Nonlinear Dynamics and Systems Theory, **16** (1) (2016) 73–85

Existence of Solutions to a New Class of Abstract Non-Instantaneous Impulsive Fractional Integro-Differential Equations

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Received: April 17, 2015; Revised: January 27, 2016

Abstract: In this paper we prove the sufficient conditions for the existence and uniqueness of piecewise continuous mild solutions to fractional integro-differential equations in a Banach space with non instantaneous impulses. The results are established by using the theory of sectorial operators and the fixed point theorem. We discuss an example to illustrate the analytical results obtained.

Keywords: *sectorial operator; solution operator; non-instantaneous impulses; Krasnoselskii's fixed point theorem.*

Mathematics Subject Classification (2010): 34G20, 34K30, 34K40, 47N20.

1 Introduction

Let $(X, \|\cdot\|)$ be a complex Banach space. The objective of this paper is to study the solutions to a new class of abstract integro-differential equations of fractional order with non-instantaneous impulses in X :

$$\left.\begin{aligned} {}^cD_t^\alpha[u(t)+\varphi(t,u(t))] &= Au(t)+J_t^{1-\alpha}f(t,u(t)),\\ &\qquad t\in(s_i,t_{i+1}],\ i=0,1,\cdots,N,\ 0<\alpha<1,\\ u(t) &= g_i(t,u(t)),\quad t\in(t_i,s_i],\quad i=1,2,\cdots,N,\\ u(0) &= u_0, \end{aligned}\right\} \tag{1}$$

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where $A : D(A) \subset X \to X$ is a sectorial operator on $(X, \|.\|)$, $u_0 \in X$, $0 = t_0 = s_0 < t_1 \le s_1 \le t_2 < \cdots < t_N \le s_N \le t_{N+1} = T_0$ are pre-fixed numbers, $g_i \in C((t_i, s_i] \times X; X)$ and $\varphi : [0, T_0] \times X \to X$, $f : [0, T_0] \times X \to X$ are suitably defined functions. The fractional derivative ${}^cD_t^\alpha$ is to be understood in Caputo sense and J_t^α denotes the Riemann-Liouville integral of order α. This paper is concerned with impulsive differential equations of fractional order, where an impulsive action starts suddenly at the points t_i and their action stays active on the interval $[t_i, s_i]$.

Fractional differential equations arise as models in many fields of engineering and science such as electrochemistry, electro-magnetics, electrical circuits control theory, visco-elasticity, porous media, neuron modelling etc. [5, 9, 13, 15, 16, 18–20, 22]. The plentiful occurrence and applications of fractional differential equations motivate the rapid developments and gained much attention in the recent years and have been studied extensively in [2–4, 6, 7, 14, 23–27, 29, 30]. But systems with non-instantaneous impulses do exist [10, 11]. For example, one can consider the hemodynamical equilibrium of a person in which impulses are non-instantaneous [10]. Such systems for the fractional differential equations are less studied. Recently, Hernández and O'regan introduced and investigated the existence of mild and classical solutions to a new class of abstract differential equations with non-instantaneous impulses in X:

$$\begin{aligned} &u'(t) = Au(t) + f(t, u(t)), \qquad t \in (s_i, t_{i+1}], \quad i = 0, 1, \cdots, N, \\ &u(t) = g_i(t, u(t)), \qquad t \in (t_i, s_i], \quad i = 1, 2, \cdots, N, \\ &u(0) = u_0. \end{aligned} \tag{2}$$

The operator A generates an infinitesimal C_0-semigroup of bounded linear operators $(X, \|.\|)$, the functions $g_i \in C((t_i, s_i] \times X; X)$ for each $i = 1, 2, \cdots, N$ and $f : [0, T_0] \times X \to X$ is a suitable function. The results are established by fixed point theorem with appropriate g_i and f [10].

Kumar et al [12] had extended the work in [10] to the following problem in a Banach space X :

$$\left.\begin{aligned} {}^cD_t^\alpha u(t) + Au(t) &= f(t, u(t), u(g(t))), \\ &t \in (s_i, t_{i+1}],\ i = 0, 1, \cdots, N,\ 0 < \alpha < 1, \\ u(t) &= g_i(t, u(t)), \quad t \in (t_i, s_i], \quad i = 1, 2, \cdots, N, \\ u(0) &= u_0, \end{aligned}\right\} \tag{3}$$

where ${}^cD_t^\alpha$ is the Caputo fractional derivative of order α, $-A$ generates an analytic semigroup. The sufficient conditions are obtained if f and h_i are Lipschitz continuous in the second variable appropriately. For more details, we refer to [12].

With the strong motivation from Hernández and O'regan [10]; and Kumar et al. [12], we establish the existence and uniqueness of piecewise continuous mild solution to the class of fractional integro-differential equations (1), where the impulses are non-instantaneous. The main results are new and complement to the existing ones that generalize some results of [10, 12, 23] to the fractional integro-differential equations.

The paper is organized as follows. We collect the basic notations, definitions, lemmas and theorems in Section 2. We prove the existence as well as uniqueness of solution of (1) in Section 3. We provide an example in Section 4 as an application of the analytical results obtained.

