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Approximate Controllability of Semilinear Stochastic Control System with Nonlocal Conditions

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Abstract: In this paper we study the approximate controllability of semilinear stochastic control system with nonlocal conditions in a Hilbert space. Nonlocal initial condition is a generalization of the classical initial condition and is motivated by physical phenomena. The results are obtained by using Sadovskii's fixed point theorem. At the end, an example is given to show the effectiveness of the result.

Keywords: *approximate controllability; semilinear systems; stochastic control system; Sadovskii's fixed point theorem.*

Mathematics Subject Classification (2010): 34K30, 34K35, 93C25.

1 Introduction

Controllability concepts play a vital role in deterministic control theory. It is well known that controllability of deterministic equation is widely used in many fields of science and technology. Kalman [23] introduced the concept of controllability for finite dimensional deterministic linear control systems. The basic concepts of control theory in finite and infinite dimensional spaces have been introduced in [31] and [24] respectively. However, in many cases, some kind of randomness can appear in the problem, so that the system should be modelled by a stochastic form. Only few authors have studied the extensions of deterministic controllability concepts to stochastic control systems. Klamka et al. [11]-[12] studied the controllability of linear stochastic systems in finite dimensional spaces with delay and without delay in control as well as in state using Rank theorem. In [17]-[22], Mahmudov et al. established results for controllability of linear and semilinear stochastic systems in Hilbert space. Instead of this, Sakthivel, Balachandran, Dauer and Bashirov et al. studied the approximate controllability of nonlinear stochastic systems

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in [25], [14], [13] and [1]. Shen and Sun [16] studied the controllability of stochastic first order nonlinear systems with delay in control in finite dimensional as well as in infinite dimensional spaces. In [26], Sakthivel et al. studied the approximate controllability of second order stochastic system with impulsive effects using Banach fixed point theorem. In [2]- [5] Anurag et al. obtained some sufficient conditions for controllability of integer and fractional order stochastic systems with delay in control and state term using different fixed point theorems.

On the other hand, Byszewski et al. [15] introduced nonlocal conditions into the initial value problems and argued that the corresponding models more accurately describe the phenomena since more information was taken into account at the oneset of the experiment, thereby reducing the ill effects incurred by a single initial measurement. Also, it has a better effect on the solution and is more precise for physical measurements than classical condition $x(0) = x_0$ alone. In [32], Y.K.Chang et al. obtained sufficient conditions for controllability of semilinear differential systems with nonlocal conditions in Banach spaces using Sadovskii fixed-point theorem.

Kumar [28]- [29] studied on the controllability of second order and fractional order systems with delays in Banach spaces using Sadovskii's Fixed point theorem. Also Farahi et al. [30] studied on the approximate controllability of fractional neutral stochastic evolution equations with nonlocal conditions using Sadovskii's fixed point theorem. Sanjukta [27] studied approximate controllability of a functional differential equation with deviated argument using fixed point theory.

Up to now, to the best of our knowledge , there are no results on the approximate controllability of semilinear stochastic control systems with nonlocal conditions using Sadovskii's fixed point theorem in the literature. So, the present paper is devoted to the study of approximate controllability of the semilinear stochastic control systems with nonlocal conditions using Sadovskii's fixed point theorem.

2 Problem Formulation and Preliminaries

Let $(\Omega, \Im, P)$ be a complete space equipped with a normal filtration $\Im_t$, $t \in J = [0, b]$. Let H, U and E be the separable Hilbert spaces and ω be a Q-Weiner process on $(\Omega, \Im_b, P)$ with the covariance operator Q such that $trQ < \infty$. We assume that there exists a complete orthonormal system e_n in E, a bounded sequence of nonnegative real numbers λ_n such that $Qe_n = \lambda_n e_n$, $n = 1, 2, \cdots$ and a sequence β_n of independent Brownian motions such that

$$w(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta_n(t) e_n, \quad t \in J,$$

