2, ..., \sigma_r]$ and $\Sigma_2 = diag[\sigma_{r+1}, \sigma_{r+2}, ..., \sigma_n]$. According to [3], the order selection of the slow mode is based on the ratio of Hankel singular values that is $\frac{\sigma_r}{\sigma_{r+1}} \gg 1$, then, the reduced bilinear system where order r is chosen. Furthermore, the balanced truncation method for bilinear systems has been developed to the singular perturbation method for bilinear systems in [22]. Denote

$$K = A_{b_{12}} + \sum_{i=1}^{m} N_{b_{12_i}} u_i(t), L = A_{b_{22}} + \sum_{i=1}^{m} N_{b_{22_i}} u_i(t), M = A_{b_{21}} + \sum_{i=1}^{m} N_{b_{21_i}} u_i(t),$$

and assume that the velocity of the fast mode is zero, then
$x_{b_2}(t) = -L^{-1}Mx_{b_1}(t) - L^{-1}B_{b_2}u(t)$. Therefore, the reduced bilinear system is given by

$$\dot{x}_{b_1}(t) = (A_{b_{11}} - KL^{-1}M)x_{b_1} + \sum_{i=1}^{m} N_{b_{11_i}} x_{b_1}(t) u_i(t) + (B_{b_1} - KL^{-1})u(t),$$

$$y(t) = (C_{b_1} - C_{b_2}L^{-1}M)x_{b_1}(t).$$

The reduced order bilinear system using the balanced truncation or the singular perturbation methods can be presented by

$$\mathfrak{B}_r : \begin{array}{l} \dot{x}_r(t) = A_r x_r(t) + \sum_{i=1}^{m} N_{ri}u_i(t)x_r(t) + B_r u(t), \\ y_r(t) = C_r x_r(t), \end{array} \tag{5}$$

where $x_r \in \Re^r, r < n, y_r \in \Re^p$, A_r is stable, (A_r, B_r) is controllable and r is order of the reduced bilinear systems.

4 The Least Upper Bounds of the Difference Bilinear Systems

Consider the full order model (1) and the reduced order model (5) of the bilinear system. The difference bilinear system is defined as a system in which the transfer function is the difference of transfer function between the full order system (1) and the reduced order system (5) of a bilinear system. The difference of the transfer matrix k-variate of the full order model and the reduced order model of the bilinear system is obtained as follows:

$$h_{i_1,\ldots,i_k}(t_1,\ldots,t_k) - h_{ri_1,\ldots,ri_k}(t_1,\ldots,t_k) = \begin{bmatrix} C & -C_r \end{bmatrix} e^{\begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix} t_k}$$

$$\begin{bmatrix} N_{i1} & 0 \\ 0 & N_{ri1} \end{bmatrix} e^{\begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix} t_{k-1}} \begin{bmatrix} N_{i2} & 0 \\ 0 & N_{ri2} \end{bmatrix}$$

$$\cdots \begin{bmatrix} N_{i_k-1} & 0 \\ 0 & N_{ri_k-1} \end{bmatrix} e^{\begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix} t_1} \begin{bmatrix} b_{i_k} \\ b_{ri_k} \end{bmatrix}.$$

The difference of the transfer matrix k-variate leads to the difference bilinear system given by

$$\mathfrak{B}_d : \begin{array}{l} \begin{bmatrix} \dot{x} \\ \dot{x}_r \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix} \begin{bmatrix} x \\ x_r \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} N_i & 0 \\ 0 & N_{r_i} \end{bmatrix} \begin{bmatrix} x \\ x_r \end{bmatrix} u_i + \begin{bmatrix} B \\ B_r \end{bmatrix} u, \\ y - y_r = \begin{bmatrix} C & -C_r \end{bmatrix} \begin{bmatrix} x \\ x_r \end{bmatrix}. \end{array} \tag{6}$$

