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Approximate Controllability of Nonlinear Fractional Impulsive Stochastic Differential Equations with Nonlocal Conditions and Infinite Delay

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Abstract: This paper is concerned with the approximate controllability of nonlinear fractional impulsive stochastic differential equations with nonlocal conditions and infinite delay in Hilbert spaces. By using the Krasnoselskii-Schaefer-type fixed point theorem and stochastic analysis theory, some sufficient conditions are given for the approximate controllability of the system. At the end, an example is given to illustrate the application of our result.

Keywords: *approximate controllability; fixed point principle; fractional impulsive stochastic differential equations; mild solution; nonlocal conditions.*

Mathematics Subject Classification (2010): 65C30, 93B05, 34K40, 34K45.

1 Introduction

The controllability is one of the fundamental concepts in linear and nonlinear control theory, and plays a crucial role in both deterministic and stochastic control systems (see e.g. Zabczyk, [27]).The controllability of nonlinear systems represented by evolution equations or inclusions in abstract spaces and qualitative theory of fractional differential equations has been extensively considered in many publications and monographs, an

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extensive list of these publications can be found in Mahmudov [16] and the references contained therein.

On the other hand, the study of stochastic differential equations has attracted great interest due to their applications in characterizing many problems in physics, biology, chemistry, mechanics, and so on (see [6,7,9,12,17]) and the references contained therein). In practice, deterministic systems often fluctuate due to environmental noise. So it is important and necessary for us to discuss the stochastic control problems.

The problem with nonlocal condition, which is a generalization of the problem of classical condition, was motivated by physical problems. The pioneering work on nonlocal conditions is due to Byszewski (see [3–5]). Since it is demonstrated that the nonlocal problems have better effects in applications than the classical Cauchy problems, stochastic differential equations with nonlocal conditions were studied by many authors and some basic results on nonlocal problems have been obtained. Slama and Boudaoui [26] obtained sufficient conditions for the existence of mild solutions for the fractional impulsive stochastic differential equation with nonlocal conditions and infinite delay. (For more details see [1, 24] and the references contained therein).

The approximate controllability of stochastic or deterministic systems has received extensive attention where a pioneering work has been reported by Bashirov and Mahmudov [2]. Mahmudov [15] investigated the controllability of infinite dimensional linear stochastic systems, and in [10] Dauer and Mahmudov extended the results to semilinear stochastic evolution equations with finite delay. Sakthivel et al. [23] studied the approximate controllability of nonlinear deterministic and stochastic evolution systems with unbounded delay in abstract spaces. Kumar and Sukavanam [13] established sufficient conditions of the approximate controllability for a class of fractional order semilinear control systems with bounded delay. Shukla et al. [25] studied the approximate controllability of semilinear stochastic control system with nonlocal conditions in a Hilbert space, the results are obtained by using Sadovskii's fixed point theorem.

Recently, the approximate controllability of fractional stochastic differential systems has been investigated. Sakthivel et al. [22] studied a class of control systems described by nonlinear fractional stochastic differential equations in Hilbert spaces. Sufficient conditions for approximate controllability of fractional stochastic differential equations are formulated by using fixed point technique, fractional calculus, and stochastic analysis technique. Rajiv Ganthi and Muthukumar [20] discussed the approximate controllability of fractional stochastic integral equation with finite delays in Hilbert spaces, and the results are obtained by using the assumption that the corresponding linear integral equation is an approximate controllable and a stochastic version of the Banach fixed point theorem. Muthukumar and Rajivganthi [18] studied the approximate controllability of fractional order neutral stochastic integro-differential system with nonlocal conditions and infinite delay in Hilbert spaces under the assumptions that the corresponding linear system is approximately controllable. Guendouzi [11] discussed the existence and approximate controllability for impulsive fractional-order stochastic infinite delay integro-differential equations in Hilbert space, sufficient conditions for the approximate controllability of impulsive fractional stochastic system are derived by using Krasnoselskii's fixed point theorem with stochastic analysis theory. Zang and Li [28] studied the approximate controllability of fractional impulsive neutral stochastic differential equations with nonlocal conditions and infinite delay. Sufficient conditions are given for the approximate controllability of the system by using the Krasnoselskii-Schaefer-type fixed point theorem and stochastic analysis theory.

For the best of our knowledge, there is no work reported on approximate controllability of nonlinear fractional impulsive stochastic differential equations with nonlocal conditions and infinite delay. Motivated by this consideration, in this paper we will study the approximate controllability of nonlinear fractional impulsive stochastic differential equations with nonlocal conditions and infinite delay in Hilbert space. Our approach is based on the fixed point theorem. The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries such as definitions of fractional calculus and some useful lemmas. In Section 3, we prove our main results. Finally in Section 4, an example is given to demonstrate the application of our results.

2 Preliminaries and Basic Properties

In this section, we introduce some notations and preliminary results, needed to establish our results. Throughout this paper, $\mathbb{H}, \mathbb{U}$ are two separable Hilbert spaces and $L(\mathbb{U}, \mathbb{H})$ is the space of bounded linear operators from $\mathbb{U}$ into $\mathbb{H}$. For convenience, we will use the same notation $\| \ . \ \|$ to denote the norms in $\mathbb{H}, \mathbb{U}$ and $L(\mathbb{U}, \mathbb{H})$, and use $\langle ., . \rangle$ to denote the inner product of $\mathbb{H}$ and $\mathbb{U}$ without any confusion. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space satisfying the usual conditions (i.e., it is increasing and right continuous, while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$). Let $W = (W_t)_{t\geq 0}$ be a Q-Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with the covariance operator Q such that $TrQ < \infty$. Let $W = W(t)_{t\geq 0}$ be a Q-Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with the covariance operator Q, that is

$$E\langle W(t), x\rangle\langle W(s), y\rangle = (t \wedge s)\langle Qx, y\rangle \quad \forall x, y \in \mathbb{U} \quad \text{and} \quad t, s \in [0, T],$$

where Q is a positive, self-adjoint, trace class operator on $\mathbb{U}$.