and $\Im_t = {\Im_t}^{\omega}$, where ${\Im_t}^{\omega}$ is the σ-algebra generated by ω. Let ${L_2}^0 = L_2(Q^{1/2}E; H)$ be the space of all Hilbert-Schmidt operators from $Q^{1/2}E$ to H with the norm $||\psi||_Q^2 = tr[\psi Q \psi^*]$. Let $L_2^{\Im}(J, H)$ be the space of all $\Im_t$-adapted, H-valued measurable square integrable processes on $J \times \Omega$.Let $C([0, b]; L^2(\Im, H))$ be the Banach space of continuous maps from $[0, b]$ into $L^2(\Im, H)$ satisfying the condition $\sup_{t \in J} \mathbb{E}||x(t)||^2 < \infty$.

Let $H_2 = C_2([0, b]; H)$. Now H_2 is the closed subspace of $C([0, b]; L^2(\Im, H))$ consisting of measurable and $\Im_t$ - adapted H valued processes $\phi \in C([0, b]; L^2(\Im, H))$ endowed with the norm

$$||\phi||_{H_2} = \left(\sup_{t \in [0,b]} \mathbb{E}||\phi(t)||_H^2 \right)^{1/2}.$$

In this paper, we consider a mathematical model given by the following nonlinear second order stochastic differential equations with variable delay in control and with nonlocal conditions of the form

$$\left.\begin{array}{rcl} dx(t) & = & [Ax(t) + Bu(t) + f(t, x(t))]dt + \sigma(t, x(t))d\omega(t), \quad t \in J \\ x(0) & = & x_0 + g(x), \end{array}\right\} \tag{1}$$

where $A : D(A) \subset H \to H$ is a closed, linear and densely defined operator on H which generates a compact semigroup $\{S(t) : t \in J\}$ on H. B is a bounded linear operator from the Hilbert space U into H. The control $u \in L^2_{\Im}([0, b], U)$; $f : J \times H \to H$; $\sigma : J \times H \to L^0_2$ are nonlinear suitable functions; x_0 is $\Im_0$ measurable H valued random variable independent of ω; g is continuous function from $C(J, H) \to H$.

For simplicity of considerations, we generally assume that the set of admissible controls is $U_a d = L^2_{\Im}(J, U)$.

Definition 2.1 A stochastic process $x \in H_2$ is a mild solution of (1) if for each $u \in L^2_{\Im}([0, b], U)$, it satisfies the following integral equation:

$$\begin{aligned} x(t) &= S(t)(x_0 + g(x)) + \int_0^t S(t-s)[Bu(s) + f(s, x(s))]ds \\ &\quad + \int_0^t S(t-s)\sigma(s, x(s))d\omega(s). \end{aligned}$$

Let us introduce the following operators and sets (see [15])
$L_b \in \mathfrak{L}(L^{\Im}_2(J \times \Omega, U), L_2(\Omega, \Im_b, H))$ is defined by

$$L_b u = \int_0^b S(b-s)Bu(s)ds,$$

where $\mathfrak{L}(X, Y)$ denotes the set of bounded linear operators from X to Y.

Then its adjoint operator $L^*_b : L_2(\Omega, \Im_b, H) \to L^{\Im}_2(J \times \Omega, U)$ is given by

$$L^*_b z = B^* S^*(b-t)\mathbb{E}\{z|\Im_t\}.$$

The set of all states reachable in time b from initial state $x(0) = x_0 \in L_2(\Omega, \Im_0, X)$, using admissible controls is defined as

$$\begin{aligned} R_b(U_{ad}) &= \{x(b; x_0, u) \in L_2(\Omega, \Im_b, H) : u \in U_{ad}\}, \\ x(b; x_0, u) &= S(b)(x_0 + g(x)) + \int_0^b S(b-s)Bu(s)ds + \int_0^b S(b-s)f(s, x(s))ds, \\ &\quad + \int_0^T S(T-s)\sigma(s, x(s)d\omega(s). \end{aligned}$$

