Nonlinear Dynamics and Systems Theory, **15** (4) (2015) 370–382

Existence Results for a Fractional Integro-Differential Equation with Nonlocal Boundary Conditions and Fractional Impulsive Conditions

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Received: September 13, 2014; Revised: October 31, 2015

Abstract: In this paper, we have established the existence and uniqueness of solution for a class of impulsive fractional integro-differential equations with nonlocal boundary conditions. The existence results are proved by applying the theory of fractional calculus and fixed point theorems. At last an application is given to verify our results.

Keywords: *fractional derivatives and integrals; differential equations with impulses; boundary value problems with impulses; equations with impulses; nonlocal and multi-point boundary value problems.*

Mathematics Subject Classification (2010): 26A33, 34A37, 34B37, 34K45, 34B10.

1 Introduction

Fractional differential equations are the corner stone for description of memory and hereditary properties of many materials and processes. Its useful applications include mathematical modeling in many engineering and science disciplines like physics, chemistry, biophysics, biology etc. Its non local behavior is the vital characteristic that makes it vary from its rival in classical calculus. For more details one can see the papers [1, 6, 8, 10, 13, 15, 22, 24, 25] and the references therein.

Integro-differential equations occur in probability theory, nonlinear viscoelastic bodies, acoustic scattering theory and bio-logical population models and systems with substantially distributed parameters. All these problems end up with boundary value problems of integro-differential equations. For details see the paper [21].

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In recent years, the theory of impulsive differential equations for integer order comes in various applications of mathematical modeling of phenomena and practical situations. For instance, the impulsive differential equations captured from real world problems describe the dynamics of processes in which sudden, discontinuous jumps occur. For more details one can see the papers [2, 3, 6, 7, 12, 19, 20, 23, 26] and references therein.

C. Bai [4] has investigated the existence of solutions of multi-point boundary value problem of nonlinear impulsive fractional differential equations at resonance. Further in his subsequent study in [5] the author has extended the results for the boundary value problem of nonlinear impulsive differential equations at resonance. The author obtained the result of existence by using the coincidence degree theory due to Mawhin.

In [20] L. Yang et al. have proved the existence and uniqueness of solution for the following nonlocal boundary value problem of impulsive fractional differential equations:

$$\begin{cases} {}^cD^q u(t) = F(t, u(t), u'(t)),\ q \in (1, 2],\ t \in [0, 1], \\ \Delta u(t_k) = I_k(u(t_k^-)),\ \Delta(u'(t_k)) = J_k(u(t_k^-)),\ k = 1, 2...., p, \\ \alpha u(0) + \beta u'(0) = g_1(u),\ \alpha u(1) + \beta u'(1) = g_2(u),\ \alpha > 0,\ \beta \geq 0, \end{cases} \tag{1}$$

by means of a fixed point theorem due to ORegan, the authors established the sufficient conditions for the existence of at least one solution of the problem. In [7] J. Cao et al. have established the existence and uniqueness results for the impulsive fractional differential inclusions with a fractional order multi-point boundary condition and with fractional order impulses and proved the results by using the multi-valued analysis of topological fixed point theory.

In [11] X. Fu et al. concerned with the fractional separated boundary value problem of the following fractional differential equations with fractional impulsive conditions:

$$\begin{cases} {}^cD^\alpha x(t) = F(t, x(t),\ t \in J = [0, T],\ t \neq t_k,\ \alpha \in (1, 2), \\ \Delta x(t_k) = I_k(x(t_k^-)),\ \Delta({}^cD^\gamma x(t_k)) = I_k{}^*(x(t_k^-)),\ k = 1, 2...., m, \\ a_1 x(0) + b_1({}^cD^\gamma x(0)) = c_1,\ a_2 x(T) + b_2({}^cD^\gamma x(T)) = c_2,\ \gamma \in (0, 1), \end{cases} \tag{2}$$

where $a_i, b_i, c_i, \in \mathbb{R},\ i = 1, 2$, with $a_i \neq 0$ and $a_2 T^\gamma \Gamma(2-\gamma) \neq -b_2$. By using the Schaefer fixed point theorem, Banach fixed point theorem, and nonlinear alternative of Leray Schauder type, the authors obtained the existence results.

