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Multivalued Homogeneous Neumann Problem Involving Diffuse Measure Data and Variable Exponent

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Abstract: We study a nonlinear elliptic problem with homogeneous Neumann boundary condition, governed by a general Leray-Lions operator with variable exponents and Radon measure data which does not charge the sets of zero $p(.)$-capacity. We prove an existence and uniqueness result of weak solution.

Keywords: *Neumann boundary condition; diffuse measure; biting lemma of Chacon; maximal monotone graph; Radon measure data; weak solution; entropic solution; Leray-Lions operator.*

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1 Introduction and Main Results

Our aim is to study the existence and uniqueness of a solution for nonlinear homogeneous Neumann boundary value problem of the form

$$N(\beta,\mu)\begin{cases} -\nabla\cdot a(x,\nabla u)+\beta(u)\ni\mu & \text{in } \Omega,\\ a(x,\nabla u).\eta=0 & \text{on } \partial\Omega,\end{cases}$$

where η is the unit outward normal vector on $\partial\Omega$, β is a maximal monotone graph on $\mathbb{R}$ such that $0\in\beta(0)$, a is a Leray-Lions operator, μ is a diffuse measure such that $\mu=\mu\lfloor\Omega$

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and $\Omega \subset \mathbb{R}^N$ is a smooth open bounded domain ($N \geq 1$). We set $\overline{\text{dom}(\beta)} = [m, M] \subset \mathbb{R}$ with $m \leq 0 \leq M$.

Recall that a Leray-Lions operator which involves variable exponents is a Carathéodory function $a(x,\xi) : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ (i.e. $a(x,\xi)$ is continuous in ξ for a.e. $x \in \Omega$ and measurable in x for every $\xi \in \mathbb{R}^N$) such that:

• There exists a positive constant C_1 such that

$$|a(x,\xi)| \leq C_1(j(x) + |\xi|^{p(x)-1}) \tag{1}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$ where j is a nonnegative function in $L^{p'(.)}(\Omega)$, with $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$.

• The following inequalities hold

$$(a(x,\xi) - a(x,\eta)).(\xi - \eta) > 0 \tag{2}$$

for almost every $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^N$, with $\xi \neq \eta$, and there exists $C > 0$ such that

$$\frac{1}{C}|\xi|^{p(x)} \leq a(x,\xi).\xi, \tag{3}$$

for almost every $x \in \Omega$, and for every $\xi \in \mathbb{R}^N$.

In this paper, we make the following assumption on the variable exponent:

$$p(.) : \overline{\Omega} \to \mathbb{R} \text{ is a continuous function such that } 1 < p_- \leq p_+ < +\infty, \tag{4}$$

where $p_- := \operatorname{ess}\inf_{x\in\Omega} p(x)$ and $p_+ := \operatorname{ess}\sup_{x\in\Omega} p(x)$.

We denote by $\mathcal{L}^N$ the N-dimensional Lebesgue measure of $\mathbb{R}^N$ and by $\mathcal{M}_b(X)$ the space of bounded Radon measure in X, equipped with its standard norm $||.||_{\mathcal{M}_b(X)}$. Given $\nu \in \mathcal{M}_b(X)$, we say that ν is diffuse with respect to the capacity $W^{1,p(.)}(X)$($p(.)-$capacity for short) if $\nu(B) = 0$ for every set B such that $Cap_{p(.)}(B, X) = 0$, where the Sobolev $p(.)-$capacity of B is defined by

$$Cap_{p(.)}(B, X) = \inf_{u \in S_{p(.)}(B)} \int_X \left(|u|^{p(x)} + |\nabla u|^{p(x)}\right) dx,$$

with

$$S_{p(.)}(B) = \{u \in W_0^{1,p(.)}(X) : u \geq 1 \text{ in an open set containing } B \text{ and } \ u \geq 0 \text{ in } X\}.$$

In the case $S_{p(.)}(B) = \emptyset$, we set $Cap_{p(.)}(B, X) = +\infty$.

The set of bounded Radon diffuse measure in the variable exponent setting is denoted by $\mathcal{M}_b^{p(.)}(X)$.

Elliptic problems with measures data in the context of constant exponent was studied by many authors (see [4–6,10,12]). The multivalued case for Dirichlet boundary condition with constant exponent was studied by some authors among whose papers one can cite the most recent one by Igbida *et als* [14]. The study of multivalued elliptic problems with measure data in the context of variable exponent was carried out for the first time by Nyanquini *et als* [16] under homogeneous Dirichlet Boundary condition. In [16], the authors first proved a decomposition theorem for the measure data (more precisely, as a sum of a function in $L^1(\Omega)$ and of a measure in $W^{-1,p'(.)}(\Omega)$) and used it to prove, following [14], a result on existence and uniqueness of entropy solution of the problem considered.

In this paper, we consider Neumann homogeneous boundary condition. Since the boundary condition is the Neumann condition, we cannot work with the common space $W_0^{1,p(.)}(\Omega)$ in which, we can use the Poincaré inequality but also, when one uses the integration by parts formula, the term which appears at the boundary due to the part of the measure in $W^{-1,p'(.)}(\Omega)$, vanishes. We have to work in the space $W^{1,p(.)}(\Omega)$. The first main difficulty which appears in this case is that for the proof of some a priori estimates, the famous Poincaré inequality doesn't apply, and neither do the Poincaré-Wirtinger inequality and the Poincaré-Sobolev inequality (since we have homogeneous Neumann condition). A second main difficulty is that, when one uses the integration by parts formula in the Yosida approximated problem (see problem $N(\beta_\epsilon, \mu_\epsilon)$ below), a term which cannot vanish appears at the boundary, for the part of the measure data which is in $W^{-1,p'(.)}(\Omega)$. In order to treat this difficulty, we consider a smooth domain Ω in order to work with the space $W_0^{1,\widetilde{p}(.)}(U_\Omega)$, where $\widetilde{p}(.) : U_\Omega \to (1, \infty)$ is continuous such that $\widetilde{p}(x) = p(x)$ for all $x \in \overline{\Omega}$, and to go back later to the space $W^{1,p(.)}(\Omega)$. More precisely, Ω is assumed to be a bounded domain in $\mathbb{R}^N$ with a boundary $\partial\Omega$ of class C^1. Then, Ω is an extension domain (see [8]), so we can fix an open bounded subset U_Ω of $\mathbb{R}^N$ such that $\overline{\Omega} \subset U_\Omega$, and there exists a bounded linear operator