Let $\mathscr{L}_2^0 = \mathscr{L}_2(\mathbb{U}, \mathbb{H})$ be the space of all Hilbert-Schmidt operators from $\mathbb{U}$ to $\mathbb{H}$ with the inner product $< \varphi, \psi >_{\mathscr{L}_2^0} = Tr[\varphi Q \psi^*]$. We consider the following fractional stochastic impulsive integro-differential systems with nonlocal conditions:

$$\begin{cases} D_t^\alpha x(t) & = Ax(t) + Bu(t) + f(t, x_t, B_1 x(t)) \\ & \quad +\sigma(t, x_t, B_2 x(t))dW(t), t \in J = [0, T], T > 0, t \neq t_k, \\ \Delta x(t_k) & = I_k(x(t_k^-)), \quad k = 1, \cdots, m, \\ x(0) + g(x) & = x_0 = \phi, \quad \phi \in B_h, \end{cases} \tag{1}$$

where D_t^α is the Caputo fractional derivative of order α, $0 < \alpha < 1$, the state variable $x(.)$ takes the value in the separable Hilbert space $\mathbb{H}$; $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of a strongly continuous semigroup of a bounded linear operators $T(t), t \geq 0$ in the Hilbert space $\mathbb{H}$. The control function $u(.)$ is given in $L^2(J; \mathbb{U})$, $\mathbb{U}$ is a Hilbert space; B is a bounded linear operator from $\mathbb{U}$ into $\mathbb{H}$. The history $x_t : (-\infty, 0] \to \mathbb{H}, x_t(\theta) = x(t + \theta), \quad \theta \leq 0$ belongs to an abstract phase space $\mathcal{B}_h$; $f : J \times \mathcal{B}_h \times \mathbb{H} \to \mathbb{H}$, $\sigma : J \times \mathcal{B}_h \times \mathbb{H} \to \mathscr{L}_2^0$ and $g : B_h \to \mathbb{H}$ are appropriate functions to be specified later; $I_k : \mathbb{H} \to \mathbb{H}$, $(k = 1, 2, \cdots, m)$, are appropriate functions. The terms $B_1 x(t)$ and $B_2 x(t)$ are given by $B_1 x(t) = \int_0^t K(t, s)x(s)ds$ and $B_2 x(t) = \int_0^t P(t, s)x(s)ds$ respectively, where $K, P \in C(D, \mathbb{R}^+)$ are the set of all positive continuous functions on $D = \{(t, s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$. Here $0 = t_0 \leq t_1 \leq \cdots \leq t_m \leq t_{m+1} = T$, $\Delta x(t_k) = I_k(x(t_k^-)) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{h\to 0} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0} x(t_k - h)$ represent the right and left limits of $x(t)$ at $t = t_k$ respectively. The initial data $\phi = \{\phi(t); t \in (-\infty, 0]\}$ is an $\mathcal{F}_0$-measurable, $\mathcal{B}_h$-valued random variable independent of $W(t)$ with finite second moments.

Now, we present the abstract space phase $\mathcal{B}_h$. Assume that $h : (-\infty, 0] \to (0, +\infty)$ with $l = \int_{-\infty}^{0} h(t)dt < \infty$ is a continuous function. We define the abstract phase space $\mathcal{B}_h$ by

$$\mathcal{B}_h := \Big\{ \phi : (-\infty, 0] \to \mathbb{H}, \text{for any} \quad a > 0, (E \mid \phi(\theta \mid^2)^{\frac{1}{2}} \\ \text{is bounded and measurable function on} \\ [-a, 0] \quad \text{and} \quad \int_{-\infty}^{0} h(s) \sup_{s \le \theta \le 0} (E \mid \phi(\theta \mid^2)^{\frac{1}{2}} < +\infty \Big\}.$$

If $\mathcal{B}_h$ is endowed with the norm

$$\|\phi\|_{\mathcal{B}_h} := \int_{-\infty}^{0} h(s) \sup_{s \le \theta \le 0} (E \mid \phi(\theta) \mid^2)^{\frac{1}{2}}, \phi \in \mathcal{B}_h,$$

then $(\mathcal{B}_h, \|.\|_{\mathcal{B}_h})$ is a Banach space [19,21].

Now we consider the space

$$\mathcal{B}_b := \{ x : (-\infty, T] \to \mathbb{H}, \quad \text{such that} \quad x|_{J_k} \in C(J_k, \mathbb{H}) \\ \text{and there exist} \quad x(t_k^+), \quad \text{and} \quad x(t_k^-) \\ \text{with} \quad x(t_k) = x(t_k^-), x_0 = \phi \in \mathcal{B}_h, k = 1, \cdots, m \},$$

where $x|_{J_k}$ is the restriction of x to $J_k = (t_k, t_{k+1}]$, $k = 0, 1, 2, \cdots, m$. We endow a seminorm $\|.\|_{\mathcal{B}_b}$on $\mathcal{B}_b$, it is defined by

$$\|x\|_{\mathcal{B}_b} = \|\phi\|_{\mathcal{B}_h} + \sup_{0 \le s \le T} (E\|x(s)\|^2)^{\frac{1}{2}}, x \in \mathcal{B}_b.$$

We recall the following lemma.

Lemma 2.1 *[21] Assume that $x \in \mathcal{B}_b$; then for $t \in J, x_t \in \mathcal{B}_h$. Moreover*

$$l(E\|x(t)\|^2)^{\frac{1}{2}} \le l \sup_{s \in [0,t]} E\|x(s)\|^2)^{\frac{1}{2}} + \|x_0\|_{\mathcal{B}_h},$$

where $l = \int_{-\infty}^{0} h(s)ds < \infty$.

Definition 2.1 [8] The Caputo derivative of order α for a function $f : [0, \infty) \to \mathbb{R}$, which is at least n-times differentiable can be defined as

$$D_a^{\alpha} f(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - s)^{n - \alpha - 1} f^{(n)}(s)ds = I_a^{(n-\alpha)} \left(\frac{d^n f}{dt^n} \right)(t) \tag{2}$$

for $n - 1 \le \alpha < n, n \in \mathbb{N}$. If $0 < \alpha \le 1$, then

$$D_a^{\alpha} f(t) = \frac{1}{\Gamma(1 - \alpha)} \int_a^t (t - s)^{-\alpha} \left(\frac{df(s)}{ds} \right) ds. \tag{3}$$

Obviously, the Caputo derivative of a constant is equal to zero. The Laplace transform of the Caputo derivative of order $\alpha > 0$ is given as

$$L\{D_t^{\alpha} f(t); \lambda\} = \lambda^{\alpha} \widehat{f}(\lambda) - \sum_{k=0}^{n-1} \lambda^{\alpha - k - 1} f^{(k)}(0) \quad n - 1 \le \alpha < n.$$

Definition 2.2 The fractional integral of order α with the lower limit 0 for a function f is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (s-t)^{\alpha-1} f(s) ds \tag{4}$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where Γ is the gamma function.