In [14] N. Kosmatov considered the following two impulsive problems:

$$\begin{cases} {}^cD^\delta x(t) = F(t, x(t)),\ t \in (0, 1]\backslash\{t_1, t_2, ..., t_m\}, \\ {}^cD^\gamma x(t_k^+)) -{}^c D^\gamma x(t_k^-)) = J_k(x(t_k)),\ k = 1, 2...., m, \\ x(0) = x_0,\ x'(0) = x_1, \end{cases} \tag{3}$$

where ${}^cD^\delta$ is the Caputo fractional derivative of order $\delta \in (1, 2)$ with the lower limit zero, $0 < \gamma < 1$, and

$$\begin{cases} {}^LD^\delta x(t) = F(t, x(t),\ t \in (0, 1]\backslash\{t_1, t_2, ..., t_m\}, \\ {}^LD^\gamma x(t_k^+)) -{}^L D^\gamma x(t_k^-)) = J_k(x(t_k)),\ k = 1, 2...., m, \\ I^{1-\alpha} x(0) = x_0, \end{cases} \tag{4}$$

where ${}^LD^\delta$ is the Riemann-Liouville fractional derivative of order $\delta \in (0, 1)$ with lower limit zero and $0 < \gamma < \delta$.

Motivated by the works [4,5,7,11,14,20] we investigate the existence and uniqueness solutions for the following impulsive fractional integro-differential equation with nonlocal boundary conditions:

$$\begin{cases} {}^cD^\alpha u(t) = f(t, u(t), \int_0^t K(t,s)u(s)ds),\ t \in [0,T],\ t \neq t_k,\ \alpha \in (1,2), \\ \Delta u(t_k) = I_k(u(t_k^-)), \\ \Delta({}^cD^q u(t_k)) = J_k(u(t_k^-)),\ q \in (0,1),\ k = 1,2,\dots,m, \\ u(0) = a_1 - g(u), \quad u(T) = a_2 - h(u),\ a_1, a_2 \in \mathbb{R}, \end{cases} \tag{5}$$

where ${}^cD^\alpha$ is the Caputo's derivative, functions $f : [0,T] \times X \times X \to X$ for $K : [0,T] \times [0,T] \to [0,\infty)$ and $g, h \in X \to X$ are continuous. The impulsive conditions for $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $I_k, J_k \in C(X,X)$, are bounded functions. We have $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$ and $\Delta({}^cD^q u(t_k)) = ({}^cD^q u(t_k^+)) - ({}^cD^q u(t_k^-)), u(t_k^+) = \lim_{h\to 0} u(t_k + h)$ and $u(t_k^-) = \lim_{h\to 0} u(t_k - h)$ represent the right and left-hand limits of $u(t)$ at $t = t_k$ respectively with $u(t_i^-) = u(t_i)$, where $K \in C(D, \mathbb{R}^+)$, the set of all positive functions which are continuous on $D = \{(t,s) \in \mathbb{R}^2 : 0 \le s \le t < T\}$ and $K^* = \sup_{t\in[0,T]} \int_0^t K(t,s)ds < \infty$.

In all the above cited papers except [4,5,7,11,14,20] the authors established the existence and uniqueness results of the fractional order boundary value problems by applying the standard fixed point theorems with the integer order impulsive conditions. In this paper, we show the existence and uniqueness solutions for the fractional integro differential equation with fractional impulsive conditions and nonlocal boundary conditions. The boundary value problems like (5) arise in many applications such as electromagnetic waves in dielectric media, the mathematical modeling of various phenomena of transport theory, the transfer of neutrons through thin plates and membranes in nuclear reactors, in the propagation of radiation through the atmosphere of planets and stars, and in several other transport problems.

In Section 2, we present some notations and preliminary results about fractional calculus and differential equations to be used in the following sections. In Section 3, we discuss existence and uniqueness results for solutions of the system (5) by using the Banach and Schauder fixed point theorems.

2 Preliminaries

Let $(X, \|\cdot\|_X)$ be a complex Banach space of functions with the norm $\|y\|_X = \sup_{t\in[0,T]}\{|y(t)| : y \in X\}$. To treat the impulsive conditions, define the following space

$$PC_t = PC([0,t] : X),\ 0 \le t \le T,$$

be a Banach space of all such functions $y : [0,T] \to X$, which are continuous everywhere except for a finite number of points $t_i,\ i = 1,2,\dots,m$, at which $y(t_i^+)$ and $y(t_i^-)$ exist with $y(t_i^-) = y(t_i)$ and are endowed with the norm

$$\|y\|_{PC_t} = \sup_{t\in[0,T]} \{\|y(t)\|_X, y \in PC_t\},$$

and

$$PC_t^1 = PC^1([0,t] : X),\ 0 \le t \le T,$$

be a Banach space of all such functions $y : [0,T] \to X$, which are continuously differentiable everywhere except for a finite number of points $t_i,\ i = 1,\dots,m$, at which $y'(t_i^+)$ and $y'(t_i^-)$ exist with $y'(t_i^-) = y'(t_i)$ and are endowed with the norm $\|y\|_{PC_t^1} = \sup_{t\in[0,T]}\{\|y(t)\|_{PC_t}, \|y'(t)\|_{PC_t}, y \in PC_t\}$. All other notations in the paper have their usual meanings.

