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A New Approach To Synchronize Different Dimensional Chaotic Maps Using Two Scaling Matrices

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Abstract: In this paper, a new type of synchronization, called $\Theta-\Phi$ synchronization, is introduced for different chaotic discrete-time systems using two scaling matrices. The proposed synchronization approach allows us to study synchronization between two different dimensional discrete-time chaotic systems in different dimensions. By using Lyapunov stability theory and stability property of linear discrete-time systems, some control schemes are proposed and new synchronization results are derived. To verify the effectiveness of our approach, numerical example and simulations are given.

Keywords: *synchronization; chaotic maps; hyperchaotic maps; different dimensions; scaling matrices.*

Mathematics Subject Classification (2010): 74H55, 74H60, 74H65, 93C55.

1 Introduction

Over the last two decades, many scholars have proposed various control schemes in chaos synchronization [1–6], but the most of works have concentrated on continuous-time rather than discrete-time chaotic systems. Recently, synchronization of chaotic and hyperchaotic maps has attracted a great deal of interest of applied scientists and engineers due to it's potential applications in cryptology and secure communication [7–10]. Different methods have been developed to study the synchronization in discrete-time chaotic dynamical systems [11–13].

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Until now, a variety of approaches have been proposed for the synchronization of discrete chaotic such as synchronization and anti-synchronization [14,15], adaptive function projective synchronization [16,17], full-state hybrid projective synchronization [18], Lag synchronization [19], impulsive synchronization [20], function cascade synchronization [21], generalized synchronization [22, 23] and Q-S synchronization [24]. Among all types of synchronization, matrix projective synchronization (MPS) is effective approach for achieving the synchronization of chaotic and hyperchaotic discrete-time systems [25, 26]. In (MPS), the drive chaotic system and the response chaotic system are synchronized up to scaling constant matrix.

In this paper, we generalize the (MPS) type to a new type of synchronization using two scaling constants matrices ($\Theta-\Phi$ synchronization). The aim of this work is to present constructive schemes to synchronize n-dimensional drive system and m-dimensional response system in m-D and n-D, respectively. The derived results are based on Lyapunov stability theory, stability property of linear discrete-time systems and nonlinear control laws. To verify the validity and the feasibility of the new synchronization results, the proposed control schemes are applied to 2D Lorenz discrete time system and 3D discrete-time Rössler system in different dimensions.

This paper is organized as follows. In Section 2, the problem of $\Theta - \Phi$ synchronization is formulated. In section 3, the $\Theta - \Phi$ synchronization is studied in m-D. The n-dimensional $\Theta - \Phi$ synchronization is investigated in Section 4. In Section 5, numerical simulations are given to illustrate the effectiveness of the main results. Finally, conclusions are drawn in Section 6.

2 $\Theta - \Phi$ Synchronization in Discrete-Time Systems

The drive and the response chaotic systems are in the following forms

$$X(k+1) = AX(k) + f(X(k)), \tag{1}$$

$$Y(k+1) = BY(k) + g(Y(k)) + U, \tag{2}$$

where $X(k) \in \mathbf{R}^n$, $Y(k) \in \mathbf{R}^m$ are state vectors of the drive system and the response system, respectively, $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{m\times im}$ are linear parts of the drive system and the response system, respectively, $f : \mathbf{R}^{\ n} \rightarrow \mathbf{R}^n$, $g : \mathbf{R}^m \rightarrow \mathbf{R}^m$ are nonlinear parts of the drive system and the response system, respectively, and $U \in \mathbf{R}^m$ is a vector controller.

Definition 2.1 The drive system (1) and the response system (2) are said to be synchronized in dimension d, with respect to scaling matrices Θ and Φ, respectively, if there exists a controller $U = (u_i)_{1\leq i\leq m} \in \mathbf{R}^m$ and given matrices $\Theta = (\Theta)_{d\times m}$ and $\Phi = (\Phi)_{d\times n}$ such that the synchronization error

$$e(k) = \Theta Y(k) - \Phi X(k) \tag{3}$$

satisfies the condition $\lim_{k\longrightarrow+\infty} \|e(k)\| = 0$.