Definition 2.3 A stochastic process $x : J \times \Omega \to \mathbb{H}$ is called a mild solution of the system (1) if

(i) $x(t)$ is measurable and $\mathcal{F}_t$-adapted, for each $t \geq 0$;

(ii) $x(t) \in \mathbb{H}$ has càdlàg paths on $t \in [0, T]$ a.s., and satisfies the following integral equation

$$\begin{aligned} x(t) = \;& T_\alpha(t)(\phi(0) - g(x)) + \textstyle\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) Bu(s) ds \\ &+ \textstyle\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s, x_s, B_1 x(s)) ds \\ &+ \textstyle\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \sigma(s, x_s, B_2 x(s)) dW(t) \\ &+ \textstyle\sum_{0<t_k<t} T_\alpha(t-t_k) I_k(x(t_k^-)), \quad t \in J; \end{aligned} \tag{5}$$

(iii) $x_0 = \phi \in \mathcal{B}_h$ on $(-\infty, 0]$ satisfying $\|\phi\|_{\mathcal{B}_h} < \infty$, where

$$T_\alpha(t) = \int_0^\infty \xi_\alpha(\theta) T(t^\alpha \theta) d\theta, \quad S_\alpha(t) = \alpha \int_0^\infty \theta \xi_\alpha(\theta) T(t^\alpha \theta) d\theta$$

$$\xi_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \varpi_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0,$$

$$\varpi(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-n\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \quad \theta \in (0, \infty),$$

ξ_α is a probability density function defined on $(0, \infty)$, that is,

$$\xi_\alpha \geq 0, \quad \theta \in (0, \infty), \quad \text{and} \quad \int_0^\infty \xi_\alpha(\theta) d\theta = 1.$$

Lemma 2.2 *[29] The operators T_α and S_α have the following properties:*

(i) For any fixed $t \geq 0$, $T_\alpha(t)$ and $S_\alpha(t)$ are linear and bounded operators, i.e., for any $x \in X$,

$$\|T_\alpha(t)x\| \leq M\|x\|, \quad \|S_\alpha(t)x\| \leq \frac{\alpha M}{\Gamma(1+\alpha)} \|x\|.$$

(ii) $\{T_\alpha(t), t \geq 0\}$ and $\{S_\alpha(t), t \geq 0\}$ are strongly continuous, which means that for every $x \in \mathbb{H}$ and for $0 \leq t' < t'' \leq T$, we have

$$\|T_\alpha(t'')x - T_\alpha(t')x\| \to 0 \quad \textit{and} \quad \|S_\alpha(t'')x - S_\alpha(t')x\| \to 0, \quad \textit{as} \quad t' \to t''.$$

(iii) For every $t \geq 0$, $T_\alpha(t)$ and $S_\alpha(t)$ are also compact operators if $T(t)$ is compact for every $t > 0$.

In order to study the approximate controllability for the fractional control system (1), we introduce the following linear fractional differential system

$$\begin{cases} D_t^{\alpha} x(t) = & Ax(t) + Bu(t), \quad t \in J, \\ x(0) = x_0. \end{cases} \tag{6}$$

The controllability operator associated with (6) is defined by

$$\Gamma_0^T = \int_O^T (T-s)^{\alpha-1} S_\alpha(t-s) BB^* S_\alpha^*(T-s) ds,$$

where B^* and S_α^* denote the adjoint of B and S_α, respectively.

Let $x(T; \phi, u)$ be the state value of (1) at terminal time T, corresponding to the control u and the initial value ϕ. Denote by $R(T, \phi) = \{x(T; \phi, u) : u \in L^2(J, \mathbb{U})\}$ the reachable set of system (1) at terminal time T, its closure in $\mathbb{H}$ is denoted by $\overline{R(T, \phi)}$.

Definition 2.4 The system (1) is said to be approximately controllable on J if $\overline{R(T, \phi)} = L^2(\Omega, \mathbb{H})$.

Lemma 2.3 *[14] The linear fractional control system* (6) *is approximately controllable on J if and only if $\lambda(\lambda I + \Gamma_0^T) \to 0$ as $\lambda \to 0^+$ in the strong operator topology.*

Lemma 2.4 *[29] (Krasnoselskii's fixed point theorem) Let E be a Banach space, let $\hat{E}$ be a bounded closed and convex subset of E, and let F_1, F_2 be maps of $\hat{E}$ into E such that $F_1 x + F_2 y \in \hat{E}$ for every pair $x, y \in \hat{E}$. If F_1 is a contraction and F_2 is completely continuous, then the equation $F_1 x + F_2 x = x$ has a solution on $\hat{E}$.*

3 Main Results

In this section, we formulate sufficient conditions for the approximate controllability of system (1). For this purpose, we first prove the existence of solutions for system (1). Second, in Theorem 3.2, we shall prove that system (1) is approximately controllable under certain assumptions.

In order to establish the results, we impose the following conditions

(H1) $f : J \times \mathcal{B}_h \times \mathbb{H} \to \mathbb{H}$ is continuous and there exist μ_1, $\mu_2 > 0$ such that

$$E\|f(t, \gamma, x) - f(t, \psi, y)\|_{\mathbb{H}}^2 \le \mu_1 \|\gamma - \psi\|_{\mathcal{B}_h}^2 + \mu_2 E\|x - y\|_{\mathbb{H}}^2,$$

and there exist two continuous functions $\mu_1, \mu_2 : J \to (0, \infty)$ such that

$$E\|f(t, \psi, x)\|_{\mathbb{H}}^2 \le \mu_1(t) \|\psi\|_{\mathcal{B}_h}^2 + \mu_2(t) E\|x\|_{\mathbb{H}}^2, \quad (t, \psi, x) \in J \times \mathcal{B}_h \times \mathbb{H},$$

where $\mu_1^* = \sup_{s \in [0,t]} \mu_1(s)$ and $\mu_2^* = \sup_{s \in [0,t]} \mu_2(s)$.