Definition 2.1 [15] The Riemann-Liouville fractional integral operator for order $\alpha > 0$, of a function $f : \mathbb{R}^+ \to \mathbb{R}$ and $f \in L^1(\mathbb{R}^+, X)$ is defined by

$$J_t^0 f(t) = f(t),\ J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,\ t > 0, \tag{6}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2 [15] The Riemann Liouville fractional derivative of order α with lower limit zero for a function $f : [0,\infty) \to \mathbb{R}$ can be written as

$${}^L D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds,\ t > 0,\ n-1 < \alpha < n. \tag{7}$$

Definition 2.3 [15] The Caputo's derivative of order α for a function $f : [0,\infty) \to \mathbb{R}$ can be written as

$${}^c D_t^\alpha f(t) = {}^L D_t^\alpha \Big[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \Big],\ t > 0,\ n-1 < \alpha < n. \tag{8}$$

Remark 2.1 [15] If $f(t) \in C^n[0,\infty)$, for order $n-1 < \alpha < n$ then

$${}^c D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds = I_t^{n-\alpha} f^{(n)}(t),\ t > 0. \tag{9}$$

The Caputo's derivative of constant is equal to zero.

The following results are needed to prove the existence results of the paper, relevant references are cited.

Theorem 2.1 *[18] If U is a closed, bounded, convex subset of a Banach space X and the mapping $A : U \to U$ is completely continuous, then A has a fixed point in U.*

Lemma 2.1 *[1] Let $\alpha > 0$, then the differential equation*

$${}^c D^\alpha h(t) = 0 \tag{10}$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},\ c_i \in \mathbb{R},\ i = 0, 1, \ldots, n-1,\ n = [\alpha]+1$.

Lemma 2.2 *[1] Let $\alpha > 0$, then*
$I^\alpha D^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},\ c_i \in \mathbb{R},\ i = 0, 1, \ldots, n-1,\ n = [\alpha]+1.$

To investigate the nonlinear impulsive fractional integro differential equation (5), we first consider the associated linear system and obtain its solution.

Lemma 2.3 *Let $\alpha < (1,2)$, $q < (0,1)$ and $\sigma \in [0,T] \to \mathbb{R}$ be continuous. A function $u(t) \in PC_t^1$ is a solution of the following fractional integral equation:*

$$u(t) = \begin{cases} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + a_1 - g(u) - \frac{t}{T}\Big[a_1 - a_2 + h(u) - g(u) \\ + \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + \sum_{i=1}^m I_i(u(t_i^-)) \\ + \sum_{i=1}^m (T-t_i)\left(\frac{\Gamma(2-q)}{t_i{}^{1-q}} J_i(u(t_i^-))\right)\Big], & t \in [0,t_1), \\ \ldots\ldots. \\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + \sum_{i=1}^k I_i(u(t_i^-)) + a_1 - g(u) - \frac{t}{T}\Big[a_1 - a_2 \\ + h(u) - g(u) + \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + \sum_{i=1}^m I_i(u(t_i^-)) \\ + \sum_{i=1}^m (T-t_i)\left(\frac{\Gamma(2-q)}{t_i{}^{1-q}} J_i(u(t_i^-))\right)\Big] \\ + \sum_{i=1}^k (t-t_i)\left(\frac{\Gamma(2-q)}{t_i{}^{1-q}} J_i(u(t_i^-))\right), & t \in (t_k, t_{k+1}], \end{cases} \tag{11}$$

iff $u(t)$ is a solution of the following BVP

$$\begin{cases} {}^cD^\alpha u(t) = \sigma(t),\ \alpha \in (1,2), \\ \Delta u(t_k) = I_k(u(t_k^-)), \Delta({}^cD^q u(t_k)) = J_k(u(t_k^-)),\ q \in (0,1), \\ u(0) = a_1 - g(u), \quad u(T) = a_2 - h(u). \end{cases} \tag{12}$$