Suppose $\bar{P}$ and $\bar{Q}$ are the controllability gramian and the observability gramian of the difference bilinear system (6), respectively. Therefore, $\bar{P}$ and $\bar{Q}$ are nonnegative matrices and the two following generalized Lyapunov equations are satisfied

$$F\bar{P} + \bar{P}F^T + \sum_{i=1}^{m} H_i \bar{P} H_i^T + S = 0, \tag{7}$$

$$F^T\bar{Q} + \bar{Q}F + \sum_{i=1}^{m} H_i^T \bar{Q} H_i + M = 0, \tag{8}$$

where $F = \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix}$, $H_i = \begin{bmatrix} N_i & 0 \\ 0 & N_{r_i} \end{bmatrix}$, $S = \begin{bmatrix} BB^T & BB_r^T \\ B_rB^T & B_rB_r^T \end{bmatrix}$, and $M = \begin{bmatrix} C^TC & -C^TC_r \\ -C_r^TC & C_r^TC_r \end{bmatrix}$.

Furthermore, the least upper bounds of the error transfer function between the full order (1) and the reduced order (5) of the bilinear time invariant systems in the H_2-norm are given by the following theorem.

Theorem 4.1 *Consider the order of the bilinear system (1) is n and the order of the reduced bilinear system (5) is $r, r = 1, 2, ..., n-1$. Suppose A and A_r are locally stable. If there exists the controllability gramian $\bar{P}$ of the difference bilinear system (6) then*

$$\|\mathfrak{B} - \mathfrak{B}_r\|_2 \leq \sqrt{\lambda_{max}(\bar{P})}\sqrt{\lambda_{max}(M)}, \forall r.$$

If there exist the observability gramian $\bar{Q}$ of the difference bilinear system (6) then

$$\|\mathfrak{B} - \mathfrak{B}_r\|_2 \leq \sqrt{\lambda_{max}(\bar{Q})}\sqrt{\lambda_{max}(S)}, \forall r.$$

Proof. Because A and A_r are locally stable then $F = \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix}$ is locally stable. By using Lemma 2.2 and the controllability gramian $\bar{P}$ of the difference bilinear system (6) (the observability gramian $\bar{Q}$ of the difference bilinear system (6)), the least upper bounds as on the right hand side are obtained. □

The results for the linear time invariant systems (LTIS) as a special case of the bilinear time invariant systems when $N_i = 0, \forall i$ is given by the following

Corollary 4.1 *If $N_i = 0, \forall i$, then (1) will become the linear time invariant system (LTIS). The least upper bound of the transfer function of the LTIS in the H_2-norm is*

$$\sqrt{\lambda_{max}(P)}\sqrt{\lambda_{max}(C^TC)},$$

where P is the controllability gramian of the LTIS. The least upper bound of the H_2-norm of the difference of the transfer function for the difference of LTIS is

$$\sqrt{\lambda_{max}(\bar{P})}\sqrt{\lambda_{max}(M)},$$

where $\bar{P}$ is the controllability gramian of the difference of LTIS.

5 Procedure to Select the Reduced Order Bilinear System

The following algorithm is used to show that the least upper bounds of the H_2-norm of the transfer function of the difference bilinear systems are valid. The algorithm can also be used to choose the reduced order bilinear system which is similar to the full order bilinear system. The input of the algorithm is a bilinear system (1), where $A, B, N_i, C, i = 1, 2, 3, ..., m$ are matrices of suitable dimensions and the order of the bilinear system is n.

- Step 1: Choose the method to obtain the reduced order bilinear system.

  1. Reduce the bilinear system (1) by using the balance truncation method.

2. Reduce the bilinear system (1) by using the singular perturbation method.

- Step 2: Calculate the H_2-norm and the least upper bounds of the difference bilinear system.

1. Suppose $\beta_{rBT} = \|\mathfrak{B} - \mathfrak{B}_r\|_2$ denotes H_2-norm of the transfer function of the difference bilinear systems with the reduced r-th order bilinear systems, $r = 1, 2, ..., n-1$ using the balanced truncation method. Calculate β_{rBT} by Lemma 2.1 where the gramian matrix $\bar{P}$ satisfies (7). Next, calculate the least upper bounds γ_{rBT} by using Theorem 4.1. It is clear that $\beta_{rBT} < \gamma_{rBT}, \forall r = 1, 2, ..., n-1$. The index BT denotes the balanced truncation method.