3 $\Theta - \Phi$ Synchronization in m-D

In this case, we assume that the synchronization dimension $d = m$. The error system between the drive system (1) and the response system (2) can be derived as

$$\begin{aligned} e(k+1) &= \Theta Y(k+1) - \Phi X(k+1) \\ &= \Theta BY(k) + \Theta g(Y(k)) + \Theta U - \Phi AX(k) - \Phi f(X(k)), \end{aligned} \tag{4}$$

where $\Theta = (\Theta_{ij}) \in \mathbf{R}^{m\times m}$ and $\Phi = (\Phi_{ij}) \in \mathbf{R}^{n\times m}$ are the scaling matrices.

Theorem 3.1 *The drive system (1) and the response system (2) are globally synchronized, with respect to scaling matrices* Θ *and* Φ, *if the following conditions are satisfied:*

(i) $U = -\Theta^{-1} \times [(L_1 - B)\, e(k) + \Theta BY(k) + \Theta g(Y(k)) - \Phi AX(k) - \Phi f(X(k))]$, *where* Θ^{-1} *is the inverse of the matrix* Θ.

(ii) $(B - L_1)^T(B - L_1) - I$ *is a negative definite matrix, where* $L_1 \in \boldsymbol{R}^{m\times m}$ *is a control matrix.*

Proof. Then, the error system (4) can be described as

$$\begin{aligned} e(k+1) &= (B - L_1)\, e(k) + \Theta U + (L_1 - B)\, e(k) + \Theta BY(k) \\ &\quad + \Theta g(Y(k)) - \Phi AX(k) - \Phi f(X(k)), \end{aligned} \tag{5}$$

where $L_1 \in \mathbf{R}^{\,m\times m}$ is a control matrix. By substituting (i) into equation (5), the error system can be written as

$$e(k+1) = (B - L_1)\, e(k)\,. \tag{6}$$

Construct the candidate Lyapunov function in the form $V(e(k)) = e^T(k)e(k)$, we obtain

$$\begin{aligned} \Delta V(e(k)) &= e^T(k+1)e(k+1) - e^T(k)e(k) \\ &= e^T(k)(B - L_1)^T(B - L_1)e(k) - e^T(k)e(k) \\ &= e^T(k)\left[(B - L_1)^T(B - L_1) - I\right], \end{aligned}$$

and by using (ii) we get $\Delta V(e(k)) < 0$. Thus, from the Lyapunov stability theory, it is immediate that $\lim_{k\to\infty} e_i(k) = 0,\ \ i = 1, 2, ..., n$. That is the zero solution of the error system (6) is globally asymptotically stable and therefore, the systems (1) and (6) are globally $\Theta - \Phi$ synchronized in m-D.

4 $\Theta - \Phi$ Synchronization in n-D

Now, the synchronization dimension $d = n$. The error system between the drive system (1) and the response system (2) can be derived as

$$\begin{aligned} e(k+1) &= (A - L_2)\, e(k) + \Theta U + (L_2 - A)\, e(k) \\ &\quad + \Theta BY(k) + \Theta g(Y(k)) - \Phi AX(k) - \Phi f(X(k)), \end{aligned} \tag{7}$$

where $\Theta = (\Theta_{ij}) \in \mathbf{R}^{n\times m}$ and $\Phi = (\Phi_{ij}) \in \mathbf{R}^{n\times n}$ are the scaling matrices. In this case, we assume that $m > n$ and we take the controller components v_i, where $i > n$, as

$$u_i = 0, \quad i = n+1, n+2, ..., m. \tag{8}$$

Then, the error system (7) can be written as

$$e(k+1) = (A - L_2)\, e(k) + \hat{\Theta}\hat{U} + R, \tag{9}$$

where $\hat{\Theta} = (\Theta_{ij})_{m\times m}$, $\hat{U} = (u_i)_{1\le i\le n}$,

$$R = (L_2 - A)\, e(k) + \Theta BY(k) + \Theta g(Y(k)) - \Phi AX(k) - \Phi f(X(k)), \tag{10}$$

and $L_2 \in \mathbf{R}^{n\times n}$ is a control matrix.