2 Preliminaries and Assumptions

In this section, we will introduce some basic definitions, notations and lemmas that are useful throughout this paper. For more details, we refer to [13, 15–20]. For the Banach space X, we denote the Banach space of all bounded linear operator from X into X by $L(X)$. We denote a ball in X of radius r centered at y as $B_r(y,X)$. The set of all m^{th} order continuously differentiable functions from $J(J\subset\mathbb{R})$ into X is denoted by $C^m(J,X)$ for $m\in\mathbb{N}$. We begin with the following definition of sectorial operator.

Definition 2.1 A closed linear operator A is said to be sectorial of type ω if there exist constants $\omega\in\mathbb{R}$, $\theta\in[\frac{\pi}{2},\pi]$, and $M>0$ such that

(a) $\rho(A)\subset\Sigma_{\theta,\omega}=\{\lambda\in\mathbb{C}:|\arg(\lambda-\omega)|<\theta,\ \lambda\neq\omega\}$,

(b) $\|R(\lambda,A)\|_{L(X)}\leq\frac{M}{|\lambda-\omega|},\quad \lambda\in\Sigma_{\theta,\omega}$.

Definition 2.2 For $f\in L^1((0,T),X)$ and $\alpha\geq 0$, we define the Reimann-Liouville integral of order α of f as

$$J_t^\alpha f(t)=(f*\Theta_\alpha)(t)=\frac{1}{\Gamma\alpha}\int_0^t(t-s)^{\alpha-1}f(s)ds,\ t>0,\ \alpha>0, \tag{1}$$

where $J_t^0 f(t)=f(t)$ and

$$\Theta_\alpha(t)=\begin{cases}\dfrac{1}{\Gamma\alpha}t^{\alpha-1}, & t>0,\\ 0, & t\leq 0,\end{cases}$$

and $\Theta_0(t)=0$.

Definition 2.3 If $f\in C^{m-1}((0,T),X)$ and $(\Theta_{m-\alpha}*f)\in W^{m,1}((0,T),X)$, $0\leq m-1<\alpha<m$, $m\in\mathbb{N}$, then the the Caputo fractional derivative of order α of f is defined as

$${}^cD_t^\alpha f(t)=D_t^m J_t^{m-\alpha}\Big(f(t)-\sum_0^{m-1}f^i(0)\Theta_{i+1}(t)\Big), \tag{2}$$

where $D_t^m=\frac{d^m}{dt^m}$ and

$$W^{m,1}((0,T);X)=\Big\{f\in X: f^m\in L^1((0,T);X)\ f(t)=\sum_{j=0}^{m-1}f^j(0)\frac{t^j}{j!}+\frac{t^{m-1}}{(m-1)!}*f^m(t)\Big\}.$$

We note the following properties of J_t^α

Lemma 2.1 [28, Proposition 2.4] *For $\alpha,\beta>0$, we have*

(i) $J_t^\alpha J_t^\beta f(t)=J_t^{\alpha+\beta}f(t)$ *for all* $f\in L^1(J;X)$*;*

(ii) $J_t^\alpha(f*g)=J_t^\alpha f*g$ *for all* $f,g\in L^p(J;X)(1\leq p<+\infty)$*;*

(iii) The Caputo fractional derivative ${}^cD_t^\alpha$ *is a left inverse of* J_t^α*:*

$${}^cD_t^\alpha J_t^\alpha f=f,\quad \textit{for all } f\in L^1(J;X),$$

but in general not a right inverse, in fact, for all $f(t) \in C^{m-1}(J;X)$ *with* $\Theta_{m-\alpha} * f \in W^{m,1}(J,X)(m \in \mathbb{N},\ 0 \le m-1 < \alpha < m)$, *one has*

$$J_t^\alpha({}^cD_t^\alpha)f(t) = f(t) - \sum_{i=0}^{m-1} f^{(i)}(0)\Theta_{i+1}(t). \tag{3}$$

We consider the following Cauchy problem

$$\begin{aligned} {}^cD_t^\alpha u(t) + \lambda u(t) &= 0, \qquad t > 0, \\ u(0) &= u_0, \quad 0 < \alpha < 1. \end{aligned} \tag{4}$$

Then the solution of (4) is $u(t) = S(t)u_0$, where $S(t) = E_{\alpha,1}(-\lambda t^\alpha) = E_\alpha(-\lambda t^\alpha)$ [8], where $E_{\alpha,\beta}$ is the generalized Mittag-Leffler function. The generalized Mittag-Leffler function $E_{\alpha,\beta}$ is defined as

$$E_{\alpha,\beta} := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i}\int_\chi \frac{\lambda^{\alpha-\beta}e^\lambda}{\lambda^\alpha - z} d\lambda \quad \text{for} \quad \alpha, \beta > 0, z \in \mathbb{C},$$

where χ is a contour that starts and ends at $-\infty$ and encircles the disc $|\lambda| \le |z|^{1/\alpha}$ counterclockwise.