(H2) There exist ν_1, $\nu_2 > 0$ such that

$$E\|\sigma(t, \gamma, x) - f(t, \psi, y)\|_{\mathscr{L}_2^0}^2 \le \nu_1 \|\gamma - \psi\|_{\mathcal{B}_h}^2 + \nu_2 E\|x - y\|_{\mathbb{H}}^2,$$

and there exist two continuous functions $\nu_1, \nu_2 : J \to (0, \infty)$ such that

$$E\|\sigma(t, \psi, x)\|_{\mathscr{L}_2^0}^2 \le \nu_1(t) \|\psi\|_{\mathcal{B}_h}^2 + \nu_2(t) E\|x\|_{\mathbb{H}}^2, \quad (t, \psi, x) \in J \times \mathcal{B}_h \times \mathscr{L}_2^0,$$

where $\nu_1^* = \sup_{s \in [0,t]} \nu_1(s)$ and $\nu_2^* = \sup_{s \in [0,t]} \nu_2(s)$.

(H3) g is continuous, and there exist some positive constants δ_1 such that

$$E\|g(x)\|_{\mathbb{H}}^2 \leq \delta_1 \|x\|_{\mathcal{B}_h}^2.$$

(H4) The function $I_k : \mathbb{H} \to \mathbb{H}$ is continuous and there exist continuous nondecreasing functions L_k such that, for each $x \in \mathbb{H}$,

$$E\|I_k(x)\|_{\mathbb{H}}^2 \leq L_k E\|x\|_{\mathbb{H}}^2 \quad \text{and} \quad \lim_{r \longrightarrow +\infty} \frac{L_k(r)}{r} = \beta_k < \infty, \quad k = \cdots, n.$$

(H5) The linear stochastic system (6) is approximately controllable on $[0, T]$.

The following lemma is required to define the control function.

Lemma 3.1 *[15] For any $\overline{x}_T \in L^2(\mathcal{F}_T, H)$, there exists $\eta(.) \in L^2_{\mathcal{F}}(\Omega; L^2(J; L_2^0))$ such that $\overline{x}_T = E\overline{x}_T + \int_0^T \eta(s)dW(s)$.*

Now, for any $\lambda > 0$ and $\overline{x}_T \in L^2(\mathcal{F}_T, H)$, we define the control function

$$\begin{aligned} u^\lambda(t) \quad = \quad & B^* S_\alpha^*(T-t)(\lambda I + \Gamma_0^T)^{-1} \\ & \times \Big[E\overline{x}_T + \int_0^t \eta(s)dW(s) + T_\alpha(T)(\phi(0) - g(x)) \Big] \\ & - B^* S_\alpha^*(T-t) \int_0^t (\lambda I + \Gamma_s^T)^{-1} (T-s)^{\alpha-1} S_\alpha(T-t) f(s, x_s, B_1 x(s)) ds \\ & - B^* S_\alpha^*(T-t) \int_0^t (\lambda I + \Gamma_s^T)^{-1} (T-s)^{\alpha-1} S_\alpha(T-t) g(s, x_s, B_2 x(s)) dW(s) \\ & - B^* S_\alpha^*(T-t)(\lambda I + \Gamma_0^T)^{-1} \sum_{0<t_k<T} T_\alpha(T-t_k) I_k(x(t_k^-)). \end{aligned}$$

Theorem 3.1 *Assume that the conditions $(H1)-(H4)$ hold. Then for each $\lambda > 0$, the system (1) has a mild solution on $[0, T]$, provided that*

$$\begin{aligned} \Big[4l^2 M^2 \delta_1 + \quad & \Big(\tfrac{MT^\alpha}{\Gamma(1+\alpha)}\Big)^2 (4l^2 \mu_1^* + \mu_2^* B_1^*) + \tfrac{T^{2\alpha-1}}{2\alpha-1} \Big(\tfrac{\alpha M}{\Gamma(1+\alpha)}\Big)^2 (4l^2 \nu_1^* + \nu_2^* B_2^*) \\ & + 4l^2 m M^2 \textstyle\sum_{k=1}^m \beta_k \Big] . \Big[5 + \tfrac{30 T^{2\alpha}}{\lambda^2 \alpha^2} \Big(\tfrac{\alpha M M_B}{\Gamma(1+\alpha)}\Big)^4 \Big] \leq 1 \end{aligned}$$

and

$$2 \left[\frac{T^{2\alpha}}{\alpha^2} \Big(\frac{\alpha M}{\Gamma(1+\alpha)}\Big)^2 (\mu_1 l + \mu_2 B_1^*) + \frac{T^{2\alpha-1}}{2\alpha-1} \Big(\frac{\alpha M}{\Gamma(1+\alpha)}\Big)^2 (\nu_1 l + \nu_2 B_2^*) \right] < 1,$$

where $B_1^* = \sup_{t\in[0,T]} \int_0^t K(t,s)ds < \infty$ and $B_2^* = \sup_{t\in[0,T]} \int_0^t P(t,s)ds < \infty$.

Proof. For any $\lambda > 0$, define the operator $\Psi : \mathcal{B}_b \to \mathcal{B}_b$ by

$\Psi x(t) = \phi(t), \quad t \in (-\infty, 0],$

$$\begin{aligned}\Psi x(t) = T_\alpha(t)(\phi(0) - g(x)) &+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) B u^\lambda(s) ds \\ &+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s, x_s, B_1(x(s))) ds \\ &+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \sigma(s, x_s, B_2(x(s))) dW(t) \\ &+ \sum_{0<t_k<t} T_\alpha(t-t_k) I_k(x(t_k^-)), t \in J.\end{aligned}$$

We shall show that the operator Ψ has a fixed point in the space $\mathcal{B}_b$, which is the mild solution of (1).

For $\phi \in \mathcal{B}_h$, we define $\widehat{\phi}$ by

$$\widehat{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ T_\alpha(t)\phi(0), & t \in J; \end{cases} \quad \text{then } \widehat{\phi} \in \mathcal{B}_b.$$

Let $x(t) = y(t) + \widehat{\phi}(t), -\infty < t < T$. It is evident that y satisfies $y_0 = 0, t \in (-\infty, 0]$ and

$$\begin{aligned} y(t) =& T_\alpha(t)(-g(y + \widehat{\phi})) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) B u^\lambda(s) ds \\ &+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s, y_s + \widehat{\phi}_s, B_1(y(s) + \widehat{\phi}(s))) ds \\ &+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \sigma(s, y_s + \widehat{\phi}_s, B_2(y(s) + \widehat{\phi}(s))) dW(s) \\ &+ \sum_{0<t_k<t} T_\alpha(t-t_k) I_k(y(t_k^-) + \widehat{\phi}(t_k^-)), \ t \in J. \end{aligned}$$

Set $\mathcal{B}_b^0 = \{y \in \mathcal{B}_b, \quad \text{such that} \quad y_0 = 0 \in \mathcal{B}_h\}$ and for any $y \in \mathcal{B}_b^0$ we have

$$\|y\|_{\mathcal{B}_b^0} = \|y_0\|_{\mathcal{B}_h} + \sup_{t \in J}(E\|y(t)\|^2)^{\frac{1}{2}} = \sup_{t \in J}(E\|y(t)\|^2)^{\frac{1}{2}},$$

thus $(\mathcal{B}_b^0, \|.\|_{\mathcal{B}_b^0})$ is a Banach space.