Proof. Let for $t \in [0,t_1)$, $u(t)$ be the solution of (12), we have

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds - c_0 - c_1 t, \tag{13}$$

using the condition $u(0) = a_1 - g(u)$ we compute $c_0 = -(a_1 - g(u))$, then we have

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + a_1 - g(u) - c_1 t. \tag{14}$$

If $t \in (t_1, t_2]$, we may write the solution as

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds - c_2 - c_3 t, \tag{15}$$

on applying first impulsive condition $\Delta u(t_1) = I_1(u(t_1^-))$, we get

$$-c_2 = I_1(u(t_1^-)) + c_3 t_1 + a_1 - g(u) - c_1 t_1. \tag{16}$$

Using the value of c_2 in (15), we obtain

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + I_1(u(t_1^-)) + a_1 - g(u) - c_1 t_1 + c_3(t_1 - t).$$

From (17) and (14), we get

$$D^q u(t) = \frac{1}{\Gamma(\alpha-q)}\int_0^t (t-s)^{\alpha-q-1}\sigma(s)ds - c_3\frac{t^{1-q}}{\Gamma(2-q)}, \tag{17}$$

$$D^q u(t) = \frac{1}{\Gamma(\alpha-q)}\int_0^t (t-s)^{\alpha-q-1}\sigma(s)ds - c_1\frac{t^{1-q}}{\Gamma(2-q)}. \tag{18}$$

Using the second impulsive condition $\Delta(D^q u(t_1)) = J_1(u(t_1^-))$, we have

$$c_3 = -\frac{\Gamma(2-q)}{{t_1}^{1-q}} J_1(u(t_1^-)) + c_1. \tag{19}$$

Put c_3 in (17), we get

$$\begin{aligned} u(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + I_1(u(t_1^-)) \\ &\quad + a_1 - g(u) + (t-t_1) \frac{\Gamma(2-q)}{{t_1}^{1-q}} J_1(u(t_1^-)) - c_1 t. \end{aligned} \tag{20}$$

For $t \in (t_2, t_3]$, we have

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds - c_4 - c_5 t. \tag{21}$$

Applying the similar pattern we obtain the following form of the solution

$$\begin{aligned} u(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + I_1(u(t_1^-)) + I_2(u(t_2^-)) + a_1 - g(u) \\ &\quad + \frac{\Gamma(2-q)}{{t_1}^{1-q}} J_1(u(t_1^-))(t-t_1) + \frac{\Gamma(2-q)}{{t_2}^{1-q}} J_2(u(t_2^-))(t-t_2) - c_1 t. \end{aligned} \tag{22}$$

For generality, when $t \in (t_k, t_{k+1}]$, we may write the solution in the following form

$$\begin{aligned} u(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + \sum_{i=1}^{k} I_i(u(t_i^-)) + a_1 - g(u) - c_1 t \\ &\quad + \sum_{i=1}^{k} (t-t_i) \left(\frac{\Gamma(2-q)}{{t_i}^{1-q}} J_i(u(t_i^-)) \right). \end{aligned} \tag{23}$$

On using the second boundary condition, $u(T) = a_2 - h(u)$, we compute the following value of the constant c_1:

$$\begin{aligned} c_1 &= \frac{1}{T} \Big[a_1 - a_2 + h(u) - g(u) + \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds \\ &\quad + \sum_{i=1}^{m} I_i(u(t_i^-)) + \sum_{i=1}^{m} (T-t_i) \left(\frac{\Gamma(2-q)}{{t_i}^{1-q}} J_i(u(t_i^-)) \right) \Big], \end{aligned} \tag{24}$$

by summarizing the above computation, we get the required result. Conversely, assume that u satisfies the impulsive fractional integral equation (11), then by direct computation, it can be seen that the solution given by (11) satisfies (12). This completes the proof of the lemma.

3 Existence and Uniqueness Results

The following result is based on Lemma 2.3.

Definition 3.1 The function $u : [0,T] \to X$ such that $u \in PC_t^1([0,T] : X)$ is said to be the solution of the system (5) if it satisfies the following integral equation