$$E : W^{1,p(.)}(\Omega) \to W_0^{1,\widetilde{p}(.)}(U_\Omega),$$

for which
(i) $E(u) = u$ a.e. in Ω for each $u \in W^{1,p(.)}(\Omega)$,
(ii) $\|E(u)\|_{W_0^{1,\widetilde{p}(.)}(U_\Omega)} \leq C\|u\|_{W^{1,p(.)}(\Omega)}$, where C is a constant depending only on Ω.

We define

$$\mathfrak{M}_b^{p(.)}(\Omega) := \{\mu \in \mathcal{M}_b^{\widetilde{p}(.)}(U_\Omega) : \ \mu \text{ is concentrated on } \Omega\}.$$

This definition is independent of the open set U_Ω. Note that for $u \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ and $\mu \in \mathfrak{M}_b^{p(.)}(\Omega)$, we have

$$\langle \mu, E(u) \rangle = \int_\Omega u \, d\mu.$$

On the other hand, as μ is diffuse (cf. Theorem 3.1 below), there exist $f \in L^1(U_\Omega)$ and $F \in (L^{\widetilde{p}'(.)}(U_\Omega))^N$ such that $\mu = f -$ div(F) in $\mathcal{D}'(U_\Omega)$. Therefore, we can also write

$$\langle \mu, E(u) \rangle = \int_{U_\Omega} f E(u) \, dx + \int_{U_\Omega} F.\nabla E(u) \, dx.$$

Now, define the following spaces which are similar to that introduced in [1, 3] (see also [7]). We note

$$\mathcal{T}^{1,p(.)}(\Omega) := \big\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable}; \ \ T_k(u) \in W^{1,p(.)}(\Omega) \text{ for all } k > 0\big\}.$$

As in [3], we can prove that for $u \in \mathcal{T}^{1,p(.)}(\Omega)$, there exists a unique measurable function $w : \Omega \longrightarrow \mathbb{R}$ such that $\nabla T_k(u) = w\chi_{\{|u|<k\}}$ $\forall k > 0$. This function w will be denoted by ∇u.

We define $\mathcal{T}_{\mathcal{H}}^{1,p(.)}(\Omega)$ (see [7]) as the set of functions $u \in \mathcal{T}^{1,p(.)}(\Omega)$ such that there exists a sequence $(u_\delta)_\delta \subset W^{1,p(.)}(\Omega)$ satisfying the following conditions:

(i) $u_\delta \longrightarrow u$ a.e. in Ω as $\delta \to 0$.
(ii) $\nabla T_k(u_\delta) \longrightarrow \nabla T_k(u)$ in $L^1(\Omega)$ for any $k > 0$ as $\delta \to 0$.
The symbol $\mathcal{H}$ in the notation is related to the fact that we consider here homogeneous Neumann boundary condition.

Our main results are the following theorems.

Theorem 1.1 *For any $\mu \in \mathfrak{M}_b^{p(\cdot)}(\Omega)$, the problem $N(\beta, \mu)$ has at least one solution (u, w, ν) in the sense that*

$$(u, w, \nu) \in W^{1,p(\cdot)}(\Omega) \times L^1(\Omega) \times \mathcal{M}_b^{p(\cdot)}(\Omega)$$

such that
(i) $u \in dom(\beta)$ $\mathcal{L}^N -$ a.e. in Ω,
(ii) $w \in \beta(u)$ $\mathcal{L}^N -$ a.e. in Ω,
(iii) $\nu \perp \mathcal{L}^N$, ν^+ is concentrated on $[u = M]$, ν^- is concentrated on $[u = m]$,
(iv) for any $\varphi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega a(x, \nabla u).\nabla\varphi\, dx + \int_\Omega w\varphi\, dx + \int_\Omega \varphi\, d\nu = \int_\Omega \varphi\, d\mu. \tag{5}$$

The uniqueness of the solution is given in the following theorem.

Theorem 1.2 *Let (u_1, w_1, ν_1) and (u_2, w_2, ν_2) be two solutions of $N(\beta, \mu)$. Then*

$$\begin{cases} u_1 - u_2 = c \ \text{a.e. in } \Omega, \\ w_1 = w_2 \ \text{a.e. in } \Omega, \\ \nu_1 = \nu_2. \end{cases} \tag{6}$$

Moreover,

$$\nu^+ \le \mu_s \ \lfloor [u = M] \tag{7}$$

and

$$\nu^- \le -\mu_s \ \lfloor [u = m]. \tag{8}$$

2 Preliminary

As the exponent $p(.)$ appearing in (1) and (3) depends on the variable x, we must work with Lebesgue and Sobolev spaces with variable exponents. We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable function $u : \Omega \longrightarrow \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)}\, dx$$

is finite. If the exponent is bounded, i.e., if $p_+ < +\infty$, then the expression

$$|u|_{p(.)} := \inf\{\lambda > 0 : \rho_{p(.)}(u/\lambda) \le 1\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxembourg norm. The space $(L^{p(\cdot)}(\Omega), |.|_{p(.)})$ is a separable Banach space. Moreover, if $1 < p_- \le p_+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\dfrac{1}{p(x)}+\dfrac{1}{p'(x)}=1$. Finally, we have the Hölder type inequality:

$$\Big|\int_\Omega uv\,dx\Big| \le \Big(\frac{1}{p_-}+\frac{1}{(p')_-}\Big)|u|_{p(\cdot)}|v|_{p'(\cdot)}, \tag{9}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.