Let $\mathcal{B}_r = \left\{y \in \mathcal{B}_b^0, \quad \|y\|^2_{\mathcal{B}_b^0} \leq r, \ r > 0\right\}$. The set $\mathcal{B}_r$ is clearly a bounded closed convex set in $\mathcal{B}_b^0$ for each $r > 0$ and for each $y \in \mathcal{B}_r$. By Lemma 2.1 we have

$$\begin{aligned} \|y_t + \widehat{\phi}_t\|^2_{\mathcal{B}_h} \leq \ & 2(\|y_t\|^2_{\mathcal{B}_h} + \|\widehat{\phi}_t\|^2_{\mathcal{B}_h}) \\ & \leq 4(l^2 \sup_{s\in[0,t]} E\|y(s)\|^2_{\mathbb{H}} + \|y_0\|^2_{\mathcal{B}_h}) \\ & +4(l^2 \sup_{s\in[0,t]} E\|\widehat{\phi}(s)\|^2_{\mathbb{H}} + \|\widehat{\phi}_0\|^2_{\mathcal{B}_h}) \\ & \leq 4(\|\phi\|^2_{\mathcal{B}_h} + l^2(r + M^2 E\|\phi(0)\|^2_{\mathbb{H}})). \end{aligned}$$

For the sake of convenience, we divide the proof into several steps.

Step 1. We claim that there exists a positive number r such that $\Psi(\mathcal{B}_r) \subset \mathcal{B}_r$. If this is not true, then, for each positive integer r, there exists $y^r \in \mathcal{B}_r$ such that $E\|\Psi(y^r)(t)\|^2 > r$ for $t \in (-\infty, T]$, t may depending upon r. However, on the other hand, we have

$$\begin{aligned}
r &\leq E\|\Psi(y^r)(t)\|^2\\
&\leq 5E\|T_\alpha(t)[-g(y^r+\widehat{\phi})\|^2\\
&\quad +5E\|\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)Bu^\lambda(s)ds\|^2\\
&\quad +5E\|\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)f(s,y_s^r+\widehat{\phi}_s,B_1(y_s^r+\widehat{\phi}_s))ds\|^2\\
&\quad +5E\|\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)\sigma(s,y_s^r+\widehat{\phi}_s,B_2(y_s^r+\widehat{\phi}_s))dW(t)\|^2\\
&\quad +5E\|\sum_{0<t_k<t}\|T_\alpha(t-t_k)I_k(y(t_k^-)+\widehat{\phi}(t_k^-))\|^2, \quad t\in J.
\end{aligned}$$

By using (H1)-(H4), Lemma 2.1 and Hölder's inequality, we obtain

$$\begin{aligned}
r &\leq E\|(\Psi y^r)(t)\|^2\\
&\leq 5M^2\delta_1\|y^r+\widehat{\phi}\|^2_{\mathcal{B}_h}+5\frac{T^\alpha}{\alpha}\left(\frac{\alpha MM_B}{\Gamma(1+\alpha)}\right)^2\int_0^t(t-s)^{\alpha-1}E\|u^\lambda(s)\|^2ds\\
&\quad +5\frac{T^\alpha}{\alpha}\left(\frac{\alpha M}{\Gamma(1+\alpha)}\right)^2\int_0^t(t-s)^{\alpha-1}E\|f(s,y_s^r+\widehat{\phi}_s,B_1(y_s^r+\widehat{\phi}_s))\|^2ds\\
&\quad +5\left(\frac{\alpha M}{\Gamma(1+\alpha)}\right)^2\int_0^t(t-s)^{2(\alpha-1)}E\|\sigma(s,y_s^r+\widehat{\phi}_s,B_2(y_s^r+\widehat{\phi}_s))\|^2_{\mathcal{L}_2^0}ds\\
&\quad +5mM^2\sum_{0=1}^m E\|I_k(y(t_k^-)+\widehat{\phi}(t_k^-))\|^2\\
&\leq 5M^2\delta_1r'+\frac{30T^{2\alpha}}{\lambda^2\alpha^2}\left(\frac{\alpha MM_B}{\Gamma(1+\alpha)}\right)^4\delta_2\\
&\quad +5\left(\frac{MT^\alpha}{\Gamma(1+\alpha)}\right)^2(\mu_1^*r'+\mu_2^*B_1^*(\sup_{s\in[0,T]}E\|x\|^2)\\
&\quad +5\frac{T^{2\alpha-1}}{2\alpha-1}\left(\frac{\alpha M}{\Gamma(1+\alpha)}\right)^2(\nu_1^*r'+\nu_2^*B_2^*(\sup_{s\in[0,T]}E\|x\|^2)\\
&\quad +5mM^2\sum_{0=1}^m L_k(r'),\\
&\leq 5M^2\delta_1r'+\frac{30T^{2\alpha}}{\lambda^2\alpha^2}\left(\frac{\alpha MM_B}{\Gamma(1+\alpha)}\right)^4\delta_2\\
&\quad +5\left(\frac{MT^\alpha}{\Gamma(1+\alpha)}\right)^2(\mu_1^*r'+\mu_2^*B_1^*r)+5\frac{T^{2\alpha-1}}{2\alpha-1}\left(\frac{\alpha M}{\Gamma(1+\alpha)}\right)^2(\nu_1^*r'+\nu_2^*B_2^*r)\\
&\quad +5mM^2\sum_{k=1}^m L_kr',
\end{aligned}$$

where $r'=4(\|\phi\|^2_{\mathcal{B}_h}+l^2(r+M^2E\|\phi(0)\|^2_{\mathbb{H}}))$,$\|B\|\leq M_B$ and

$$\begin{aligned}
\delta_2 &= 2E\|\bar{x}_T\|^2+2\int_0^tE\|\eta(s)\|^2_{\mathcal{L}_2^0}ds+M^2\|\phi\|^2_{\mathcal{B}_h}+M^2\delta_1r'\\
&\quad +\left(\frac{MT^\alpha}{\Gamma(1+\alpha)}\right)^2(\mu_1^*r'+\mu_2^*B_1^*r)+\frac{T^{2\alpha-1}}{2\alpha-1}\left(\frac{\alpha M}{\Gamma(1+\alpha)}\right)^2(\nu_1^*r'+\nu_2^*B_2^*r)\\
&\quad +mM^2\sum_{k=1}^m L_kr'.
\end{aligned}$$