$$u(t) = \begin{cases} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),\int_0^s K(s,\tau)u(\tau)d\tau)ds \\ +a_1 - g(u) - \frac{t}{T}\Big[a_1 - a_2 + h(u) - g(u) \\ + \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),\int_0^s K(s,\tau)u(\tau)d\tau)ds \\ + \sum_{i=1}^m I_i(u(t_i^-)) + \sum_{i=1}^m (T-t_i)\left(\frac{\Gamma(2-q)}{t_i{}^{1-q}} J_i(u(t_i^-))\right)\Big], & t \in [0,t_1), \\ .... \\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),\int_0^s K(s,\tau)u(\tau)d\tau)ds + \sum_{i=1}^k I_i(u(t_i^-)) \\ +a_1 - g(u) - \frac{t}{T}\Big[a_1 - a_2 + h(u) - g(u) \\ + \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),\int_0^s K(s,\tau)u(\tau)d\tau)ds \\ + \sum_{i=1}^m I_i(u(t_i^-)) + \sum_{i=1}^m (T-t_i)\left(\frac{\Gamma(2-q)}{t_i{}^{1-q}} J_i(u(t_i^-))\right)\Big] \\ + \sum_{i=1}^k (t-t_i)\left(\frac{\Gamma(2-q)}{t_i{}^{1-q}} J_i(u(t_i^-))\right), & t \in (t_k, t_{k+1}]. \end{cases} \tag{25}$$

Our first result is based on Banach fixed point theorem.

Theorem 3.1 *Let the functions f, g, h, I_k and J_k satisfy the Lipchitz condition with positive constants L_1, L_2, L_3, L_4, L_5 and L_6, such that*

$$\begin{aligned} &\|f(t,u,v) - f(t,x,y)\|_X \le L_1\|u-x\|_X + L_2\|v-y\|_X, \\ &\|g(u) - g(x)\|_X \le L_4\|u-x\|_X, \ \ \|h(u) - h(x)\|_X \le L_6\|u-x\|_X, \\ &\|I_k(x) - I_k(y)\|_X \le L_3\|x-y\|_X, \ \ \|J_k(x) - J_k(y)\|_X \le L_5\|x-y\|_X, \end{aligned}$$

$t \in [0,T]$, $\forall\, x,y,u,v \in X$. If the following inequality holds

$$\Delta = \Big[\frac{(L_1 + L_2K^*)}{\Gamma(\alpha+1)}2T^\alpha + 2mL_3 + 2L_4 + L_6 + 2mT^q\Gamma(2-q)L_5\Big] < 1,$$

then the system (5) has a unique solution.

Proof. We transform the system (5) into a fixed point problem. Consider an operator $N : PC_t^1 \to PC_t^1$, defined by

$$(Nu)t = \begin{cases} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),\int_0^s K(s,\tau)u(\tau)d\tau)ds \\ +a_1 - g(u) - \frac{t}{T}\Big[a_1 - a_2 + h(u) - g(u) \\ + \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),\int_0^s K(s,\tau)u(\tau)d\tau)ds \\ + \sum_{i=1}^m I_i(u(t_i^-)) + \sum_{i=1}^m (T-t_i)\left(\frac{\Gamma(2-q)}{t_i{}^{1-q}} J_i(u(t_i^-))\right)\Big], & t \in [0,t_1), \\ \dots \\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),\int_0^s K(s,\tau)u(\tau)d\tau)ds + \sum_{i=1}^k I_i(u(t_i^-)) \\ +a_1 - g(u) - \frac{t}{T}\Big[a_1 - a_2 + h(u) - g(u) \\ + \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),\int_0^s K(s,\tau)u(\tau)d\tau)ds \\ + \sum_{i=1}^m I_i(u(t_i^-)) + \sum_{i=1}^m (T-t_i)\left(\frac{\Gamma(2-q)}{t_i{}^{1-q}} J_i(u(t_i^-))\right)\Big] \\ + \sum_{i=1}^k (t-t_i)\left(\frac{\Gamma(2-q)}{t_i{}^{1-q}} J_i(u(t_i^-))\right), & t \in (t_k, t_{k+1}]. \end{cases} \tag{26}$$

To show that N has fixed point consider $u_1, u_2 \in PC_t^1$. For $t \in [0, t_1)$, we have the following estimate

$$\begin{aligned}
&\|N(u_1) - N(u_2)\|_X \leq \\
&\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|f(s, u_1(s), \int_0^s K(s,\tau)u_1(\tau)d\tau) - f(s, u_2(s), \int_0^s K(s,\tau)u_2(\tau)d\tau)\|_X ds \\
&+\|g(u_1) - g(u_2)\|_X + \frac{|t|}{T}\Big[\|h(u_1) - h(u_2)\|_X + \|g(u_1) - g(u_2)\|_X \\
&+ \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \|f(s, u_1(s), \int_0^s K(s,\tau)u_1(\tau)d\tau) \\
&-f(s, u_2(s), \int_0^s K(s,\tau)u_2(\tau)d\tau)\|_X ds + \sum_{i=1}^m \|I_i(u_1(t_i^-)) - I_i(u_2(t_i^-))\|_X \\
&+ \sum_{i=1}^m |(T - t_i)| \frac{\Gamma(2-q)}{|t_i|^{1-q}} \|J_i(u_1(t_i^-)) - J_i(u_2(t_i^-))\|_X\Big],
\end{aligned}$$