Now, let

$$W^{1,p(\cdot)}(\Omega) := \Big\{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\Big\},$$

which is a Banach space equipped with the following norm

$$||u||_{1,p(.)} = |u|_{p(.)} + |(|\nabla u|)|_{p(.)}.$$

The space $\Big(W^{1,p(\cdot)}(\Omega), ||u||_{1,p(.)}\Big)$ is a separable and reflexive Banach space. For the interested reader, more details about Lebesgue and Sobolev spaces with variable exponent can be found in [11, 15].

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(.)}$ of the space $L^{p(\cdot)}(\Omega)$. We have the following result (cf. [13]):

Lemma 2.1 *If $u_n, u \in L^{p(\cdot)}(\Omega)$ and $p_+ < +\infty$, then the following properties hold:*
i) $|u|_{p(.)} > 1 \Longrightarrow |u|^{p_-}_{p(.)} \le \rho_{p(.)}(u) \le |u|^{p_+}_{p(.)}$;
ii) $|u|_{p(.)} < 1 \Longrightarrow |u|^{p_+}_{p(.)} \le \rho_{p(.)}(u) \le |u|^{p_-}_{p(.)}$;
iii) $|u|_{p(.)} < 1$ *(respectively* $= 1; > 1$*)* $\Longleftrightarrow \rho_{p(.)}(u) < 1$ *(respectively* $= 1; > 1$*);*
iv) $|u_n|_{p(.)} \longrightarrow 0$ *(respectively* $\longrightarrow +\infty$*)* $\Longleftrightarrow \rho_{p(.)}(u_n) \longrightarrow 0$ *(respectively* $\longrightarrow +\infty$*);*
v) $\rho_{p(.)}\big(u/|u|_{p(.)}\big) = 1$.

For a measurable function $u : \Omega \to \mathbb{R}$, we introduce the functional

$$\rho_{1,p(.)}(u) := \int_\Omega |u|^{p(x)}\,dx + \int_\Omega |\nabla u|^{p(x)}\,dx.$$

Then, we have the following lemma (see [17, 18]).

Lemma 2.2 *If $u_n, u \in W^{1,p(\cdot)}(\Omega)$ and $p_+ < +\infty$, then the following properties hold:*
(i) $\|u\|_{1,p(.)} > 1 \Longrightarrow \|u\|^{p_-}_{1,p(.)} \le \rho_{1,p(.)}(u) \le \|u\|^{p_+}_{1,p(.)}$;
(ii) $\|u\|_{1,p(.)} < 1 \Longrightarrow \|u\|^{p_+}_{1,p(.)} \le \rho_{1,p(.)}(u) \le \|u\|^{p_-}_{1,p(.)}$;
(iii) $\|u\|_{1,p(.)} < 1$ *(respectively* $= 1; > 1$*)* $\Longleftrightarrow \rho_{1,p(.)}(u) < 1$ *(respectively* $= 1; > 1$*);*
(iv) $\|u_n\|_{1,p(.)} \longrightarrow 0$ *(respectively* $\longrightarrow +\infty$*)* $\Longleftrightarrow \rho_{1,p(.)}(u_n) \longrightarrow 0$ *(respectively* $\longrightarrow +\infty$*).*

For any given $l, k > 0$, we define the function h_l by $h_l(r) = \min\big((l+1-|r|)^+, 1\big)$ and the truncation function $T_k : \mathbb{R} \to \mathbb{R}$ by $T_k(s) = \max\{-k, \min(k, s)\}$.

For any $l_0 > 0$, we consider a function h_0 such that
(i) $h_0 \in C^1_c(\mathbb{R})$, $h_0(r) \ge 0$, for all $r \in \mathbb{R}$,
(ii) $h_0(r) = 1$ if $|r| \le l_0$ and $h_0(r) = 0$ if $|r| \ge l_0 + 1$.

Let γ be a maximal monotone operator defined on $\mathbb{R}$. We recall the definition of the main section γ_0 of γ:

$$\gamma_0(s) = \begin{cases} \text{the element of minimal absolute value of } \gamma(s), & \text{if } \gamma(s) \neq \phi, \\ +\infty, & \text{if } [s, +\infty) \cap D(\gamma) = \phi, \\ -\infty, & \text{if } (-\infty, s] \cap D(\gamma) = \phi. \end{cases}$$

We write for any $u : \Omega \to \mathbb{R}$ and $k \geq 0, \{|u| \leq k(< k, > k, \geq k, = k)\}$ for the set $\{x \in \Omega/|u(x)| \leq k(< k, > k, \geq k, = k)\}$.

To end this section, we give a useful convergence result.

Lemma 2.3 *(Lebesgue generalized convergence theorem) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions and f be a measurable function such that $f_n \to f$ a.e. in Ω. Let $(g_n)_{n\in\mathbb{N}} \subset L^1(\Omega)$ such that for all $n \in \mathbb{N},\ |f_n| \leq g_n$ a.e. in Ω and $g_n \to g$ in $L^1(\Omega)$. Then*

$$\int_\Omega f_n\, dx \to \int_\Omega f\, dx.$$

3 Decomposition of a Measure in $\mathcal{M}_b^{p(.)}(X)$

Let X be an open subset of $\mathbb{R}^N$. We have the following result.