Replacing λ by $-A$, we rewrite $S(t)$ as

$$S(t) = \frac{1}{2\pi i}\int_{B_\gamma} e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha, A)d\lambda,$$

where B_γ denotes the Bromwich path. Moreover, if A is a sectorial operator of type ω then A is the generator of a solution operator given by

$$S_\alpha(t) = \frac{1}{2\pi i}\int_\Upsilon e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}d\lambda,$$

where Υ is suitable path lying on $\Sigma_{\theta,\omega}$. For more details, we refer the reader to [3, 6, 15, 16, 23, 27–29].

We consider the following Cauchy problem

$$\left.\begin{aligned} {}^cD_t^\alpha[u(t) + \Phi(t)] = Au(t) + J_t^{1-\alpha}f(t), \quad 0 < \alpha < 1, \\ u(0) = u_0 \in X, \end{aligned}\right\} \tag{5}$$

where $f : [0,\infty) \to X$ and A is a sectorial operator. The solution of (5) is given by the following theorem.

Theorem 2.1 *If* f *and* Φ *satisfy the uniform Hölder condition with exponent* $\beta \in (0,1]$ *and* A *is a sectorial operator, then the unique solution of the Cauchy problem (5) is given by*

$$u(t) = S_\alpha(t)[u_0 + \Phi(0)] - \Phi(t) - \int_0^t T_\alpha(t-s)\Phi(s)ds + \int_0^t S_\alpha(t-s)f(s)ds,$$

where

$$S_\alpha(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha, A)d\lambda, \quad T_\alpha(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R(\lambda^\alpha, A)d\lambda,$$

for a suitable path Γ *lying on* $\Sigma_{\theta,\omega}$.

Proof. Applying the Riemann-Liouville fractional integral operator J_t^α to both sides of equation (5), we get

$$J_t^\alpha({}^cD_t^\alpha)[u(t)+\Phi(t)] = J_t^\alpha Au(t) + J_t^1 f(t).$$

Using (1) and (3), we get

$$\begin{aligned} u(t)+\Phi(t) &= [u_0+\Phi(0)] + \frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}Au(s)ds + \int_0^t f(s)ds \\ &= [u_0+\Phi(0)] + \frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}[Au(s)+\Phi(s)]ds \\ &\quad -\frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}\Phi(s)ds + \int_0^t f(s)ds. \end{aligned} \tag{6}$$

Applying the Laplace transform to equation (6), we get

$$(\mathcal{L}(u+\Phi))(\lambda) = \frac{1}{\lambda}[u_0+\Phi(0)] + \frac{1}{\lambda^\alpha}A(\mathcal{L}(u+\Phi)(\lambda) - \frac{1}{\lambda^\alpha}(\mathcal{L}\Phi)(\lambda) + \frac{1}{\lambda}(\mathcal{L}f)(\lambda).$$

Since $(\lambda^\alpha I - A)^{-1}$ exists, i.e., $\lambda^\alpha \in \rho(A)$, we obtain

$$(\mathcal{L}(u+\Phi))(\lambda) = (\lambda^\alpha I - A)^{-1}\Big[\lambda^{\alpha-1}(u_0+\Phi(0)) - (\mathcal{L}\Phi)(\lambda) + \lambda^{\alpha-1}(\mathcal{L}f)(\lambda)\Big].$$

Applying the inverse Laplace transform, we get

$$u(t) = S_\alpha(t)[u_0+\Phi(0)] - \Phi(t) - \int_0^t T_\alpha(t-s)\Phi(s)ds + \int_0^t S_\alpha(t-s)f(s)ds.$$

□

We define the set $\mathcal{PC}(X)$ for the solution space as follows

$$\mathcal{PC}(X) = \{u:[0,T_0]\to X : u(\cdot) \text{ is continuous at } t\neq t_i, u(t_k^-)=u(t_k), u(t_k^+) \text{ exists for all } i=1,2,\cdots,N\}.$$

We note that $\mathcal{PC}(X)$ is a Banach space endowed with the supremum norm

$$\|u\|_{\mathcal{PC}} := \sup_{t\in[0,T_0]}\|u(t)\|.$$

Now, we define the functions $\widetilde{u}_i \in C([t_i,t_{i+1}];X)$ given by

$$\widetilde{u}_i(t) = \begin{cases} u(t), & \text{for } t\in(t_i,t_{i+1}], \\ u(t_i^+), & \text{for } t=t_i. \end{cases}$$

For a ball $B_r \subseteq \mathcal{PC}(X)$, we define

$$\widetilde{B}_i = \{\widetilde{u}_i : u\in B_r\}.$$

The following Arzela-Ascoli type lemma will be used to establish the main result.