Dividing both sides by r and taking the limit as $r\longrightarrow\infty$, we obtain

$$\begin{aligned}
1\leq\Big[4l^2M^2\delta_1+ &\left(\frac{MT^\alpha}{\Gamma(1+\alpha)}\right)^2(4l^2\mu_1^*+\mu_2^*B_1^*)+\frac{T^{2\alpha-1}}{2\alpha-1}\left(\frac{\alpha M}{\Gamma(1+\alpha)}\right)^2(4l^2\nu_1^*+\nu_2^*B_2^*)\\
&+4l^2mM^2\sum_{k=1}^m\beta_k\Big].\Big[5+\frac{30T^{2\alpha}}{\lambda^2\alpha^2}\left(\frac{\alpha MM_B}{\Gamma(1+\alpha)}\right)^4\Big]
\end{aligned}$$

which is a contradiction to our assumption. Thus, for each $\lambda>0$, there exists some positive number r such that $\Psi(\mathcal{B}_r)\subset\mathcal{B}_r$.

Next, we show that the operator Ψ is condensing, for convenience, we decompose Ψ as $\Psi=\Psi_1+\Psi_2$, where

$$(\Psi_1y)(t)=\begin{cases}\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)f(s,y_s+\widehat{\phi}_s,B_1(y(s)+\widehat{\phi}(s)))ds\\ +\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)\sigma(s,y_s+\widehat{\phi}_s,B_2(y(s)+\widehat{\phi}(s)))dW(t),\end{cases} \tag{7}$$

and

$$(\Psi_2 y)(t) = \begin{cases} T_\alpha(t)(-g(y+\widehat{\phi})) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) B u^\lambda(s) ds \\ \quad + \sum_{0<t_k<t} T_\alpha(t-t_k) I_k(y(t_k^-) + \widehat{\phi}(t_k^-)), \qquad t \in [0,T]. \end{cases} \tag{8}$$

Step 2. We prove that Ψ_1 is a contraction on $\mathcal{B}_r$. Let $t \in J$ and $y,\, y^* \in \mathcal{B}_r$, we have

$$\begin{aligned}
&\|(\Psi_1 y)(t) - (\Psi_1 y^*)(t)\|^2_{\mathbb{H}} \\
&\leq 2E\| \int_0^t (T-s)^{\alpha-1} S_\alpha(T-s) \Big[f(s, y_s + \widehat{\phi}_s, B_1(y(s) + \widehat{\phi}(s))) \\
&-f(s, y^*_s + \widehat{\phi}_s, B_1(y^*(s) + \widehat{\phi}(s))) \Big] ds \|^2_{\mathbb{H}} \\
&+2E\| \int_0^t (T-s)^{\alpha-1} S_\alpha(T-s) \Big[\sigma(s, y_s + \widehat{\phi}_s, B_2(y(s) + \widehat{\phi}(s))) \\
&-\sigma(s, y^*_s + \widehat{\phi}_s, B_2(y^*(s) + \widehat{\phi}(s))) \Big] dW(t) \|^2_{\mathbb{H}} \\
&\leq 2 \frac{T^\alpha}{\alpha} \left(\frac{\alpha M}{\Gamma(1+\alpha)} \right)^2 \int_0^t (T-s)^{\alpha-1} \big[\mu_1 \|y(s) - y^*(s)\|^2_{\mathcal{B}_h} \\
&+\mu_2 E\|B_1(y(s) + \widehat{\phi}(s)) - B_1(y^*(s) + \widehat{\phi}(s)\|^2_{\mathbb{H}} \Big] ds \\
&+2 \left(\frac{\alpha M}{\Gamma(1+\alpha)} \right)^2 \int_0^t (T-s)^{2(\alpha-1)} \big[\nu_1 \|y_s - y^*_s\|^2_{\mathcal{B}_h} \\
&+\nu_2 E\|B_2(y(s) + \widehat{\phi}(s)) - B_2(y^*(s) + \widehat{\phi}(s))\|^2_{\mathbb{H}} \Big] ds \\
&\leq 2 \left[\frac{T^{2\alpha}}{\alpha^2} \left(\frac{\alpha M}{\Gamma(1+\alpha)} \right)^2 (\mu_1 l + \mu_2 B_1^*) \right. \\
&\left. + \frac{T^{2\alpha-1}}{2\alpha-1} \left(\frac{\alpha M}{\Gamma(1+\alpha)} \right)^2 (\nu_1 l + \nu_2 B_2^*) \right] \|y - y^*\|^2_{\mathcal{B}_b^0},
\end{aligned}$$

where $2\left[\frac{T^{2\alpha}}{\alpha^2} \left(\frac{\alpha M}{\Gamma(1+\alpha)} \right)^2 (\mu_1 l + \mu_2 B_1^*) + \frac{T^{2\alpha-1}}{2\alpha-1} \left(\frac{\alpha M}{\Gamma(1+\alpha)} \right)^2 (\nu_1 l + \nu_2 B_2^*) \right] < 1$, hence Ψ_1 is a contraction.