Theorem 3.1 *Let $p(.) : \overline{X_1} \subset X \longrightarrow [1, +\infty]$ with $1 < p_- \leq p_+ < +\infty$ be a continuous function and $\mu \in \mathcal{M}_b(X)$. Then $\mu \in \mathcal{M}_b^{p(.)}(X)$ if and only if $\mu \in L^1(X) + W^{-1,p'(.)}(X)$.*

Proof. The proof of Theorem 3.1 is carried out in the same way as in [16], Theorem 1.2.

4 Proof of Theorem 1.1

For every $\epsilon > 0$, we consider the Yosida regularisation β_ϵ of β given by

$$\beta_\epsilon = \frac{1}{\epsilon}\big(I - (I + \epsilon\beta)^{-1}\big).$$

In accordance to [9], there exists a nonnegative, convex and l.s.c. function j defined on $\mathbb{R}$, such that $\beta = \partial j$. To regularize β, we consider

$$j_\epsilon(s) = \min_{r\in\mathbb{R}} \Big\{\frac{1}{2\epsilon}|s - r|^2 + j(r)\Big\},\ \forall s \in \mathbb{R},\ \forall \epsilon > 0.$$

According to ([9], Proposition 2.11) we have

(i) $\mathrm{dom}(\beta) \subset \mathrm{dom}(j) \subset \overline{\mathrm{dom}(j)} \subset \overline{\mathrm{dom}(\beta)}$.

(ii) $j_\epsilon(s) = \frac{\epsilon}{2}\big|\beta_\epsilon(s)\big|^2 + j(J_\epsilon)$ where $J_\epsilon = (I + \epsilon\beta)^{-1}$,

(iii) j_ϵ is convex, Frechet-differentiable and $\beta_\epsilon = \partial j_\epsilon$,

(iv) $j_\epsilon \uparrow j$ as $\epsilon \downarrow 0$.

Note that β_ϵ is a nondecreasing and Lipschitz-continuous function.

Since $\mu \in \mathcal{M}_b^{\widetilde{p}(.)}(U_\Omega)$, recall that (cf. Theorem 3.1) $\mu = f - \mathrm{div}(F)$ in $\mathcal{D}'(U_\Omega)$ with $f \in L^1(U_\Omega)$ and $F \in (L^{\widetilde{p}'(.)}(U_\Omega))^N$ where U_Ω is the open bounded subset of $\mathbb{R}^N$ which extends Ω via the operator E.

We regularize μ as follows: $\forall\epsilon > 0,\ \forall x \in U_\Omega$ we define

$$f_\epsilon(x) = T_{\frac{1}{\epsilon}}(f(x))\chi_\Omega(x).$$

Let $(F_\epsilon)_{\epsilon>1} \subset C_0^\infty(U_\Omega)$ be a sequence such that $F_\epsilon \to F$ strongly in $(L^{\widetilde{p}'(.)}(U_\Omega))^N$. For any $\epsilon > 0$, we set $\tilde{F}_\epsilon = \chi_\Omega F_\epsilon$ and $\mu_\epsilon = f_\epsilon - \mathrm{div}(\tilde{F}_\epsilon)$. For any $\epsilon > 0$, one has

$\mu_\epsilon \in \mathfrak{M}_b^{p(\cdot)}(\Omega)$, $\mu_\epsilon \rightharpoonup \mu$ in $\mathcal{M}_b(U_\Omega)$ and $\mu_\epsilon \in L^\infty(\Omega)$. Furthermore, for any $k > 0$ and any $\xi \in \mathcal{T}^{1,p(\cdot)}(\Omega)$,

$$\left| \int_\Omega T_k(\xi)\, d\mu_\epsilon \right| \leq kC(\mu, \Omega).$$

Lemma 4.1 *The Yosida regularisation β_ϵ is a surjective operator.*

Proof. Since dom$(\beta) \subset [m, M]$, we have $\forall\ r \in\ \mathbb{R},\ J_\epsilon(r) = \big(I + \epsilon\beta\big)^{-1}(r) \in [m, M]$. Consequently

$$\lim_{r\to+\infty} \beta_\epsilon(r) = \lim_{r\to+\infty} \frac{r - J_\epsilon(r)}{\epsilon} = +\infty$$

and

$$\lim_{r\to-\infty} \beta_\epsilon(r) = \lim_{r\to-\infty} \frac{r - J_\epsilon(r)}{\epsilon} = -\infty.$$

As β_ϵ is a maximal monotone graph, according to ([9], Corollaire 2.3), we conclude that β_ϵ is surjective.

Now, we consider the following approximating scheme problem

$$N(\beta_\epsilon, \mu_\epsilon) \begin{cases} -div\ a(x, \nabla u_\epsilon) + \beta_\epsilon(u_\epsilon) = \mu_\epsilon & \text{in } \Omega, \\ a(x, \nabla u_\epsilon).\eta = 0 & \text{on } \partial\Omega. \end{cases}$$

We have the following results (see [16]).

Proposition 4.1
(i) There exists a unique weak solution u_ϵ for problem $N(\beta_\epsilon, \mu_\epsilon)$ in the sense that $u_\epsilon \in W^{1,p(\cdot)}(\Omega), \beta_\epsilon(u_\epsilon) \in L^\infty(\Omega)$ and $\forall \varphi \in W^{1,p(\cdot)}(\Omega)$,

$$\int_\Omega a(x, \nabla u_\epsilon).\nabla\varphi\, dx + \int_\Omega \beta_\epsilon\big(u_\epsilon\big)\varphi\, dx = \int_\Omega \varphi d\, \mu_\epsilon. \tag{10}$$

(ii) Moreover, for any $k > 0$,

$$\int_\Omega |\nabla T_k(u_\epsilon)|^{p(x)} dx \ \leq\ kC(\mu, \Omega) \tag{11}$$

and

$$\int_\Omega \beta_\epsilon(u_\epsilon) T_k(u_\epsilon) dx \ \leq\ kC(\mu, \Omega), \tag{12}$$

where $C(\mu, \Omega)$ is a positive constant.