Theorem 4.1 *The drive system (1) and the response system (2) are globally synchronized, with respect to the scaling matrices Θ and Φ, if the following conditions are satisfied:*
(i) $\hat{U} = -\hat{\Theta}^{-1} \times R$, where $\hat{\Theta}^{-1}$ is the inverse of the matrix $\hat{\Theta}$.
(ii) All the eigenvalues of $A - L_2$ lie inside the unit disk.

Proof. By substituting (i) into equation (9), the error system can be written as

$$e(k+1) = (A - L_2)\, e(k). \tag{11}$$

With respect to the asymptotic stability property of linear discrete-time systems, if all eigenvalues of $A - L_2$ are strictly inside the unit disk, it is immediate that all solutions of error system (11) go to zero as $k \to \infty$. Therefore, the systems (1) and (2) are globally $\Theta - \Phi$ synchronized in n-D.

5 Numerical Application and Simulations

In this section, a numerical example is given to illustrate the effectiveness of the theoretical results derived in the previous sections. Thus, we consider the 2D Lorenz discrete time system as the drive system and the controlled 3D discrete-time Rössler system as the response system. The Lorenz discrete time system is described by

$$\begin{aligned} x_1(k+1) &= (1+ab)\, x_1(k) - bx_1(k)\, x_2(k), \\ x_2(k+1) &= (1-b)\, x_2(k) + bx_1^2(k), \end{aligned} \tag{12}$$

which has a chaotic attractor, for example, when $(a,\ b) = (1.25,\ 0.75)$ [27].

The controlled discrete-time Rössler system can be described as:

$$\begin{aligned} y_1(k+1) &= \alpha y_1(k)\,(1 - y_1(k)) - \beta\,(y_3(k) + \gamma)\,(1 - 2y_2(k)) + u_1, \\ y_2(k+1) &= \delta y_2(k)\,(1 - y_2(k)) + \varsigma y_3(k) + u_2, \\ y_3(k+1) &= \eta\,((y_3(k) + \gamma)\,(1 - 2y_2(k)) - 1)\,(1 - \theta y_1(k)) + u_3, \end{aligned} \tag{13}$$

where $U = (u_1, u_2, u_3)^T$ is the vector controller. When $\alpha = 3.8$, $\beta = 0.05$, $\gamma = 0.35$, $\delta = 3.78$, $\varsigma = 0.2$, $\eta = 0.1$ and $\theta = 1.9$, the discrete-time Rössler system (i.e., the system map (18) with $u_1 = 0$, $u_2 = 0$ and $u_3 = 0$) has a hyperchaotic attractor [28].

The linear part A and the nonlinear part f of the Lorenz discrete time system are given by

$$A = \begin{pmatrix} 1+ab & 0 \\ 0 & 1-b \end{pmatrix},\ f = \begin{pmatrix} -bx_1(k)\, x_2(k) \\ bx_1^2(k) \end{pmatrix}.$$

The linear part B and the nonlinear part g of the discrete-time Rössler system are given by

$$\begin{aligned} B &= \begin{pmatrix} \alpha & 2\beta\gamma & -\beta \\ 0 & \delta & \varsigma \\ \eta\theta(1-\gamma) & -2\gamma\eta & \eta \end{pmatrix}, \\ g &= \begin{pmatrix} 2\beta y_3(k)\, y_2(k) - \alpha y_1^2(k) - \beta\gamma \\ -\delta y_2^2(k) \\ \eta(\gamma - 1) - \eta y_3(k)\,(\theta y_1(k) + 2y_2(k)) + 2\theta y_1(k)\, y_2(k)\,(\gamma + \eta y_3(k)) \end{pmatrix}. \end{aligned}$$

5.1 Synchronization of the Lorenz discrete time system and the discrete-time Rössler system in 3D

In this case, the scaling matrices are chosen as

$$\Theta = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \ \Phi = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 1 \end{pmatrix},$$

so,

$$\Theta^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

The control matrix L_1 is selected as

$$L_1 = \begin{pmatrix} \frac{3\alpha}{4} & 2\beta\gamma & -\beta \\ 0 & \frac{4\delta}{5} & \varsigma \\ \eta\theta(1-\gamma) & -2\gamma\eta & 0 \end{pmatrix}. \tag{14}$$