2. Suppose γ_{rBT} denotes the least upper bound of the difference bilinear systems with the reduced r-th order bilinear system which is reduced by using the balanced truncation method. Suppose the index SP denotes the singular perturbation method. Calculate β_{rSP} by Lemma 2.1, where the gramian matrix $\bar{P}$ satisfies (8). Next, calculate the least upper bounds γ_{rSP} by using Theorem 4.1. It is also clear that $\beta_{rSP} < \gamma_{rSP}, \forall r$.

- Step 3: Choose the smallest r of the reduced order bilinear systems $\mathfrak{B}_r$ such that $\frac{\gamma_{(r-1)BT}}{\gamma_{rBT}} \approx 1$, or $\frac{\gamma_{(r-1)SP}}{\gamma_{rSP}} \approx 1$, where γ_{rBT} is the least upper bound of the transfer function of the difference of the bilinear systems with the reduced r-th order bilinear system using the balanced truncation method, $\gamma_{(r-1)BT}$ for order $r-1$. The index SP is for the singular perturbation method.

6 Simulation Results

Consider the circuit bilinear time invariant system as in [17] as follows

$$
\mathfrak{B}: \begin{aligned}
\dot{x}(t) &= \begin{bmatrix} -5 & 2 & 0 & \dots & 0 \\ 2 & -5 & 2 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 2 & -5 & 2 \\ 0 & 0 & 0 & 2 & -5 \end{bmatrix} + \begin{bmatrix} 0 & -3 & 0 & \dots & 0 \\ 3 & 0 & -3 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 3 & 0 & -3 \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix} u_1(t)x(t) \\
&+ \begin{bmatrix} 1 & 3 & 0 & \dots & 0 \\ -3 & 1 & 3 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & -3 & 1 & 3 \\ 0 & 0 & 0 & -3 & 1 \end{bmatrix} u_2(t)x(t) + \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{bmatrix} u(t), \\
y(t) &= \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \end{bmatrix} x(t).
\end{aligned}
$$

Furthermore, the simulation of circuit bilinear system with order 25 and 15 is presented. The H_2-norm and the least upper bounds of the difference bilinear system with order 25 and 15 are obtained by using the proposed algorithm as shown in Figures 1 and 2. It is found that $\beta_{rBT} < \gamma_{rBT}$ for each r. When the order of the reduced bilinear system is increased, the value of $\|\mathfrak{B} - \mathfrak{B}_r\|_2$ is decreased and the least upper bounds of the difference of bilinear system are increased. According to Definition 3.1, the Hankel singular values of the circuit bilinear system and the ratio of the Hankel singular values are presented in Table 1. The ratio of the Hankel singular value of each order of the reduced bilinear system from 2 up to 14 is near to 1. Therefore, the order of the reduced bilinear system is not easy to be determined because it depends on knowledge of the decision makers.

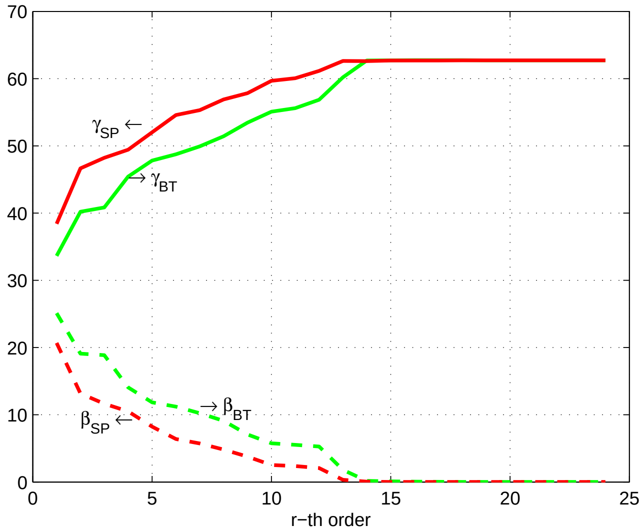

Figure 1: The H_2-norm β and least upper bound γ of the difference bilinear system.

For the order of the circuit system is 25, the order of the reduced bilinear systems can be chosen to the 10-th order when the balance truncation method is used to obtain the reduced bilinear system and to the 13-th order when the singular perturbation method is used. The output of the circuit bilinear system is presented in Figures 3 and 4. For the 11-th order reduced bilinear system, the response of the reduced bilinear system is not similar to that of the full order, so it is not recommended as the reduced order model.