Proposition 4.2 *The sequences $\Big(\beta_\epsilon(u_\epsilon)\Big)_{\epsilon>0}$ and $\Big(\beta_\epsilon(T_k(u_\epsilon))\Big)_{\epsilon>0}$ are uniformly bounded in $L^1(\Omega)$.*

Proposition 4.3 *Let u_ϵ be a solution of $N(\beta_\epsilon, \mu_\epsilon)$, then*

$$meas\{|u_\epsilon| > k\} \ \leq\ \frac{C(\mu, \Omega)}{\min\Big(\beta_\epsilon(k), \big|\beta_\epsilon(-k)\big|\Big)} \quad \textit{for } \ k > 0 \textit{ large enough} \tag{13}$$

and

$$meas\{|\nabla u_\epsilon| > k\} \;\leq\; \frac{(k+1)C}{k^{p_-}} + \frac{C(\mu,\Omega)}{\min\Big(\beta_\epsilon(k), \big|\beta_\epsilon(-k)\big|\Big)} \quad \text{for} \;\; k>0 \text{ large enough}, \quad (14)$$

where C *is a positive constant.*

Proposition 4.4 *For all* $k>0, T_k(u_\epsilon) \to T_k(u)$ *in* $L^{p_-}(\Omega)$ *and a.e. in* Ω*, as* $\epsilon \to 0$. *Moreover,* $u : \Omega \to \mathbb{R}$ *is such that* $u \in dom(\beta)$ *a.e. in* Ω *and* $u_\epsilon \to u$ *in measure and a.e. in* Ω*, as* $\epsilon \to 0$.

Proposition 4.5 *For any* $k>0$*, as* ϵ *tends to* 0*, we have*
(i) $a(x, \nabla T_k(u_\epsilon)) \rightharpoonup a(x, \nabla T_k(u))$ *weakly in* $\Big(L^{p'(\cdot)}(\Omega)\Big)^N$.
(ii) $\nabla T_k(u_\epsilon) \longrightarrow \nabla T_k(u)$ *a.e. in* Ω.
(iii) $a(x, \nabla T_k(u_\epsilon)).\nabla T_k(u_\epsilon) \longrightarrow a(x, \nabla T_k(u)).\nabla T_k(u)$ *a.e. in* Ω *and strongly in* $L^1(\Omega)$.
(iv) $\nabla T_k(u_\epsilon) \longrightarrow \nabla T_k(u)$ *strongly in* $\big(L^{p(\cdot)}(\Omega)\big)^N$.

Proof. The proof can be carried out in the same way as the proof of Proposition 4.5 in [16]. The following lemmas are useful for the subsequent presentation.

Lemma 4.2 *For any* $h \in C_c^1(\mathbb{R})$ *and* $\varphi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$,

$$\nabla[h(u_\epsilon)\varphi] \longrightarrow \nabla[h(u)\varphi] \text{ strongly in } (L^{p(\cdot)}(\Omega))^N \text{ as } \epsilon \to 0.$$

Proof. For any $h \in C_c^1(\mathbb{R})$ and $\varphi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, we have

$$\begin{aligned} &\nabla[h(u_\epsilon)\varphi] - \nabla[h(u)\varphi] = (h(u_\epsilon) - h(u))\nabla\varphi + h'(u_\epsilon)\varphi[\nabla u_\epsilon - \nabla u] \\ &+(h'(u_\epsilon) - h'(u))\varphi\nabla u := \psi_1^\epsilon + \psi_2^\epsilon + \psi_3^\epsilon. \end{aligned} \quad (15)$$

For the term ψ_1^ϵ, we consider $\rho_{p(.)}(\psi_1^\epsilon) = \int_\Omega |(h(u_\epsilon) - h(u))\nabla\varphi|^{p(x)}\,dx$.

Set $\Theta_1^\epsilon(x) = |(h(u_\epsilon) - h(u))\nabla\varphi|^{p(x)}$. We have $\Theta_1^\epsilon(x) \to 0$ a.e. $x \in \Omega$ as $\epsilon \to 0$ and $|\Theta_1^\epsilon(x)| \leq C(h, p_-, p_+)|\nabla\varphi|^{p(x)} \in L^1(\Omega)$. Then, by the Lebesgue dominated convergence theorem, we get that $\lim_{\epsilon\to 0} \rho_{p(.)}(\psi_1^\epsilon) = 0$. Hence,

$$\|\psi_1^\epsilon\|_{L^{p(\cdot)}(\Omega)} \to 0 \text{ as } \epsilon \to 0. \quad (16)$$

For the term ψ_2^ϵ we consider $\rho_{p(.)}(\psi_2^\epsilon) = \int_\Omega |h'(u_\epsilon)\varphi(\nabla T_l(u_\epsilon) - \nabla T_l(u))|^{p(x)}\,dx$ for some $l>0$ such that $\text{supp}(h) \subset [-l, l]$.