Lemma 2.2 [10, Lemma 1.1] *A set $B_r \subseteq \mathcal{PC}(X)$ is relatively compact in $\mathcal{PC}(X)$ if and only if $\widetilde{B}_i$ is relatively compact in $C([t_i,t_{i+1}];X])$ for every $i=0,1,2,\cdots,N$.*

Definition 2.4 A function $u \in \mathcal{PC}(X)$ is said to be a mild solution of the problem (1) if $u(0) = u_0$, $u(t) = g_i(t, u(t))$ for all $t \in (t_i, s_i]$ and each $i = 1, \cdots, N$, and

$$\begin{aligned} u(t) &= S_\alpha(t)[u_0 + \varphi(0, u_0)] - \varphi(t, u(t)) - \int_0^t T_\alpha(t-s)\varphi(s, u(s))ds \\ &\quad + \int_0^t S_\alpha(t-s)f(s, u(s))ds, \text{for all } \ t \in [0, t_1], \end{aligned}$$

and

$$\begin{aligned} u(t) &= S_\alpha(t-s_i)g_i(s_i, u(s_i)) - \varphi(t, u(t)) - \int_{s_i}^t T_\alpha(t-s)\varphi(s, u(s))ds \\ &+ \int_{s_i}^t S_\alpha(t-s)f(s, u(s))ds, \text{for all } \ t \in [s_i, t_{i+1}] \ \ i = 1, \cdots, N. \end{aligned}$$

3 The Main Results

In this section, we prove the existence of solution to problem (1). The idea of the proof is based on [10, 23]. We need the following hypothesis on f, φ and g_i. Let V be an open subset of X. For each $v \in V$, there is a ball $B(v, r)$ such that $B(v, r) \subset V$ for $r > 0$.

(H1) There exist constants $L_f > 0$, $L_\varphi > 0$ such that the nonlinear maps $f, \varphi : [0, T_0] \times V \to X$, will satisfy the following conditions,

$$\|f(t, u) - f(t, u_1)\| \le L_f \|u - u_1\|, \tag{1}$$

$$\|\varphi(t, u) - \varphi(t, u_1)\| \le L_\varphi \|u - u_1\|, \tag{2}$$

for all $u, u_1 \in V$ and $t > 0$.

(H2) The functions $g_i : [t_i, s_i] \times X \to X$ are continuous and there are positive constants L_{g_i} such that

$$\|g_i(t, x) - g_i(t, y)\| \le L_{g_i} \|x - y\|,$$

for all $x, y \in X$, $t \in [t_i, s_i]$ and each $i = 0, 1, \cdots, N$.

(H3) The solution operators $S_\alpha, T_\alpha : \mathbb{R}_+ \to L(X)$ are bounded i.e., there exist constants $\mathcal{M}_1$ and $\mathcal{M}_2$ such that

$$\|S_\alpha(t)\|_{L(X)} \le \mathcal{M}_1, \ \|T_\alpha(t)\|_{L(X)} \le \mathcal{M}_2 \ \text{ for } \quad t > 0.$$

And the operators $(S_\alpha(t))_{t \ge 0}$, $(\overline{T_\alpha(t)})_{t \ge 0}$ are compact, where $(\overline{T_\alpha(t)}) = t^{1-\alpha} T_\alpha(t)$.

Theorem 3.1 *Let $u_0 \in X$. Also let the assumptions (H1)-(H2) hold such that*

$$L = \max\{\mathcal{M}_1(L_{g_i} + L_f \ T_0) + L_\varphi(1 + \mathcal{M}_2 T_0), L_{g_i} : \ i = 1, \cdots, N\} < 1. \tag{3}$$

Then there exists a unique mild solution $u \in \mathcal{PC}(X)$ of the problem (1).

Proof. We define a map $\digamma : \mathcal{PC}(X) \to \mathcal{PC}(X)$, given by $\digamma u(0) = u_0$, $\digamma u(t) = g_i(t, u(t))$ for each $t \in (t_i, s_i]$, $i = 1, \cdots, N$ and

$$\begin{aligned}\digamma u(t) &= S_\alpha(t)[u_0 + \varphi(0, u_0)] - \varphi(t, u(t)) - \int_0^t T_\alpha(t-s)\varphi(s, u(s))ds \\ &\quad + \int_0^t S_\alpha(t-s)f(s, u(s))ds, \quad \text{for all } \ t \in [0, t_1],\end{aligned}$$

and

$$\begin{aligned}\digamma u(t) &= S_\alpha(t-s_i)g_i(s_i, u(s_i)) - \varphi(t, u(t)) - \int_{s_i}^t T_\alpha(t-s)\varphi(s, u(s))ds \\ &\quad + \int_{s_i}^t S_\alpha(t-s)f(s, u(s))ds, \quad \text{for all } \ t \in [s_i, t_{i+1}] \text{ and } \quad i = 1, \cdots, N.\end{aligned}$$

Then $\digamma$ is well defined. Next we show that $\digamma$ is a contraction map on $\mathcal{PC}(X)$. For $u, v \in \mathcal{PC}(X)$, $i = 1, \cdots, N$ and $t \in [s_i, t_{i+1}]$, we have

$$\begin{aligned}\|\digamma u(t) - \digamma v(t)\| &\leq \|S_\alpha(t-s_i)\| \, \|g_i(s_i, u(s_i)) - g_i(s_i, v(s_i))\| \\ &\quad + \|\varphi(t, u(t)) - \varphi(t, v(t))\| \\ &\quad + \int_{s_i}^t \|T_\alpha(t-s)\| \, \|\varphi(s, u(s)) - \varphi(s, v(s))\|ds \\ &\quad + \int_{s_i}^t \|S_\alpha(t-s)\| \, \|f(s, u(s)) - f(s, v(s))\|ds \\ &\leq [\mathcal{M}_1(L_{g_i} + L_f T_0) + L_\varphi(1 + \mathcal{M}_2 T_0)]\|u - v\|_{\mathcal{PC}(X)}.\end{aligned}$$