Step 3. Ψ_2 maps bounded sets into bounded sets in $\mathcal{B}_r$, Let us prove that for $r > 0$ there exists a $\widehat{r} > 0$ such that for each $y \in \mathcal{B}_r$ we have $E\|(\Psi_2 y)(t)\|^2_{\mathbb{H}} < \widehat{r}$ for $t \in J$. Now we have

$$\begin{aligned}
E\|\Psi_2 y(t)\|^2_{\mathbb{H}} &\leq 3E\|T_\alpha(t)(-g(y+\widehat{\phi}))\|^2 \\
&\quad +3E\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) B u^\lambda(s) ds \|^2 \\
&\quad +3E\| \sum_{0<t_k<t} T_\alpha(t-t_k) I_k(y(t_k^-) + \widehat{\phi}(t_k^-)) \|^2 \\
&\leq 3M^2 \delta_1 r' + \frac{18}{\lambda^2} \frac{T^{2\alpha}}{\alpha^2} \left(\frac{\alpha M M_B}{\Gamma(1+\alpha)} \right)^4 \delta_2 + 3M^2 m^2 \sum_{k=1}^m L_k r' \\
&= \widehat{r}.
\end{aligned}$$

Step 4. The map Ψ_2 is equicontinuous. Let $u, v \in J$, $0 \le u < v \le T$, $y \in \mathcal{B}_r$, we obtain

$$\begin{aligned}
&E\|(\Psi_2 y)(v) - (\Psi_2 y)(u)\|^2_{\mathbb{H}} \le \\
&5E\|T_\alpha(v) - T_\alpha(u)(-g(y+\widehat{\phi}))\|^2 \\
&+ 5E\|\int_0^u (u-s)^{\alpha-1}[S_\alpha(v-s) - S_\alpha(u-s)]Bu^\lambda(s)ds\|^2 \\
&+ 5E\|\int_0^u [(v-s)^{\alpha-1} - (u-s)^{\alpha-1}]S_\alpha(v-s)Bu^\lambda(s)ds\|^2 \\
&+ 5E\|\int_u^v (v-s)^{\alpha-1}S_\alpha(v-s)Bu^\lambda(s)ds\|^2 \\
&+ 5E\|\sum_{0\le t_k\le T}[T_\alpha(v-t_k) - T_\alpha(u-t_k)]I_k(y(t_k^-) + \widehat{\phi}(t_k^-))\|^2.
\end{aligned}$$

Noting the fact that for every $\epsilon > 0$, there exists a $\delta > 0$ such that, whenever $|s_1 - s_2| < \delta$ for every $s_1, s_2 \in J$, $\|T_\alpha(s_1) - T_\alpha(s_2)\| < \epsilon$ and $\|S_\alpha(s_1) - S_\alpha(s_2)\| < \epsilon$. Therefore, when $|v - u| < \delta$, we have

$$\begin{aligned}
E\|(\Psi_2 y)(v) - (\Psi_2 y)(u)\|^2_{\mathbb{H}} \le & 5\epsilon^2\delta_1 r' + \frac{30\epsilon^2 M_B^2}{\lambda^2}\frac{T^{2\alpha}}{\alpha^2}\delta_2 \\
&+ \frac{30\delta_2}{\alpha^2\lambda^2}\Big(\frac{\alpha M M_B}{\Gamma(\alpha+1)}\Big)^4[v^\alpha - u^\alpha - (v-u)^\alpha]^2 \\
&+ \frac{30\delta_2}{\alpha^2\lambda^2}\Big(\frac{\alpha M M_B}{\Gamma(\alpha+1)}\Big)^4(v-u)^{2\alpha} + 5m\epsilon^2\sum_{k=1}^m L_k r'.
\end{aligned}$$

The right hand of the inequality above tends to 0 as $v \longrightarrow u$ and $\epsilon \longrightarrow 0$, hence the set $\{\Psi_1 y, y \in \mathcal{B}_r\}$ is equicontinuous.

Step 5. The set $V(t) = \{\Psi_2 y(t), y \in \mathcal{B}_r\}$ is relatively compact in $\mathcal{B}_r$. Let $0 < t \le T$ be fixed and $0 < \epsilon < t$. For $\delta > 0$, $y \in \mathcal{B}_r$, we define

$$\begin{aligned}
\Psi_2^{\epsilon,\delta}y(t) \quad \le \quad & \textstyle\int_\delta^\infty \xi_\alpha(\theta)T(t^\alpha\theta)(-g(y+\widehat{\phi}))d\theta \\
&+\alpha\textstyle\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)Bu^\lambda(s)d\theta ds \\
&+\textstyle\sum_{0<t_k<t}\int_\delta^\infty \xi_\alpha(\theta)T((t-t_k)^\alpha\theta)I_k(y(t_k^-) + \widehat{\phi}(t_k^-))d\theta \\
= \quad & T(\epsilon^\alpha\delta)\textstyle\int_\delta^\infty \xi_\alpha(\theta)T(t^\alpha\theta - \epsilon^\alpha\delta)(-g(y+\widehat{\phi})d\theta \\
&+\alpha T(\epsilon^\alpha\delta)\textstyle\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-\theta)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta - \epsilon^\alpha\delta)Bu^\lambda(s)d\theta ds \\
&+\textstyle\sum_{0<t_k<t}T(\epsilon^\alpha\delta)\int_\delta^\infty \xi_\alpha(\theta)T((t-t_k)^\alpha\theta - \epsilon^\alpha\delta)I_k(y(t_k^-) + \widehat{\phi}(t_k^-))d\theta.
\end{aligned}$$

Then from the compactness of $T(\epsilon^\alpha\delta)$, we obtain that $V_{\epsilon,\delta}(t) = \{\Psi_2^{\epsilon,\delta}y(t) : y \in \mathcal{B}_r\}$ is relatively compact in H for every ϵ, $0 < \epsilon < t$. Moreover, for $y \in \mathcal{B}_r$, we can easily prove that $\Psi_2^{\epsilon,\delta}y(t)$ is convergent to $\Psi_2 y(t)$ in $\mathcal{B}_r$ as $\epsilon \longrightarrow 0$ and $\delta \longrightarrow 0$, hence the set $V(t) = \{\Psi_2 y(t) : y \in \mathcal{B}_r\}$ is also relatively compact in $\mathcal{B}_r$. Thus, by Arzela-Ascoli theorem Ψ_1 is completely continuous. Consequently, from Lemma 2.4 Ψ has a fixed point, which is a mild solution of (1).