On simplifying, we obtain

$$\begin{aligned}
&\|N(u_1) - N(u_2)\|_{PC_t^1} \\
&\leq \Big[\frac{(L_1 + L_2K^*)}{\Gamma(\alpha+1)} 2T^\alpha + 2L_4 + L_6 + mL_3 + mT^q\Gamma(2-q)L_5\Big]\|u_1 - u_2\|_{PC_t^1}.
\end{aligned}$$

For $t \in (t_k, t_{k+1}]$, we have

$$\begin{aligned}
&\|N(u_1) - N(u_2)\|_X \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|f(s, u_1(s), \int_0^s K(s,\tau)u_1(\tau)d\tau) \\
&-f(s, u_2(s), \int_0^s K(s,\tau)u_2(\tau)d\tau)\|_X ds \\
&+ \sum_{i=1}^k \|I_i(u_1(t_i^-)) - I_i(u_2(t_i^-))\|_X + \|g(u_1) - g(u_2)\|_X + \frac{|t|}{T}\Big[\|h(u_1) - h(u_2)\|_X \\
&+\|g(u_1) - g(u_2)\|_X + \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \|f(s, u_1(s), \int_0^s K(s,\tau)u_1(\tau)d\tau) \\
&-f(s, u_2(s), \int_0^s K(s,\tau)u_2(\tau)d\tau)\|_X ds + \sum_{i=1}^m \|I_i(u_1(t_i^-)) - I_i(u_2(t_i^-))\|_X \\
&+ \sum_{i=1}^m |(T - t_i)| \frac{\Gamma(2-q)}{|t_i|^{1-q}} \|J_i(u_1(t_i^-)) - J_i(u_2(t_i^-))\|_X\Big] \\
&+ \sum_{i=1}^k |(t - t_i)| \frac{\Gamma(2-q)}{|t_i|^{1-q}} \|J_i(u_1(t_i^-)) - J_i(u_2(t_i^-))\|_X,
\end{aligned}$$

Hence we estimate as

$$\begin{aligned}
&\|N(u_1) - N(u_2)\|_{PC_t^1} \\
&\leq \Big[\frac{(L_1 + L_2K^*)}{\Gamma(\alpha+1)} 2T^\alpha + 2mL_3 + 2L_4 + L_6 + 2mT^q\Gamma(2-q)L_5\Big]\|u_1 - u_2\|_{PC_t^1} \\
&\leq \Delta\|u_1 - u_2\|_{PC_t^1}.
\end{aligned}$$

Since $\Delta < 1$, it follows that the operator N is a contraction mapping and has a fixed point $u \in PC_t^1$, hence the system (5) has a unique solution on the interval $[0, T]$. This completes the proof of the theorem.

Our second result is based on Schauder fixed point theorem.

Theorem 3.2 *Let the functions f, g, h, I_k, J_k be continuous and there exist positive constants M_1, M_2, M_3, M_4 and M_5 such that $\|f(t,u,v)\|_X \leq M_1$, $\|g(u)\|_X \leq M_2$, $\|h(u)\|_X \leq M_3$, $\|I_k(y)\|_X \leq M_4$, $\|J_k(y)\|_X \leq M_5$, $\forall u, v, y \in X$. Then the system (5) has at least one solution on $[0, T]$.*

Proof. Consider an operator $N : PC_t^1 \to PC_t^1$ defined as in (26) in Theorem 3.1. First, we shall show that N is continuous, let us consider a sequence $u_n \to u$ in PC_t^1 in the interval $(t_k, t_{k+1}]$, $(k = 1, \ldots, m)$ we have