Thus we obtain

$$\|\digamma u - \digamma v\|_{C([s_i, t_{i+1}]; X)} \leq [\mathcal{M}_1(L_{g_i} + L_f \ T_0) + L_\varphi(1 + \mathcal{M}_2 T_0)]\|u - v\|_{\mathcal{PC}(X)}. \tag{4}$$

Similarly, we obtain

$$\|\digamma u - \digamma v\|_{C([0, t_1]; X)} \leq (\mathcal{M}_1 L_f T_0 + L_\varphi(1 + \mathcal{M}_2 T_0)\|u - v\|_{\mathcal{PC}(X)}, \tag{5}$$

$$\|\digamma u - \digamma v\|_{C((t_i, s_i]; X)} \leq L_{g_i}\|u - v\|_{\mathcal{PC}(X)} \quad i = 1, 2, 3, ...N. \tag{6}$$

It follows from (4)-(6) that

$$\|\digamma u - \digamma v\|_{\mathcal{PC}(X)} \leq L\|u - v\|_{\mathcal{PC}(X)}. \tag{7}$$

By the assumption (3), the map $\digamma(\cdot)$ is a contraction and hence there exists a unique mild solution of (1). □

By a ball B_r with center at 0 and radius r, we mean the set $B_r(0, \mathcal{PC}(X)) = \{u \in \mathcal{PC}(X) : \|u\|_{\mathcal{PC}} \leq r\}$. We define

$$N_f = \sup_{s \in [s_i, t_{i+1}], v \in B_r(0, \mathcal{PC}(X))} \|f(s, v(s))\| \ \ N_\varphi = \sup_{s \in [s_i, t_{i+1}], v \in B_r(0, \mathcal{PC}(X))} \|\varphi(s, v(s))\|.$$

Theorem 3.1 can be proved with a weaker assumptions on f. We prove the theorem for the existence of mild solution to problem (1) with the following hypothesis.

(H1)´ There exists constant $L_\varphi > 0$ such that the nonlinear maps $\varphi : [0, T_0] \times V \to X$, will satisfy

$$\|\varphi(t, u) - \varphi(t, u_1)\| \leq L_\varphi \|u - u_1\|, \tag{8}$$

for all $u, u_1 \in V$ and $t > 0$.

Theorem 3.2 *Let $f : [0, T_0] \times X \to X$ be a continuous function that maps a bounded set into bounded set and $\varphi(\cdot, 0)$, $g_i(\cdot, 0)$ are bounded for each $u_0 \in X$. Let $r > 1$ and $0 < \delta < 1$ be two numbers such that*

$$\mathcal{M}_1 \|[u_0 + \varphi(0, u_0)]\| + (1 + \mathcal{M}_1) \max_{i=1,\cdots,N} \|g_i(\cdot, 0)\| \leq (1 - \delta) r, \tag{9}$$

$$\max_{i=1,\cdots,N} \Big\{ N_\varphi + L_{g_i}(1 + \mathcal{M}_1)\|u\|_{\mathcal{PC}} + (\mathcal{M}_2 N_\varphi + \mathcal{M}_1 N_f) T_0 \\ + (1 + \mathcal{M}_1)\|g_i(t, 0)\| \Big\} \leq \delta r, \tag{10}$$

$$\Big(\mathcal{M}_1 \sup_{s \in [0, t_1], v \in B_r(0, \mathcal{PC}(X))} \|f(s, v(s))\| + \mathcal{M}_2 \sup_{s \in [0, t_1], v \in B_r(0, \mathcal{PC}(X))} \|\varphi(s, v(s))\| \Big) T_0 \leq \delta r, \tag{11}$$

Also, we assume that

$$(1 + \mathcal{M}_1) L_{g_i} + L_\varphi (1 + \mathcal{M}_2 T_0) < 1. \tag{12}$$

*If assumptions **(H1)´**, **(H2)** and **(H3)** hold, then there exists a mild solution $u \in \mathcal{PC}(X)$ to problem (1).*