Let us introduce the linear controllability operator $\Pi^b_0 \in \mathfrak{L}(L_2(\Omega, \Im_b, H), L_2(\Omega, \Im_b, H))$ as follows:

$$\begin{aligned} \Pi^b_0\{.\} &= L_b(L_b)^*\{.\} \\ &= \int_0^b S(b-t)BB^*S^*(b-t)\mathbb{E}\{.|\Im_t\}dt. \end{aligned}$$

The corresponding controllability operator for deterministic model is:

$$\begin{aligned}\Gamma_s^b &= L_b(s)L_b^*(s)\\ &= \int_s^b S(b-t)BB^*S^*(b-t)dt.\end{aligned}$$

Definition 2.2 The stochastic system (1) is approximately controllable on $[0,b]$ if $\overline{\Re(b)} = L_2(\Omega, \Im_b, H)$, where $\Re(b) = \{x(b;u) : u \in L_2(\Omega, \Im_b, H) : u \in U_ad\}$ and $L^2_{\Im}([0,b],U)$ is the closed subspace of $L^2_{\Im}([0,b]\times\Omega,U)$, consisting of all $\Im_t$ adapted, U valued stochastic processes.

Lemma 2.1 *[6] Let $G : J\times\Omega \to L_2^0$ be a strongly measurable mapping such that $\int_0^b \mathbb{E}||G(t)||^p_{L_2^0} < \infty$. Then*

$$\mathbb{E}\bigg|\bigg|\int_0^t G(s)d\omega(s)\bigg|\bigg|^p \leq L_G\int_0^t \mathbb{E}||G(s)||^p_{L_2^0}ds$$

for all $t\in J$ and $p\geq 2$, where L_G is the constant involving p and b.

Lemma 2.2 *(Sadovskii's fixed point theorem [7]). Suppose that M is a nonempty, closed, bounded and convex subset of a Banach space X and $\Gamma : M\subseteq X\to X$ is a condensing operator. Then the operator Γ has a fixed point in M.*

To prove our main results, we list the following basic assumptions of this paper:

(i) A is the infinitesimal generator of a compact semigroup $\{S(t) : t\geq 0\}$ on H.

(ii) The function $f : J\times H\to H$ and $\sigma : J\times H\to L_2^0$ satisfy linear growth and Lipschitz conditions, i.e, there exist positive constants N_1, N_2, K_1 and K_2 such that

$$\begin{aligned}||f(t,x)-f(t,y)||^2 \leq N_1||x-y||^2, \quad ||f(t,x)||^2 \leq N_2(1+||x||^2),\\ ||\sigma(t,x)-\sigma(t,y)||^2_{L_2^0} \leq K_1||x-y||^2, \quad ||\sigma(t,x)||^2_{L_2^0} \leq K_2(1+||x||^2).\end{aligned}$$

(iii) The function g is continuous function and there exists some positive constants M_g such that

$$||g(x)-g(y)||^2 \leq M_g||x-y||^2, \quad ||g(x)||^2 \leq M_g(1+||x||^2)$$

for all $x, y\in C(J,H)$.

(iv) For each $0\leq t<b$, the operator $\alpha(\alpha I+\Gamma_t^b)^{-1}\to 0$ in the strong operator topology as $\alpha\to 0^+$, where

$$\Gamma_t^b = \int_t^b S(b-s)BB^*S^*(b-s)ds$$

is the controllability Grammian. Observe that the linear deterministic system corresponding to (1)

$$\left.\begin{aligned}dx'(t) &= [Ax(t)+Bu(t)]dt, \quad t\in J\\ x(0) &= x_0\end{aligned}\right\} \tag{2}$$

is approximately controllable on $[t,b]$ iff the operator $\alpha(\alpha I+\Gamma_t^b)^{-1}\to 0$ strongly as $\alpha\to 0^+$.

For simplicity, let us take $M_B = \max\{||B||\}$.

3 Main Result

Let us recall two lemmas concerning approximate controllability, which will be used in the proof.

The following lemma is required to define the control function.