Set $\Theta_2^\epsilon(x) = |h'(u_\epsilon)\varphi(\nabla T_l(u_\epsilon) - \nabla T_l(u))|^{p(x)}$. We have $\Theta_2^\epsilon(x) \to 0$ a.e. $x \in \Omega$ as $\epsilon \to 0$ and $|\Theta_2^\epsilon(x)| \leq C(h, p_-, p_+, \|\varphi\|_\infty)|\nabla T_l(u_\epsilon) - \nabla T_l(u)|^{p(x)}$. Since $\nabla T_l(u_\epsilon) \to \nabla T_l(u)$ strongly in $\big(L^{p(\cdot)}(\Omega)\big)^N$, we get $\rho_{p(.)}(\nabla T_l(u_\epsilon) - \nabla T_l(u)) \to 0$ as $\epsilon \to 0$, which is equivalent to, say

$$\lim_{\epsilon\to 0} \int_\Omega |\nabla T_l(u_\epsilon) - \nabla T_l(u)|^{p(x)}\,dx = 0.$$

Then $|\nabla T_l(u_\epsilon) - \nabla T_l(u)|^{p(\cdot)} \to 0$ strongly in $L^1(\Omega)$.

By the Lebesgue generalized convergence theorem, one has

$$\lim_{\epsilon\to 0} \int_\Omega \Theta_2^\epsilon(x)\,dx = \lim_{\epsilon\to 0} \rho_{p(.)}(\psi_2^\epsilon) = 0.$$

Hence,

$$\|\psi_2^\epsilon\|_{L^{p(.)}(\Omega)} \to 0 \text{ as } \epsilon \to 0. \tag{17}$$

For the term ψ_3^ϵ we consider $\rho_{p(.)}(\psi_3^\epsilon) = \displaystyle\int_\Omega |(h'(u_\epsilon) - h'(u))\varphi \nabla u|^{p(x)}\, dx$.

Set $\Theta_3^\epsilon(x) = |(h'(u_\epsilon) - h'(u))\varphi\nabla u|^{p(x)}$. We have $\Theta_3^\epsilon(x) \to 0$ a.e. $x \in \Omega$ as $\epsilon \to 0$ and $|\Theta_3^\epsilon(x)| \leq C(h, p_-, p_+, \|\varphi\|_\infty)|\nabla T_l(u)|^{p(x)} \in L^1(\Omega)$, with some $l > 0$ such that $\text{supp}(h) \subset [-l, l]$. Then, by the Lebesgue dominated convergence theorem, we get $\lim\limits_{\epsilon\to 0} \rho_{p(.)}(\psi_3^\epsilon) = 0$. Hence,

$$\|\psi_3^\epsilon\|_{L^{p(.)}(\Omega)} \to 0 \text{ as } \epsilon \to 0. \tag{18}$$

According to (16)-(18), we get $\left\|\psi_1^\epsilon + \psi_2^\epsilon + \psi_3^\epsilon\right\|_{L^{p(.)}(\Omega)} \to 0$ as $\epsilon \to 0$ and the lemma is proved.

Lemma 4.3 *For any $h \in C_c^1(\mathbb{R})$ and $\varphi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$,* $\displaystyle\lim_{\epsilon\to 0} \int_\Omega h(u_\epsilon)\varphi\, d\mu_\epsilon = \int_\Omega h(u)\varphi d\mu.$

Proof. We have

$$\begin{aligned}
\int_\Omega h(u_\epsilon)\varphi\, d\mu_\epsilon &= \int_\Omega E\big(h(u_\epsilon)\varphi\big)\, d\mu_\epsilon = \big\langle \mu_\epsilon, E\big(h(u_\epsilon)\varphi\big)\big\rangle \\
&= \int_{U_\Omega} f_\epsilon E\big(h(u_\epsilon)\varphi\big)\, dx + \int_{U_\Omega} \tilde{F}_\epsilon.\nabla E\big(h(u_\epsilon)\varphi\big)\, dx \\
&= \int_{U_\Omega} \chi_\Omega T_{\frac{1}{\epsilon}}(f) E\big(h(u_\epsilon)\varphi\big)\, dx + \int_{U_\Omega} (\chi_\Omega F_\epsilon).\nabla E\big(h(u_\epsilon)\varphi\big)\, dx \\
&= \int_\Omega T_{\frac{1}{\epsilon}}(f) h(u_\epsilon)\varphi\, dx + \int_{U_\Omega} F_\epsilon.\nabla E\big(\chi_\Omega h(u_\epsilon)\varphi\big)\, dx.
\end{aligned} \tag{19}$$

By the Lebesgue dominated convergence theorem, we have for the first term of the right hand side of (19),

$$\lim_{\epsilon\to 0} \int_\Omega T_{\frac{1}{\epsilon}}(f) h(u_\epsilon)\varphi\, dx = \int_\Omega f h(u)\varphi\, dx. \tag{20}$$

Furthermore, the sequence $\Big(E\big(\chi_\Omega h(u_\epsilon)\varphi\big)\Big)_{\epsilon>0}$ is bounded in $W_0^{1,\widetilde{p}(.)}(U_\Omega)$. Indeed, $\big(\chi_\Omega h(u_\epsilon)\varphi\big)_{\epsilon>0}$ is bounded in $W^{1,p(.)}(\Omega)$ and we use the inequality

$$\|E(v)\|_{W_0^{1,\widetilde{p}(.)}(U_\Omega)} \leq C\|v\|_{W^{1,p(.)}(\Omega)},\ \forall v \in W^{1,p(.)}(\Omega).$$

We also have $E\big(\chi_\Omega h(u_\epsilon)\varphi\big) = \chi_\Omega h(u_\epsilon)\varphi$ a.e. in U_Ω and $\chi_\Omega h(u_\epsilon)\varphi \to \chi_\Omega h(u)\varphi$ a.e. in U_Ω as $\epsilon \to 0$. Hence $E\big(\chi_\Omega h(u_\epsilon)\varphi\big) \to E\big(\chi_\Omega h(u)\varphi\big)$ a.e. in U_Ω as $\epsilon \to 0$. Then,

$$\nabla E\big(\chi_\Omega h(u_\epsilon)\varphi\big) \rightharpoonup \nabla E\big(\chi_\Omega h(u)\varphi\big) \text{ in } \big(L^{\widetilde{p}(.)}(U_\Omega)\big)^N.$$