The reduced circuit system by using the two methods will have nearly the same response when the order of the reduced bilinear system is 13. For the order of the circuit system is 15, the order of the reduced bilinear systems can be chosen to the 4-th order when the balanced truncation method is used to obtain the reduced bilinear system. The reduced circuit system by using the two methods will have nearly the same response when the order of the reduced bilinear system is 8. The outputs of the circuit bilinear system are shown in Figure 5 for the 4-th order reduced circuit bilinear system.

7 Conclusions

The least upper bounds of the difference bilinear time invariant systems were derived by defining the H_2-norm of the bilinear systems in terms of the error transfer function. The least upper bounds of the difference bilinear system were presented by the controllability gramian or the observability gramian of the difference bilinear system. The results were

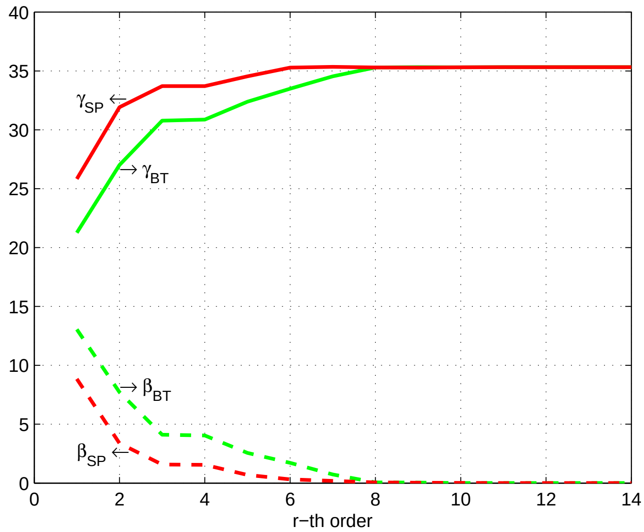

Figure 2: The H_2-norm β and least upper bound γ of the difference bilinear system.

Order	$\sigma_i, i = 1, 2, ..., n = 25$	$\mathfrak{R}_k, k = 1, 2, ..., 24$	$\sigma_i, i = 1, 2, ..., n = 15$	$\mathfrak{R}_k, k = 1, 2, ..., 14$
1	4.5536	5.8207	3.4232	4.3639
2	0.7823	1.9669	0.7844	1.9817
3	0.3977	1.0126	0.3958	1.0977
4	0.3928	1.3096	0.3606	1.1909
5	0.2999	1.2173	0.3028	1.2161
6	0.2464	1.3369	0.2490	1.3118
7	0.1843	1.2429	0.1898	1.2736
8	0.1483	1.1669	0.1490	1.3711
9	0.1271	1.3048	0.1087	1.6485
10	0.0974	1.3010	0.0659	1.8538
11	0.0749	1.2387	0.0356	2.0791
12	0.0604	1.1717	0.0171	2.3844
13	0.0516	1.1419	0.0072	2.8787
14	0.0452	1.4572	0.0025	3.9790
15	0.0310	1.6138	0.0006	
16	0.0192	1.7376		
17	0.0111	1.8585		
18	0.0059	1.9983		
19	0.0030	2.1563		
20	0.0014	2.3551		
21	0.0006	2.6129		
22	0.0002	2.9861		
23	0.0001	3.6031		
24	0.0000	5.0032		
25	0.0000			

Table 1: Hankel singular value $\sigma_i, i = 1, 2, ..., n$ for the circuit bilinear system and its ratios $\mathfrak{R}_k = \frac{\sigma_k}{\sigma_{k+1}}, k = 1, 2, ..., n-1$.