Lemma 3.1 *[19] For any $x_b \in L_2(\Omega, \Im_b, H)$, there exists $\phi \in L_2^{\Im}(J, L_2^0)$ such that $x_b = \mathbb{E}x_b + \int_0^b \phi(s)d\omega(s)$.*

Now for any $\alpha > 0$ and $x_b \in L_2(\Omega, \Im_b, H)$, we define the control function in the following form

$$\begin{aligned} u^{\alpha}(t,x) &= B^*S^*(b-t)\big[(\alpha I + \Psi_0^b)^{-1}\big(\mathbb{E}x_b - S(b)(x_0 + g(x))\big) \\ &\quad + \int_0^t (\alpha I + \Psi_s^b)^{-1}\phi(s)dw(s)\big], \\ &\quad - B^*S^*(b-t)\int_0^t (\alpha I + \Psi_s^b)^{-1}S(b-s)f(s,x(s))ds, \\ &\quad - B^*S^*(b-t)\int_0^t (\alpha I + \Psi_s^b)^{-1}S(b-s)\sigma(s,x(s))dw(s). \end{aligned}$$

Lemma 3.2 *There exists a positive constant M_u such that for all $x, y \in H_2$, we have*

$$\mathbb{E}||u^{\alpha}(t,x) - u^{\alpha}(t,y)||^2 \le \frac{M_u}{\alpha^2}||x-y||^2, \tag{3}$$

$$\mathbb{E}||u^{\alpha}(t,x)||^2 \le \frac{M_u}{\alpha^2}\left(1 + ||x||^2\right). \tag{4}$$

Proof. Let $x, y \in H_2$. From Holder's inequality, Lemma 2.1 and the assumptions on the data, we obtain

$$\begin{aligned} \mathbb{E}||u^{\alpha}(t,x) - u^{\alpha}(t,y)||^2 &\le 3\mathbb{E}\left|\left|B^*S^*(b-t)(\alpha I + {\psi_0}^b)^{-1}S(b)[g(x) - g(y)]\right|\right|^2 \\ &+ 3\mathbb{E}\left|\left|B^*S^*(b-t)\int_0^t (\alpha I + \Psi_s^b)^{-1}S(b-s)[f(s,x(s)) - f(s,y(s))]ds\right|\right|^2 \\ &+ 3\mathbb{E}\left|\left|B^*S^*(b-t)\int_0^t (\alpha I + \Psi_s^b)^{-1}S(b-s)[\sigma(s,x(s)) - \sigma(s,y(s))]dw(s)\right|\right|^2 \\ &\le \frac{3}{\alpha^2}M_B^2M^4\left[M_g||x-y||_{H_2}^2 + b\int_0^t N_1\mathbb{E}||x(s)-y(s)||_H^2ds + L_G\int_0^t K_1\mathbb{E}||x(s)-y(s)||_H^2ds\right] \\ &\le \frac{3}{\alpha^2}M_B^2M^4\left[M_g + bN_1b\sup_{s\in[0,b]}\mathbb{E}||x(s)-y(s)||_H^2 + L_GK_1b\sup_{s\in[0,b]}\mathbb{E}||x(s)-y(s)||_H^2\right] \\ &\le \frac{3}{\alpha^2}M_B^2M^4[M_g + b^2N_1 + L_GK_1b]||x-y||_{H_2}^2 = \frac{M_u}{\alpha^2}||x-y||_{H_2}^2, \end{aligned}$$

where $M_u = 3M_B^2M^4[M_g + b^2N_1 + L_GK_1b]$ and p, q are conjugate indices.

The proof of the second inequality can be verified in a similar manner by putting $u^{\alpha}(t,y) = 0$. So, the proof of the lemma is completed.

For any $\alpha > 0$, define the operator $\mathbf{P}_\alpha : H_2 \to H_2$ by

$$\begin{aligned}(\mathbf{P}_\alpha x)(t) &= S(t)(x_0 + g(x)) + \int_0^t S(t-s)[Bu^\alpha(s,x) + f(s,x(s))]ds \\ &\quad + \int_0^t S(t-s)\sigma(s,x(s))d\omega(s).\end{aligned}$$

To prove the approximate controllability, we first prove in Theorem 3.1, the existence of a fixed point of the operator $\mathbf{P}_\alpha$ defined above, using the Sadovskii's fixed point theorem. Second, in Theorem 3.2, we show that under certain assumptions the approximate controllability of system (2) is implied by the approximate controllability of the corresponding deterministic linear system.