Finally, we get for the second term in the right hand side of (19)

$$\lim_{\epsilon\to 0} \int_{U_\Omega} F_\epsilon.\nabla E\big(\chi_\Omega h(u_\epsilon)\varphi\big)\, dx = \int_{U_\Omega} F.\nabla E\big(\chi_\Omega h(u)\varphi\big)\, dx. \tag{21}$$

Using (20) and (21), we get from (19),

$$\begin{aligned}\lim_{\epsilon\to 0}\int_\Omega h(u_\epsilon)\varphi\, d\mu_\epsilon &= \int_\Omega fh(u)\varphi\, dx + \int_{U_\Omega} F.\nabla E\big(\chi_\Omega h(u)\varphi\big)\, dx\\ &= \int_{U_\Omega} fE\big(\chi_\Omega h(u)\varphi\big)\, dx + \int_{U_\Omega} F.\nabla E\big(\chi_\Omega h(u)\varphi\big)\, dx\\ &= \Big\langle \mu, E\big(\chi_\Omega h(u)\varphi\big)\Big\rangle = \int_{U_\Omega} E\big(\chi_\Omega h(u)\varphi\big)\, d\mu = \int_\Omega h(u)\varphi\, d\mu.\end{aligned}$$

We continue the proof of Theorem 1.1. So we need to pass to the limit in the second integral of (10). Since, for any $k>0$, $(h_k(u_\epsilon)\beta_\epsilon(u_\epsilon))_{\epsilon>0}$ is bounded in $L^1(\Omega)$, there exists $z_k\in\mathcal{M}_b(\Omega)$, such that

$$h_k(u_\epsilon)\beta_\epsilon(u_\epsilon) \overset{*}{\rightharpoonup} z_k \text{ in } \mathcal{M}_b(\Omega) \text{ as } \epsilon\to 0.$$

Moreover, for any $\varphi\in W^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$, we have

$$\int_\Omega \varphi\, dz_k = \int_\Omega \varphi h_k(u)\, d\mu - \int_\Omega a(x,\nabla u)\cdot\nabla(h_k(u)\varphi)dx,$$

which implies that $z_k\in\mathcal{M}_b^{p(.)}(\Omega)$ and, for any $k\le l$, $z_k=z_l$ on $[|T_k(u)|<k]$.

Let us consider the Radon measure z defined by

$$\begin{cases} z = z_k, \quad \text{on } [|T_k(u)|<k] \text{ for } k\in\mathbb{N}^*,\\ z=0 \quad \text{on } \displaystyle\bigcap_{k\in\mathbb{N}^*}[|T_k(u)|=k]. \end{cases} \tag{22}$$

For any $h\in\mathcal{C}_c^1(\mathbb{R})$, $h(u)\in L^\infty(\Omega, d|z|)$ and

$$\int_\Omega h(u)\varphi\, dz = -\int_\Omega a(x,\nabla u)\cdot\nabla(h(u)\varphi)dx + \int_\Omega h(u)\varphi d\mu,$$

for any $\varphi\in W^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$. Indeed, let $k_0>0$ be such that $\text{supp}(h)\subseteq[-k_0,k_0]$,

$$\begin{aligned}\int_\Omega h(u)\varphi\, dz &= \int_\Omega h(u)\varphi\, dz_{k_0} = -\lim_{\epsilon\to 0}\int_\Omega a(x,\nabla u_\epsilon)\cdot\nabla(h(u_\epsilon)\varphi)dx + \lim_{\epsilon\to 0}\int_\Omega h(u_\epsilon)\varphi d\mu_\epsilon\\ &= -\lim_{\epsilon\to 0}\int_\Omega a(x,\nabla T_{k_0}(u_\epsilon))\cdot\nabla(h(u_\epsilon)\varphi)dx + \lim_{\epsilon\to 0}\int_\Omega h(u_\epsilon)\varphi d\mu_\epsilon\\ &= -\int_\Omega a(x,\nabla u)\cdot\nabla(h(u)\varphi)dx + \lim_{\epsilon\to 0}\int_\Omega h(u_\epsilon)\varphi d\mu_\epsilon\\ &= -\int_\Omega a(x,\nabla u)\cdot\nabla(h(u)\varphi)dx + \int_\Omega h(u)\varphi d\mu.\end{aligned} \tag{23}$$

Moreover, we have (see [16])

Lemma 4.4 *The Radon-Nikodym decomposition of the measure z given by (22) with respect to $\mathcal{L}^N$,*

$$z = w\,\mathcal{L}^N + \nu \quad \textit{with } \nu\perp\mathcal{L}^N, \tag{24}$$

satisfies the following properties:
(i) $w\in\beta(u)$ $\mathcal{L}^N-a.e.$ in Ω, $w\in L^1(\Omega)$,
(ii) $\nu\in\mathcal{M}_b^{p(.)}(\Omega)$, ν^+ is concentrated on $[u=M]$ and ν^- is concentrated on $[u=m]$.