Using simple calculations, we can show that $(B - L_1)^T(B - L_1) - I$ is a negative definite matrix. According to our approach presented in Section 3, the vector controller $U = (u_1, u_2, u_3)^T$ can be obtained as

$$\begin{aligned} u_1 &= -\frac{\alpha}{8}e_1(k) - \alpha y_1(k) - 2\beta\gamma y_2(k) + \beta\gamma \\ &\quad +\beta y_3(k) - 2\beta y_3(k)\, y_2(k) + \alpha y_1^2(k) \\ &\quad +\frac{1}{2}(1+ab)\, x_1(k) - \frac{1}{2}bx_1(k)\, x_2(k), \\ u_2 &= -\frac{\delta}{5}e_2(k) - \delta y_2(k) - \varsigma y_3(k) + \delta y_2^2(k) \\ &\quad +3(1-b)\, x_2(k) + bx_1^2(k), \\ u_3 &= -\frac{\eta}{3}e_3(k) - \eta\theta(1-\gamma)\, y_1(k) + 2\gamma\eta y_2(k) - \eta y_3(k) \\ &\quad +\eta y_3(k)\,(\theta y_1(k) + 2y_2(k)) - 2\theta y_1(k)\, y_2(k)\,(\gamma + \eta y_3(k)) \\ &\quad +\frac{1}{3}(1+ab)\, x_1(k) - \frac{b}{3}x_1(k)\, x_2(k) + \frac{1}{3}(1-b)\, x_2(k) + \frac{b}{3}x_1^2(k) \\ &\quad -\eta(\gamma - 1), \end{aligned} \tag{15}$$

where $e_1(k) = 2y_1(k) - x_1(k) - 2x_2(k)$, $e_2(k) = y_2(k) - 2x_1(k) - 3x_2(k)$ and $e_3(k) = 3y_3(k) - x_1(k) - x_2(k)$. Therefore, the systems (12) and (13) are globally synchronized in 3D, with respect to the scaling matrices Θ and Φ. In this case, the error system can be described as: $e_1(k+1) = \frac{\alpha}{4}e_1(k)$, $e_2(k+1) = \frac{\delta}{5}e_2(k)$ and $e_3(k+1) = \eta e_3(k)$. The time evolution of errors $e_1(k), e_2(k)$ and $e_3(k)$ between the maps (12) and (13) in 3D is shown in Figure 1.

5.2 Synchronization of the Lorenz discrete time system and the discrete-time Rössler system in 2D

In this case, the scaling matrices are chosen as

$$\Theta = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 1 \end{pmatrix}, \ \Phi = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix},$$

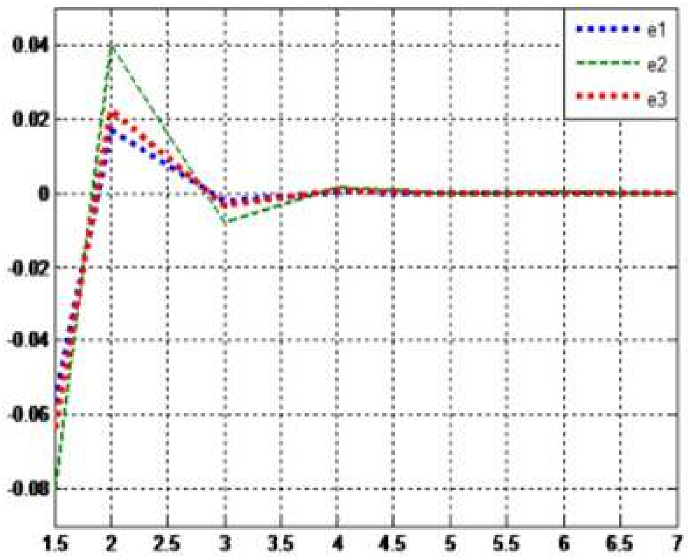

Figure 1: Time evolution of errors $e_1(k), e_2(k)$ and $e_3(k)$ between the maps (12) and (13) in 3D.

so,

$$\hat{\Theta} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}, \ \hat{\Theta}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}.$$