Theorem 3.1 *Assume hypothesis $(i) - (iv)$ are satisfied. Then the system* (1) *has a mild solution on* $[0, b]$ *provided that*

$$\begin{aligned}&8M^2M_g + 4M^2\left(\frac{M_B^2 b^2 M_u}{\alpha^2} + b^2N_2 + L_\sigma K_2 b\right) < 1, \qquad (5)\\ &\frac{3M^2M_B^2 bM_u}{\alpha^2} + 3M^2bN_1 + 3M^2L_G < 1.\end{aligned}$$

Proof. The proof of this theorem is divided into several steps.
Step 1. For any $x \in H_2$, $\mathbf{P}_\alpha(x)(t)$ is continuous on J in the L^p sense.

Proof of Step 1: Let $0 \leq t_1 \leq t_2 \leq b$. Then for any fixed $x \in H_2$, it follows from Holder's inequality, Lemma 2.1 and assumptions of the theorem that

$$\begin{aligned}&\mathbb{E}||(\mathbf{P}_\alpha x)(t_2) - (\mathbf{P}_\alpha x)(t_1)||^2 \\ &\leq 7\Bigg[\mathbb{E}||(S(t_2) - S(t_1))(x_0 + g(x))||^2 + \mathbb{E}\bigg|\bigg|\int_0^{t_1}[S(t_2-s) - S(t_1-s)]f(s,x(s))ds\bigg|\bigg|^2 \\ &+\mathbb{E}\bigg|\bigg|\int_{t_1}^{t_2} S(t_2-s)f(s,x(s))ds\bigg|\bigg|^2 + \mathbb{E}\bigg|\bigg|\int_0^{t_1}[S(t_2-s) - S(t_1-s)]\sigma(s,x(s))d\omega(s)\bigg|\bigg|^2 \\ &+\mathbb{E}\bigg|\bigg|\int_{t_1}^{t_2} S(t_2-s)\sigma(s,x(s))d\omega(s)\bigg|\bigg|^2 + \mathbb{E}\bigg|\bigg|\int_0^{t_1}[S(t_2-s) - S(t_1-s)]Bu^\alpha(s,x)ds\bigg|\bigg|^2 \\ &+\mathbb{E}\bigg|\bigg|\int_{t_1}^{t_2} S(t_2-s)Bu^\alpha(s,x)ds\bigg|\bigg|^2\Bigg] \\ &\leq 7\Bigg[2\Big(\mathbb{E}||(S(t_2) - S(t_1))x_0||^2 + \mathbb{E}||(S(t_2) - S(t_1))g(x)||^2\Big) \\ &+t_1\int_0^{t_1}\mathbb{E}||[S(t_2-s) - S(t_1-s])f(s,x(s))||^2ds + M^2(t_2-t_1)\int_{t_1}^{t_2}\mathbb{E}||f(s,x(s))||^2ds \\ &+L_G\int_0^{t_1}\mathbb{E}||(S(t_2-s) - S(t_1-s))\sigma(s,x(s))||^2ds + M^2L_G\int_{t_1}^{t_2}\mathbb{E}||\sigma(s,x(s))||^2ds \\ &+t_1\int_0^{t_1}\mathbb{E}||[S(t_2-s) - S(t_1-s])Bu^\alpha(s,x)||^2ds + ||B||^2M^2(t_2-t_1)\int_{t_1}^{t_2}\mathbb{E}||u^\alpha(s,x)||^2ds\Bigg]\end{aligned}$$

Hence using Lebesgue's dominated convergence theorem, we conclude that the right hand side of the above inequality tends to zero as $t_2 - t_1 \to 0$. Thus we conclude $\mathbf{P}_\alpha(x)(t)$ is continuous from the right in $[0, b)$. A similar argument shows that it is also continuous from the left in $(0, b]$. Thus $\mathbf{P}_\alpha(x)(t)$ is continuous on J in the L^p sense.