To finish the proof of Theorem 1.1, we consider $\varphi \in W^{1,p(.)}(\Omega) \cap L^{\infty}(\Omega)$ and $h \in C_c^1(\mathbb{R})$. Then, we take $h(u_\epsilon)\varphi$ as test function in (10). We get

$$\int_\Omega a(x, \nabla u_\epsilon).\nabla[h(u_\epsilon)\varphi]dx + \int_\Omega \beta_\epsilon(u_\epsilon)h(u_\epsilon)\varphi dx = \int_\Omega h(u_\epsilon)\varphi d\mu_\epsilon. \tag{25}$$

By Lemma 4.3, we have for the term in the right hand side of (25),

$$\lim_{\epsilon\to 0} \int_\Omega h(u_\epsilon)\varphi\, d\mu_\epsilon = \int_\Omega h(u)\varphi\, d\mu.$$

The first term of (25) can be written as

$$\int_\Omega a(x, \nabla u_\epsilon).\nabla[h(u_\epsilon)\varphi]dx = \int_\Omega a(x, \nabla T_{l_0+1}(u_\epsilon)).\nabla[h_0(u_\epsilon)\varphi]dx,$$

for some $l_0 > 0$ so that, by Proposition 4.5-(i) and Lemma 4.2, we have

$$\begin{aligned}
\lim_{\epsilon\to 0} \int_\Omega a(x, \nabla u_\epsilon).\nabla[h(u_\epsilon)\varphi]dx &= \lim_{\epsilon\to 0} \int_\Omega a(x, \nabla T_{l_0+1}(u_\epsilon)).\nabla[h_0(u_\epsilon)\varphi]dx \\
&= \int_\Omega a(x, \nabla T_{l_0+1}(u)).\nabla[h_0(u)\varphi]dx \\
&= \int_\Omega a(x, \nabla u).\nabla[h(u)\varphi]dx.
\end{aligned}$$

Due to the convergence of Lemma 4.2 and Proposition 4.5-(i) we have from (25)

$$\begin{aligned}
\lim_{\epsilon\to 0} \int_\Omega \beta_\epsilon(u_\epsilon)h(u_\epsilon)\varphi dx &= \int_\Omega h(u)\varphi d\mu - \int_\Omega a(x, \nabla u).\nabla[h(u)\varphi]dx. \\
&= \int_\Omega h(u)\varphi dz = \int_\Omega h(u)w\varphi dx + \int_\Omega h(u)\varphi d\nu.
\end{aligned}$$

Letting ϵ go to 0 in (25), we obtain

$$\int_\Omega a(x, \nabla u).\nabla[h(u)\varphi]dx + \int_\Omega h(u)w\varphi dx + \int_\Omega h(u)\varphi d\nu = \int_\Omega h(u)\varphi d\mu. \tag{26}$$

In (26), we take $h \in C_c^1(\mathbb{R})$ such that $[m, M] \subset \text{supp}(h) \subset [-l, l]$ and $h(s) = 1$ for all $s \in [m, M]$. As $u \in \text{dom}(\beta)$, then $h(u) = 1$ and it yields that (u, w, ν) is a solution of the problem $N(\beta, \mu)$. □

5 Proof of Theorem 1.2

Proof. For u_1, we choose $\varphi = u_1 - u_2$ as test function in (5) to get

$$\int_\Omega a(x, \nabla u_1).\nabla(u_1 - u_2)dx + \int_\Omega w_1(u_1 - u_2)dx \le \int_\Omega (u_1 - u_2)d\mu.$$

Similarly we get for u_2,

$$\int_\Omega a(x, \nabla u_2).\nabla(u_2 - u_1)dx + \int_\Omega w_2(u_2 - u_1)dx \le \int_\Omega (u_2 - u_1)d\mu.$$

Adding these two last inequalities yields

$$\int_\Omega \Big(a(x,\nabla u_1) - a(x,\nabla u_2)\Big).\nabla(u_1-u_2)dx + \int_\Omega (w_1-w_2)\,(u_1-u_2)dx. \tag{27}$$

From (27) it yields

$$\int_\Omega \Big(a(x,\nabla u_1) - a(x,\nabla u_2)\Big).\nabla(u_1-u_2)dx = 0 \tag{28}$$

From (28), it follows that there exists a constant c such that $u_1 - u_2 = c$ a.e. in Ω. Now, let us see that $w_1 = w_2$ a.e. in Ω and $\nu_1 = \nu_2$. Indeed, for any $\varphi \in \mathcal{D}(\Omega)$, taking φ as a test function in (5) for the solutions (u_1, w_1, ν_1) and (u_1, w_2, ν_2), after substraction, we get

$$\int_\Omega (w_1-w_2)\varphi dx + \int_\Omega \varphi d(\nu_1-\nu_2) = 0.$$

Hence

$$\int_\Omega w_1\varphi dx + \int_\Omega \varphi d\nu_1 = \int_\Omega w_2\varphi dx + \int_\Omega \varphi d\nu_2.$$

Therefore

$$w_1\mathcal{L}^N + \nu_1 = w_2\mathcal{L}^N + \nu_2.$$

Since the Radon-Nikodym decomposition of a measure is unique, we get $w_1 = w_2$ a.e. in Ω and $\nu_1 = \nu_2$.

To complete the proof of Theorem 1.2, it remains to show that (7) and (8) hold. To this aim, let us recall the following result.

Lemma 5.1 *Let $\eta \in W^{1,p(.)}(\Omega)$, $Z \in \mathcal{M}_b^{p(.)}(\Omega)$ and $\lambda \in \mathbb{R}$ be such that*

$$\begin{cases} \eta \le \lambda \text{ a.e. in } \Omega \ \ (\textit{respectively } \ \eta \ge \lambda), \\ Z = -div\ a(x,\nabla\eta) \text{ in } \mathcal{D}'(\Omega). \end{cases} \tag{29}$$

Then

$$\int_{[\eta=\lambda]} \xi dZ \ge 0 \quad \Big(\textit{respectively} \quad \int_{[\eta=\lambda]} \xi dZ \le 0\Big),$$

for any $\xi \in C_c^1(\Omega)$, $\xi \ge 0$.

Proof of Lemma 5.1 The proof of this lemma follows the same steps of [2].

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