Proof. We decompose $\mathcal{F}$ as

$$\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2,$$

where $\mathcal{F}_1 = \sum_{i=0}^N \mathcal{F}_i^1$, $\mathcal{F}_2 = \sum_{i=0}^N \mathcal{F}_i^2$ and $\mathcal{F}_i^k : \mathcal{PC}(X) \to \mathcal{PC}(X)$, $i = 0, 1, \cdots, N$, $k = 1, 2$. The map $\mathcal{F}_i^k$ is given by

$$(\mathcal{F}_i^1 u)(t) = \begin{cases} g_i(t, u(t)), & \text{for } t \in (t_i, s_i],\ i \geq 1, \\ S_\alpha(t - s_i) g_i(s_i, u(s_i)) - \varphi(t, u(t)) \\ \quad - \int_{s_i}^t T_\alpha(t - s)\varphi(s, u(s)) ds, & \text{for } t \in (s_i, t_{i+1}],\ i \geq 1, \\ 0, & \text{for } t \notin (t_i, t_{i+1}],\ i \geq 0, \\ S_\alpha(t)[u_0 + \varphi(0, u_0)] - \varphi(t, u(t)) \\ \quad - \int_0^t T_\alpha(t - s)\varphi(s, u(s)) ds, & \text{for } t \in [0, t_1],\ i = 0, \end{cases}$$

$$(\mathcal{F}_i^2 u)(t) = \begin{cases} \int_{s_i}^t S_\alpha(t - s) f(s, u(s)) ds, & \text{for } t \in [s_i, t_{i+1}], i \geq 0, \\ 0, & \text{for } t \notin [s_i, t_{i+1}], i \geq 0. \end{cases}$$

The proof is divided into four steps.

Step 1. We begin by showing $\digamma B_r(0,\mathcal{PC}(X)) \subset B_r(0,\mathcal{PC}(X))$. Let $u \in B_r(0,\mathcal{PC}(X))$. For $i \geq 1$ and $t \in (t_i, t_{i+1}]$, we have

$$\begin{aligned}
&\|(\digamma_1 u)(t) + (\digamma_2 u)(t)\| \\
&\leq \|g_i(t,u(t)) - g_i(t,0)\| + \|g_i(t,0)\| + \|\varphi(t,u(t))\| \\
&\quad + \|S_\alpha(t-s_i)\| \, \|g_i(s_i,u(s_i)) - g_i(s_i,0)\| + \|S_\alpha(t-s_i)\| \|g_i(s_i,0)\| \\
&\quad + \int_{s_i}^t \|T_\alpha(t-s)\| \, \|\varphi(s,u(s))\| ds + \int_{s_i}^t \|S_\alpha(t-s)\| \, \|f(s,u(s))\| ds \\
&\leq N_\varphi + L_{g_i}\|u(t)\| + \|g_i(t,0)\| + \mathcal{M}_1 L_{g_i}\|u(t)\| + \mathcal{M}_1\|g_i(t,0)\| \\
&\quad + \mathcal{M}_2 N_\varphi (t-s_i) + \mathcal{M}_1 N_f (t-s_i) \\
&\leq N_\varphi + L_{g_i}(1+\mathcal{M}_1)\|u\|_{\mathcal{PC}} + (1+\mathcal{M}_1)\|g_i(t,0)\| \\
&\quad + (\mathcal{M}_2 N_\varphi + \mathcal{M}_1 N_f) T_0,
\end{aligned}$$

It follows from assumption (10) that

$$\|\digamma_1 u + \digamma_1 u\|_{\mathcal{PC}} \leq r \qquad \forall i \geq 1.$$

Similarly, for each $t \in [0, t_1]$, we have

$$\begin{aligned}
&\|(\digamma_1 u)(t) + (\digamma_2 u)(t)\| \\
&\leq \|S_\alpha(t)\| \, \|u_0 + \varphi(0,u_0)\| + \int_0^t \|T_\alpha(t-s)\| \|\varphi(s,u(s))\| ds \\
&\qquad + \int_0^t \|S_\alpha(t-s)\| \|f(s,u(s))\| ds + \|\varphi(t,u(t))\| \\
&\leq N_\varphi + \mathcal{M}_1 \|[u_0 + \varphi(0,u_0)]\| + (\mathcal{M}_2 N_\varphi + \mathcal{M}_1 N_f) T_0.
\end{aligned}$$

Using (9) and (10), we can conclude that

$$\|\digamma_1 u + \digamma_2 u\|_{\mathcal{PC}} \leq r.$$

Thus, we have $\digamma_1 u + \digamma_2 u \in B_r(0,\mathcal{PC}(X))$.

Step 2. In this step, we prove that $\digamma_1 = \sum_{i=0}^N \digamma_i^1$ is a contraction on $B_r(0,\mathcal{PC}(X))$. Let $t \in (t_i, t_{i+1}]$ and $u, v \in B_r(0,\mathcal{PC}(X))$. For $i = 1, \cdots, N$, we have

$$\|(\digamma_i^1 u)(t) - (\digamma_i^1 v)(t)\| \leq \Big[(1+\mathcal{M}_1)L_{g_i} + L_\varphi(1+\mathcal{M}_2 T_0)\Big] \|u-v\|_{C((t_i,t_{i+1}],X)}.$$

Thus

$$\left\| \sum_{i=0}^N \digamma_i^1 u - \sum_{i=0}^N \digamma_i^1 v \right\|_{\mathcal{PC}} \leq \Big[(1+\mathcal{M}_1)L_{g_i} + L_\varphi(1+\mathcal{M}_2 T_0)\Big] \|u-v\|_{\mathcal{PC}}.$$

It is clear from (12) that $\digamma_1$ is a contraction on $B_r(0,\mathcal{PC}(X))$.