The control matrix L_2 is selected as

$$L_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{16}$$

Simply, we can see that all eigenvalues of $A - L_2$ are strictly inside the unit disk. According to the control scheme proposed in Section 4, the vector controller $U = (u_1, u_2, u_3)^T$ can be designed as follows

$$\begin{aligned} u_1 \ = \ & \frac{1}{2}abe_1(k) - \frac{1}{2}\eta\left(\left(y_3(k) + \gamma\right)\left(1 - 2y_2(k)\right) - 1\right)\left(1 - \theta y_1(k)\right) \\ & -\alpha y_1(k)\left(1 - y_1(k)\right) + \beta\left(y_3(k) + \gamma\right)\left(1 - 2y_2(k)\right) \\ & + (1 + ab)\,x_1(k) - \frac{1}{2}bx_1(k)\,x_2(k), \end{aligned} \tag{17}$$

$$\begin{aligned} u_2 &= -\frac{1}{5}be_2(k) - \frac{1}{5}\eta\left((y_3(k)+\gamma)(1-2y_2(k)) - 1\right)(1-\theta y_1(k)) \\ &\quad -4\frac{1}{5}\delta y_2(k)(1-y_2(k)) - \frac{1}{5}\varsigma y_3(k) + \frac{1}{5}(1+ab)x_1(k) \\ &\quad -\frac{1}{5}bx_1(k)x_2(k) + \frac{1}{5}3(1-b)x_2(k) + \frac{1}{5}3bx_1^2(k), \\ u_3 &= 0, \end{aligned}$$

where $e_1(k) = 2y_1(k) + y_3(k) - 2x_1(k)$ and $e_2(k) = 4y_2(k) + y_3(k) - x_1(k) - 3x_2(k)$. Therefore, the systems (12) and (13) are globally synchronized in 2D, with respect to the scaling matrices Θ and Φ. In this case, the error system can be written as: $e_1(k+1) = abe_1(k)$ and $e_2(k+1) = -be_2(k)$. The time evolution of errors $e_1(k)$ and $e_2(k)$ between the maps (12) and (13) in 2D is shown in Figure 2.

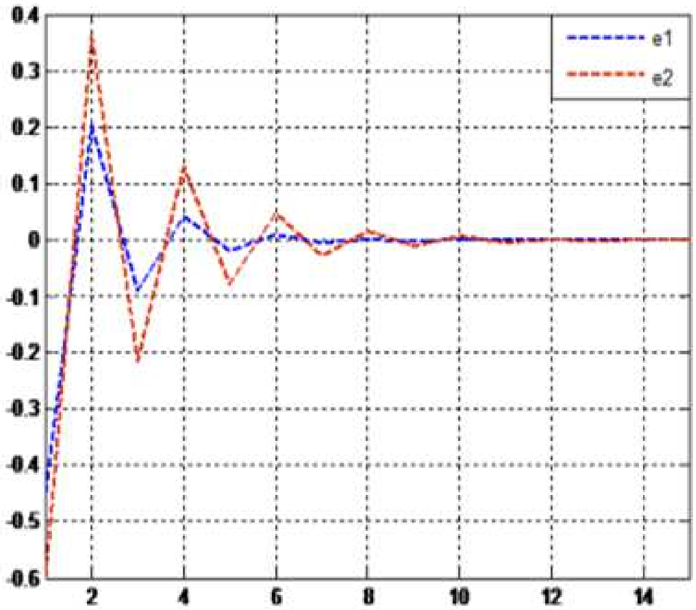

Figure 2: Time evolution of errors $e_1(k)$ and $e_2(k)$ between the maps (12) and (13) in 2D.

6 Conclusion

In this paper, the $\Theta - \Phi$ synchronization was proposed to synchronize n-dimensional drive system and m-dimensional response system. To derive new results, two control schemes were proposed using two constants scaling matrices Θ and Φ. The first scheme was presented when the synchronization dimension $d = m$, ($\Theta - \Phi$ synchronization in m-D) and the second one was constructed when the synchronization dimension $d = n$, ($\Theta - \Phi$ synchronization in n-D). Numerical example and simulation results were used to verify the effectiveness of the proposed schemes.

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