$$\begin{aligned}
&\|N(u_n) - N(u)\|_X \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big(\|f(s, u_n(s), \int_0^s K(s,\tau)u_n(\tau)d\tau) \\
&-f(s, u(s), \int_0^s K(s,\tau)u(\tau)d\tau)\|_X \Big) ds \\
&+\sum_{i=1}^{k} \|I_i(u_n(t_i^-)) - I_i(u(t_i^-))\|_X + \|g(u_n) - g(u)\|_X - \frac{|t|}{T}\Big[\|h(u_n) - h(u)\|_X \\
&+\|g(u_n) - g(u)\|_X + \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \Big(\|f(s, u_n(s), \int_0^s K(s,\tau)u_n(\tau)d\tau) \\
&-f(s, u(s), \int_0^s K(s,\tau)u(\tau)d\tau)\|_X \Big) ds + \sum_{i=1}^{m} \|I_i(u_n(t_i^-)) - I_i(u(t_i^-))\|_X \\
&+\sum_{i=1}^{m} |(T - t_i)| \left(\frac{\Gamma(2-q)}{|t_i|^{1-q}} \|J_i(u_n(t_i^-)) - J_i(u(t_i^-))\|_X \right) \Big] \\
&+\sum_{i=1}^{k} |(t - t_i)| \left(\frac{\Gamma(2-q)}{|t_i|^{1-q}} \|J_i(u_n(t_i^-)) - J_i(u(t_i^-))\|_X \right).
\end{aligned}$$

Since the functions f, g, h, I_k, J_k are continuous, $\|N(u_n) - N(u)\|_{PC_t^1} \to 0$, as $n \to \infty$ which implies that the mapping N is continuous on PC_t^1.

Now, consider the space $\mathcal{B}_r = \{u \in PC_t^1 : \|u\|_{PC_t^1} \leq r\}$. It is obvious that $\mathcal{B}_r$ is closed, bounded and convex subset of PC_t^1. Let $u \in \mathcal{B}_r$, then for $t \in (t_k, t_{k+1}]$, we have

$$\begin{aligned}
&\|Nu(t)\|_X \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|f(s, u(s), \int_0^s K(s,\tau)u(\tau)d\tau)\|_X ds \\
&+\sum_{i=1}^{k} \|I_i(u(t_i^-))\|_X + a_1 + \|g(u)\|_X + \frac{|t|}{T}\Big[a_1 + a_2 + \|h(u)\|_X + \|g(u)\|_X \\
&+\sum_{i=1}^{m} \|I_i(u(t_i^-))\|_X + \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \|f(s, u(s), \int_0^s K(s,\tau)u(\tau)d\tau)\|_X ds \qquad (27)
\end{aligned}$$

$$+\sum_{i=1}^{m}|(T-t_i)|\left(\frac{\Gamma(2-q)}{|t_i|^{1-q}}\|J_i(u(t_i^-))\|_X\right)\Big]$$
$$+\sum_{i=1}^{k}|(t-t_i)|\left(\frac{\Gamma(2-q)}{|t_i|^{1-q}}\|J_i(u(t_i^-))\|_X\right), \tag{28}$$

it can be estimated as

$$\|Nu(t)\|_{PC_t^1}\le 2M_1\frac{T^\alpha}{\Gamma(\alpha+1)}+2mM_4+2a_1+2M_2+a_2+M_3+2mT^q\Gamma(2-q)M_5.$$

Its proves that N maps bounded set into bounded set in $\mathcal{B}_r$ for all subintervals $(t_k,t_{k+1}]$, $(k=1,\dots,m)$.

Finally, we shall show that N maps bounded sets into equi-continuous sets in $\mathcal{B}_r$. Let $l_1,l_2\in(t_k,t_{k+1}]$ with $l_1<l_2$, $1\le k\le m$, we have

$$\begin{aligned}
&\|(Nu)(l_2)-(Nu)(l_1)\|_X\\
&\le\|\int_0^{l_2}\frac{(l_2-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),\int_0^s K(s,\tau)u(\tau)d\tau)ds\\
&-\int_0^{l_1}\frac{(l_1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),\int_0^s K(s,\tau)u(\tau)d\tau)ds\|_X\\
&+\frac{|(l_2-l_1)|}{T}\Big[\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\|f(s,u(s),\int_0^s K(s,\tau)u(\tau)d\tau)\|_X ds\Big]\\
&+\sum_{i=1}^{k}|(l_2-l_1)|\left(\frac{\Gamma(2-q)}{|t_i|^{1-q}}\|J_i(u(t_i^-))\|_X\right).
\end{aligned}$$

it can be estimated as

$$\begin{aligned}
&\|(Nu)(l_2)-(Nu)(l_1)\|_{PC_t^1}\\
&\le\frac{M_1}{\Gamma(\alpha+1)}\Big((l_2-l_1)^\alpha+\|-(l_2-l_1)^\alpha+(l_2-l_k)^\alpha-(l_1-l_k)^\alpha\|\Big)\\
&+\frac{(l_2-l_1)}{T}\Big[M_1\frac{T^\alpha}{\Gamma(\alpha+1)}\Big]+m(l_2-l_1)\left(\frac{\Gamma(2-q)}{T^{1-q}}M_5\right),
\end{aligned}$$

which is independent of u. Thus, N is equicontinuous. Thus all the assumptions of Sachuder's fixed point theorem are satisfied. Hence, the system (5) has at least one solution on $[0,T]$.