Step 3. We prove that the set $\{\digamma_2 u : u \in B_r\}$ is relatively compact i.e., the set $\{(\digamma_2 u)(t) : u \in B_r\}$ is uniformly bounded, equicontinuous and for any $t \in [0, T_0]$.

The continuity of f implies that $\digamma_i^2$ is continuous for each $i = 0, 1, \cdots, N$ and $t \in [s_i, t_{i+1}]$. Thus $\digamma_2 = \sum_{i=0}^N \digamma_i^2$ is continuous and we have the following estimates

$$\|(\digamma_i^2 u)(t)\| \leq \mathcal{M}_1 N_f T_0, \quad \text{for } i = 0, 1, \cdots, N$$

for any $u \in B_r(0, \mathcal{PC}(X))$. Therefore, $\{\mathcal{F}_2 u : u \in B_r\}$ is uniformly bounded on B_r. Next, we prove that the set $\bigcup \mathcal{F}_i^2 B_r(0, \mathcal{PC}(X))(t)$ for $t \in [s_i, t_{i+1}],\ i = 0, 1, \cdots, N$, is relatively compact in X, where

$$\mathcal{F}_i^2 B_r(0, \mathcal{PC}(X))(t) = \{(\mathcal{F}_i^2 u)(t) : B_r(0, \mathcal{PC}(X))\}.$$

Applying mean value theorem for Bochner integral [17] and Young inequality, we have

$$(\mathcal{F}_0^2 u)(t) \subset \frac{t^{1+\alpha}}{\alpha} \overline{co\{\overline{S_\alpha(t-s)} f(s, u(s)) : s \in [0, t_1], u \in B_r\}}.$$

Similarly, for $t \in (s_i, t_{i+1}],\ i = 1, \cdots, N$, we obtain

$$(\mathcal{F}_i^2 u)(t) \subset \frac{(t-s_i)^{1+\alpha}}{\alpha} \overline{co\{\overline{S_\alpha(t-s)} f(s, u(s)) : s \in [s_i, t_{1+i}], u \in B_r\}}.$$

It follows from assumption (**H3**) that $\{(\mathcal{F}_i^2 u)(t)\}$ is a compact subset of X, for $t \in I,\ u \in B_r$. So, $\mathcal{F}_2$ is compact.

Step 4. In this step, we prove that the set of functions $[\mathcal{F}_i^2 B_r(\widetilde{0, \mathcal{PC}(X)})]_i,\quad i = 0, 1, \cdots, N$ is an equicontinuous subset of $C([t_i, t_{i+1}], X)$.

Clearly, $[\mathcal{F}_i^2 B_r(\widetilde{0, \mathcal{PC}(X)})]_i$ is equicontinuous on $[t_i, s_i],$ for each $i = 0, 1, \cdots, N$. Let $t_1, t_2 \in (s_i, t_{i+1}],\ i = 0, 1, \cdots, N,$ with $t_1 < t_2$ and $u \in B_r(0, \mathcal{PC}(X))$, we get

$$\begin{aligned} \|\widetilde{\mathcal{F}_i^2} u(t_2) - \widetilde{\mathcal{F}_i^2} u(t_1)\| \quad \leq \quad & \int_{t_1}^{t_2} \|S_\alpha(t_2 - s)\| \, \|f(s, u(s))\| ds \\ & + \int_{s_i}^{t_1} \|S_\alpha(t_2 - s) - S_\alpha(t_1 - s)\| \|f(s, u(s))\| ds. \end{aligned} \tag{13}$$

For the first term on the right hand side of (13), we have

$$\int_{t_1}^{t_2} \|S_\alpha(t_2 - s)\| \|\varphi(s, u(s))\| ds \leq \mathcal{M}_1 N_f s(t_2 - t_1). \tag{14}$$

For $t_1 = s_i$, it is easy to see that the second term on the right hand side of (13) will be zero. If $t_1 > s_i$ and $\nu > 0$ be sufficiently small, we have

$$\begin{aligned} & \int_{s_i}^{t_1 - \nu} \|[S_\alpha(t_2 - s) - S_\alpha(t_1 - s)]\| \|f(s, u(s))\| ds \\ & \qquad + \int_{t_1 - \nu}^{t_1} \|[S_\alpha(t_2 - s) - S_\alpha(t_1 - s)]\| \|f(s, u(s))\| ds \\ \leq \quad & N_f \sup_{s \in [s_i, t_1 - \nu]} \|S_\alpha(t_2 - s) - S_\alpha(t_1 - s)\| (t_1 - \nu) + 2\mathcal{M}_1 N_f \nu. \end{aligned} \tag{15}$$

It follows from (14) and (15) that

$$\|\widetilde{\mathcal{F}_i^2} u(t_2) - \widetilde{\mathcal{F}_i^2} u(t_1)\|$$

tends to zero as $t_2 \to t_1$ and $\nu \to 0$ for any $u \in B_r(0, \mathcal{PC}(X))$. This means that $[\mathcal{F}_i^2 B_r(\widetilde{0, \mathcal{PC}(X)})]_i$ is equicontinuous. Thus $[\mathcal{F}_i^2 B_r(\widetilde{0, \mathcal{PC}(X)})]_i$ is an equicontinuous subset of $C([t_i, t_{i+1}], X)$.