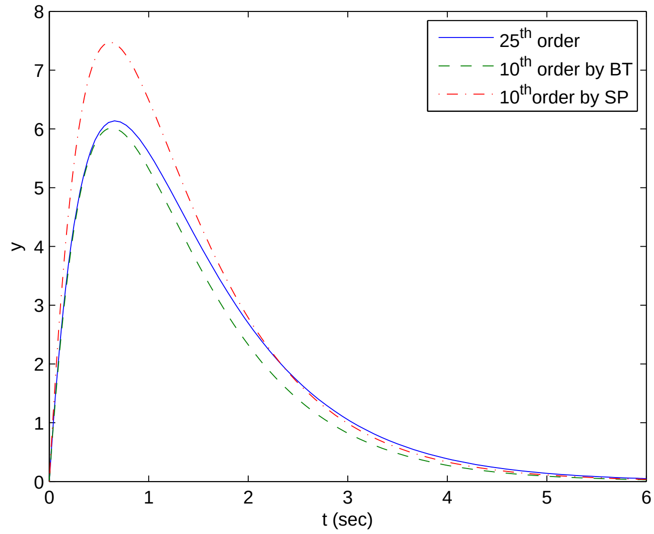

Figure 3: The output of the circuit bilinear system, BT: balanced truncation, SP: singular perturbation.

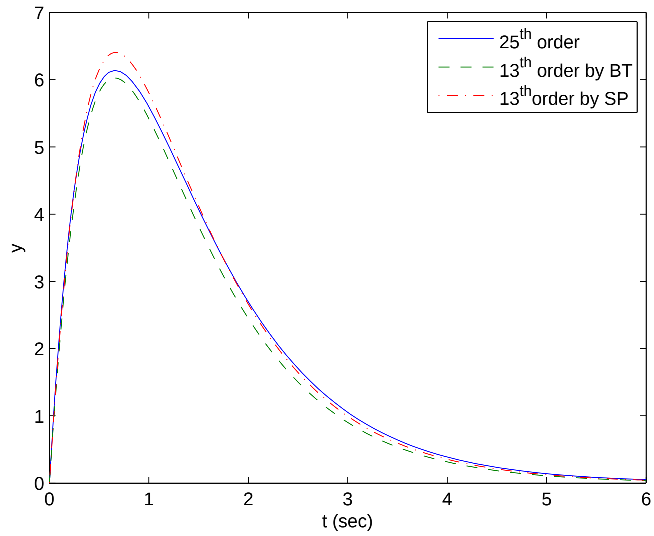

Figure 4: The output of the circuit bilinear system, BT: balanced truncation, SP: singular perturbation.

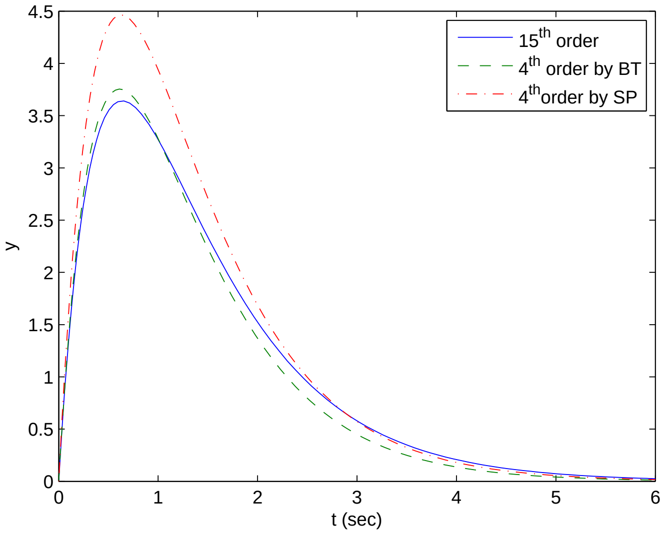

Figure 5: The output of the circuit bilinear system, BT: balanced truncation, SP: singular perturbation.

also valid for the linear time invariant systems as a special case. The value of the $\|\mathfrak{B} - \mathfrak{B}_r\|_2$ decreased as the order of the reduced bilinear system was closer to the full order bilinear system.

The order selection of the reduced bilinear system was based on the alteration value of the least upper bounds or the value alteration of $\|\mathfrak{B} - \mathfrak{B}_r\|_2$. The proposed method was easier than using the alteration of the singular Hankel values. The least upper bounds of the transfer function of the bilinear system in H_2-norm are a function of the controllability gramian or the observability gramian of the bilinear system. The simulation result showed that the balanced truncation method was better than the singular perturbation method when the system frequency is low and vice versa. Therefore, the order of the reduced bilinear system can be chosen to be smaller when using the balanced truncation method although H_2-norm of difference bilinear system was greater when using the singular perturbation method.

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