Theorem 3.2 *Assume that (H1)-(H5) are satisfied, and the conditions of Theorem 3.1 hold. Further, if the functions f and σ are uniformly bounded, and $T(t)$ is compact, then the system (1) is approximately controllable on $[0, T]$.*

Proof. Let x^λ be a solution of (1), then we can easily get that

$$\begin{aligned} x^\lambda(t) &= \bar{x}_T - \lambda(\lambda I + \Gamma_0^T)^{-1}\Big[E\overline{x}_T + \int_0^t \eta(s)dW(s) - T_\alpha(T)(\phi(0) - g(x))\Big] \\ &\quad +\lambda \int_0^T (\lambda I + \Gamma_s^T)^{-1}(T-s)^{\alpha-1}S_\alpha(T-t)f(s, x_s^\lambda, B_1x^\lambda(s))ds \\ &\quad +\lambda \int_0^T (\lambda I + \Gamma_s^T)^{-1}(T-s)^{\alpha-1}S_\alpha(T-t)\sigma(s, x_s^\lambda, B_2x^\lambda(s))dW(s) \\ &\quad +\lambda(\lambda I + \Gamma_0^T)^{-1} \sum_{0<t_k<T} T_\alpha(T-t_k)I_k(x(t_s^\lambda)). \end{aligned}$$

In view of the assumptions that f and σ are uniformly bounded on J, there is a subsequence still denoted by $f(s, x_s^\lambda, B_1x^\lambda(s))$ and $\sigma(s, x_s^\lambda, B_2x^\lambda(s))$, which converges weakly to, say $f(s)$ in H, and $\sigma(s)$ in $L(U,H)$. On the other hand, by assumption $(H5)$, the operator $\lambda(\lambda I + \Gamma_s^T)^{-1} \longrightarrow 0$ strongly as $\lambda \longrightarrow 0^+$ for all $0 \le s \le T$, and, moreover, $\|\lambda(\lambda I + \Gamma_s^T)^{-1}\| \le 1$. Thus, the Lebesgue dominated convergence theorem and the compactness of S yield

$$\begin{aligned} E\|x^\lambda(t) - \bar{x}_T\|^2 &\le 4\|\lambda(\lambda I + \Gamma_0^T)^{-1}\|^2 E\|E\overline{x}_T + \int_0^T \eta(s)dW(s) - T_\alpha(T)(\phi(0) - g(x))\|^2 \\ &+ 4E\Big(\int_0^T \|\lambda(\lambda I + \Gamma_s^T)^{-1}(T-s)^{\alpha-1}S_\alpha(T-t)f(s, x_s^\lambda, B_1x^\lambda(s))\|ds\Big)^2 \\ &+ 4E\Big\|\int_0^T \|\lambda(\lambda I + \Gamma_s^T)^{-1}(T-s)^{\alpha-1}S_\alpha(T-t)\sigma(s, x_s^\lambda, B_1x^\lambda(s))dW(s)\Big\|^2 \\ &+ 4\|\lambda(\lambda I + \Gamma_0^T)^{-1}\|^2 E\|\sum_{0<t_k<T} T_\alpha(T-t_k)I_k(x(t_s^\lambda))\|^2 \to 0, \text{as } \lambda \to 0^+. \end{aligned}$$

This gives the approximate controllability of (1), the proof is complete.

4 An Example

As an application, we consider an impulsive stochastic partial differential equation of the following form

$$\begin{cases} D_t^\alpha x(t,y) = \frac{\partial^2}{\partial y^2}x(t,y) + b(y)u(t) + \int_{-\infty}^0 H(t,y,s-t)Q(x(s,y))ds \\ +\int_0^t K(s,t)e^{-x(s,y)}ds + \Big[\int_{-\infty}^0 V(t,y,s-t)U(x(s,y))ds \\ +\int_0^t p(s,t)e^{-x(s,y)}ds\Big]dW(t)\ y \in [0,\pi], \quad t \in [0,T], T > 0, t \ne t_k, \\ I_k(x(t_k^-, y)) = x(t_k^+, y) - x(t_k^-, y), \quad k = 1, \cdots, m, \\ x(t,0) = x(t,\pi) = 0, \quad t \in [0,T], \\ x(0,y) + \int_0^\pi G(y,z)x(t,z)dz = \phi(t,y), \quad t \in (-\infty, 0]. \end{cases} \tag{9}$$

Let $\mathbb{U} = \mathbb{H} = L^2([0,\pi])$ and $h(t) = e^{2t}$, $t < 0$, Then $l = \int_{-\infty}^0 h(s)ds = \frac{1}{2}$ .To study the approximate controllability of (9), assume that H, Q, V and U are continuous; $\phi \in \mathcal{B}_h$.

We define the operator A by $Ax = \frac{\partial^2 x}{\partial y^2}$. with domain $D(A) = \{x \in \mathbb{H}, \frac{\partial x}{\partial y}, \frac{\partial^2 x}{\partial y^2} \in \mathbb{H}$ and $x(0) = x(\pi) = 0\}$. It is well known that A generates an analytic semigroup $T(t), t \ge 0$ given by $T(t)x = \sum_{n=1}^\infty e^{-n^2 t}\langle x, e_n\rangle e_n$, $x \in H$, and $e_n(y) = (2/\pi)^{1/2}\sin(ny), n = 1, 2, \cdots$, is the orthogonal set of eigenvectors of A.

Define the operators $f : J \times \mathcal{B}_h \times L^2([0,\pi]) \to \mathbb{H}$, $\sigma : J \times \mathcal{B}_h \times L^2([0,\pi]) \to \mathscr{L}_2^0(\mathbb{U},\mathbb{H})$, $g : \mathcal{B}_h \to L^2([0,\pi])$,

$$f(t,\phi,B_1x(t))(y) = \int_{-\infty}^{t} H(t,y,s-t)Q(x(s,y))ds + \int_0^t K(s,t)e^{-x(s,y)}ds,$$

$$\sigma(t,\phi,B_2x(t))(y) = \int_{-\infty}^{0} V(t,y,s-t)U(x(s,y))ds + \int_0^t p(s,t)e^{-x(s,y)}ds,$$

$$g(y) = \int_0^{\pi} G(y,z)x(t,z)dz.$$

With the choice of A, f, σ and g can be rewritten as the abstract form of system (1). Under the appropriate conditions on the functions f, σ, g and I_k as those in (H1)-(H5), system (9) is approximately controllable.

5 Conclusion

Approximate controllability of nonlinear fractional impulsive stochastic differential equations with nonlocal conditions and infinite delay in Hilbert spaces has been investigated. By employing fractional calculus, Krasnoselskii-Schaefer-type fixed point theorem and stochastic analysis theory, sufficient conditions for the approximate controllability of nonlinear fractional impulsive stochastic differential equations are formulated and proved under the assumption that the associated linear system is approximately controllable.

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