By Ascoli-Arzela theorem, $\{\mathcal{F}_2 u : u \in B_r\}$ is relatively compact. Hence $\mathcal{F}_2$ is a completely continuous operator. So by Krasnoselskii's fixed point theorem [1], $\mathcal{F}$ has a fixed point. This completes the proof of the existence of a mild solution. □

4 Application

We discuss the following problem to illustrate the results. We consider the following system with noninstantaneous impulse for fractional partial differential equations in $L^2([0,\pi])$,

$$\left.\begin{array}{rcl} {}^cD_t^\alpha[u(t,x)+\partial_x G(t,x,u(t,x))] &=& \dfrac{\partial^2}{\partial x^2}u(t,x) \\ && +\frac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t (t-s)^{-\alpha}F(s,x,u(s,x))ds, \\ && (t,x)\in\bigcup_{i=1}^N[s_i,t_{i+1}]\times(0,\pi), \\ u(t,0) &=& u(t,\pi)=0, \qquad t\in[0,T_0], \\ u(0,x) &=& u_0(x), \qquad x\in(0,\pi), \\ u(t,x) &=& H_i(t,x,u(t,x)),\ x\in(0,\pi), t\in(t_i,s_i], \end{array}\right\} \qquad (1)$$

where $0=t_0=s_0<t_1\le s_1<\cdots<t_N\le s_N<t_{N+1}=T_0$. Here T_0 is a fixed real number, $u_0\in X$, $F\in([0,T_0]\times[0,\pi]\times\mathbb{R},\mathbb{R})$ and $H_i\in C((t_i,s_i]\times[0,\pi]\times\mathbb{R},\mathbb{R})$ for all $i=1,\cdots,N$.

Let $X=L^2([0,\pi])$ and $Au=\frac{\partial^2}{\partial x^2}u$ with

$$D(A)=\{u\in X:\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2}\in X, u(0)=u(\pi)=0\}.$$

Then the operator $A:D(A)\subset X\to X$ is the infinitesimal generator of a solution operator $\{S_\alpha(t)\}_{t\ge 0}$ [3, see Theorem 3.1].

The system (1) can be formulated in the abstract form (1), where $u(t)=u(t,.)$, i.e., $u(t)(x)=u(t,x)$ and the functions $f:[0,T_0]\times X\to X$ and $g_i:(t_i,s_i]\times X\to X$ are given by

$$\begin{array}{rcl} f(t,u(t))(x) &=& F(t,x,u(t,x)), \\ \varphi(t,u(t))(x) &=& \partial_x G(t,x,u(t,x)), \\ g_i(t,u(t))(x) &=& H_i(t,x,u(t,x)). \end{array}$$

For $t\in[0,T_0],\ \ u\in X,\ x\in(0,\pi)$, we define f as

$$f(t,u(t))(x) \;=\; \frac{2e^{-t}|u(t,x)|}{(a+2e^t)(1+2|u(t,x)|)},\quad a>-1.$$

Then $f:[0,T_0]\times X\to X$ is continuous function and satisfies

$$\|f(t,u_1)-f(t,u_2)\|\le L_f\ \|u_1-u_2\|,$$

for $u_1,u_2\in X$ and $L_f=\frac{2}{a+2}$.

If we define g_i as follows

$$\begin{array}{rcl} g_i(t,u(t))(x) &=& \dfrac{(\cos(e^t)+\sin(e^{-t}))|u(t,x)|}{4(1+|u(t,x)|)}, \\ && t\in[t_i,s_i],\quad u\in X,\ x\in(0,\pi), \end{array}$$

then $g_i:[t_i,s_i]\times X\to X$ is continuous function and satisfies

$$\|g_i(t,u_1)-g_i(t,u_2)\|\le L_{g_i}\|u_1-u_2\|,$$

for $u_1,u_2\in X$ and $L_{g_i}=\frac{1}{2}$. Hence the assumptions in Theorem 3.1 are satisfied [25]. Thus we have the following theorem for the existence.

Theorem 4.1 *If φ is chosen such that*

$$\|\varphi(t,u)-\varphi(t,v)\| \leq L_\phi\|u-v\|, \quad t\in[0,T_0], \quad u,v\in X$$

and

$$L=\max\{\mathcal{M}_1(1/2+\frac{2}{a+2}\ T_0)+L_\varphi(1+\mathcal{M}_2T_0), 1/2:\ i=1,\cdots,N\}<1,$$

then problem (1) has a unique piecewise continuous mild solution.

Acknowledgements

The first author would like to thank the "International Travel Support Scheme" (ITS) committee members, Department of Science and Technology, New Delhi, India. Also, the first authors would like to acknowledge the financial aid from the Department of Science and Technology, New Delhi, India under its research project SERB/MATH/2014043.

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