4 Example

Consider the following fractional order impulsive integro- differential equation with non-local conditions:

$$\begin{cases}
{}^cD^{3/2}u(t)=\frac{e^t|u(t)|}{(9+e^t)(1+|u(t)|)}+\int_0^t\frac{e^{-(s-t)}}{10}|u(s)|ds,\ t\in[0,1],\ t\ne(1/3),\\
\Delta u(1/3)=\frac{|u(1/3)|}{17+|u(1/3)|},\ \Delta({}^cD^{1/2}u(1/3))=\frac{|u(1/3)|}{19+|u(1/3)|},\\
u(0)=-\int_0^1\frac{|u(s)|}{23+|u(s)|}ds,\quad u(T)=-\int_0^1\frac{|u(s)|}{25+|u(s)|}ds.
\end{cases}\tag{29}$$

Here $f(t,u,\int_0^t K(t,s)u(s)ds) = \frac{e^t|u(t)|}{(9+e^t)(1+|u(t)|)} + \int_0^t \frac{e^{-(s-t)}}{10}|u(s)|ds$. Let $x,y \in X$ and $t \in [0,1]$ then we have

$$\begin{aligned}
&|f(t,x,\int_0^t K(t,s)x(s)ds) - f(t,y,\int_0^t K(t,s)y(s)ds)| \\
&= |\frac{e^{-t}}{(9+e^t)}\Big[\frac{|x(t)|}{1+|x(t)|} - \frac{|y(t)|}{1+|y(t)|}\Big]| + |\int_0^t K(t,s)[x(s)-y(s)]ds| \\
&= |\frac{e^{-t}}{(9+e^t)}[\frac{|x(t)(1+|y(t)|) - |y(t)|(1+|x(t)|)|}{(1+|x(t)|)(1+|y(t)|)}]| \\
&+|\int_0^t \frac{e^{-(s-t)}}{10}(x(s)-y(s))ds| = |\frac{e^{-t}}{(9+e^t)}[\frac{|x(t)|-|y(t)|}{(1+|x(t)|)(1+|y(t)|)}]| \\
&+|\int_0^t \frac{e^{-(s-t)}}{10}(x(s)-y(s))ds|.
\end{aligned}$$

By taking sup norm we estimate it as follows

$$||f(t,x,\int_0^t K(t,s)x(s)ds) - f(t,y,\int_0^t K(t,s)y(s)ds)||_X \leq \frac{1}{10}\|x-y\|_X.$$

In similar way we can verify the following estimates

$$\|g(x)-g(y)\|_X \leq \frac{1}{23}\|x-y\|_X,\ \|h(x)-h(y)\|_X \leq \frac{1}{25}\|x-y\|_X,\ \forall\, x,y \in X,$$
$$\|I_k(x)-I_k(y)\|_X \leq \frac{1}{17}\|x-y\|_X,\ \|J_k(x)-J_k(y)\|_X \leq \frac{1}{19}\|x-y\|_X,\ \forall\, x,y \in X.$$

The rest of the parameters used in Theorem 3.1 are computed as $q=\frac{1}{2},\ \alpha=\frac{3}{2},\ (L_1+L_2K^*)=\frac{1}{10},\ L_3=\frac{1}{17},\ L_4=\frac{1}{23},\ L_5=\frac{1}{19},\ L_6=\frac{1}{25}$, and the inequality $\Big[\frac{(L_1+L_2K^*)}{\Gamma(\alpha+1)}2T^\alpha + 2mL_3 + 2L_4 + L_6 + 2mT^q\Gamma(2-q)L_5\Big] = 0.48834 < 1.$

Thus, all the conditions of Theorem 3.1 are satisfied. Hence, the impulsive fractional boundary value problem (5) has a unique solution on $[0,1]$.

5 Conclusion

At the foundation of this paper, one can consider the fractional integro-differential equation of order $\alpha \in (1,2)$ with nonlocal boundary conditions and fractional impulsive conditions. For the solution of the system (5) we follow the concept from the recent contributions on impulsive fractional differential equations by M. Feckan et al. [12, 16, 19]. The existence and uniqueness of solutions for the system (5) are treated with the help of Banachs and Schauders fixed point theorems.

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