Step 2: For each positive integer q, let $B_q = \{x \in H_2 : \mathbb{E}||x(t)||_H^2 \leq q\}$, then the set B_q is clearly a bounded, closed and convex set in H_2. From Lemma 2.1, Holder's inequality and assumption (i), we derive that

$$\begin{aligned}
\mathbb{E}\Big|\Big|\int_0^t S(t-s)f(s,x(s))ds\Big|\Big|_H^2 &\leq \mathbb{E}\Big(\int_0^t ||S(t-s)f(s,x(s))||_H ds\Big)^2 \\
&\leq M^2\mathbb{E}\Big(\int_0^t ||f(s,x(s))||_H ds\Big)^2 \\
&\leq M^2 b\int_0^t \mathbb{E}||f(s,x(s))||_H^2 ds \\
&= M^2 b\int_0^t N_2(1+\mathbb{E}||x(s)||_H^2)ds \\
&\leq M^2 b N_2\int_0^t (1+\sup_{s\in[0,b]}\mathbb{E}||x(s)||_H^2)ds \\
&\leq M^2 b N_2 b(1+||x||_{H_2}^2) \\
&\leq M^2 b^2 N_2(1+||x||_{H_2}^2),
\end{aligned}$$

which deduces that $S(t-s)f(s, x(s))$ is integrable on J, by Bochner's theorem, P_α is well defined on B_q.

Similarly from (ii), we derive that

$$\begin{aligned}
\mathbb{E}\Big|\Big|\int_0^t S(t-s)\sigma(s,x(s))dw(s)\Big|\Big|^2 &\leq L_\sigma\int_0^t \mathbb{E}||S(t-s)\sigma(s,x(s))||_{L_2^0}^2 ds \\
&\leq L_\sigma M^2\int_0^t \mathbb{E}||\sigma(s,x(s))||_{L_2^0}^2 ds \\
&\leq L_\sigma M^2\int_0^t K_2(1+\mathbb{E}||x(s)||_H^2)ds \\
&\leq L_\sigma M^2 K_2\int_0^t (1+\sup_{s\in[0,b]}\mathbb{E}||x(s)||_H^2)ds \\
&\leq L_\sigma M^2 K_2 b(1+||x||_{H_2}^2) \\
&\leq L_\sigma M^2 K_2 b(1+||x||_{H_2}^2).
\end{aligned}$$

Now, we claim that there exists a positive number q such that $P_\alpha(B_q) \subseteq B_q$.

If it is not true, then for each positive number q, there is a function $x_q(.) \in B_q$ but $P_\alpha x_q$ does not belong to B_q, that is $\mathbb{E}||P_\alpha x_q(t)||_H^2 > q$ for some $t \in J$.

On the other hand, from assumptions (ii), (iii) and Lemma 3.2, we have

$$\begin{aligned}
q \leq \mathbb{E}||P_\alpha x_q(t)||_H^2 &= \mathbb{E}\bigg|\bigg|S(t)(x_0+g(x)) + \int_0^t S(t-s)[Bu^\alpha(s,x)+f(s,x(s))]ds \\
&\quad + \int_0^t S(t-s)\sigma(s,x(s))dw(s)\bigg|\bigg|_H^2 \\
&\leq 4M^2\mathbb{E}||x_0+g(x)||^2 + 4M^2M_B^2b^2\frac{M_u}{\alpha^2}(1+||x||_{H_2}^2) \\
&\quad +4M^2b^2N_2(1+||x||_{H_2}^2) + 4M^2L_\sigma K_2 b(1+||x||_{H_2}^2) \\
&\leq 4M^2[2\mathbb{E}||x_0||^2 + 2\mathbb{E}||g(x)||^2] + 4M^2M_B^2b^2\frac{M_u}{\alpha^2}(1+||x||_{H_2}^2) \\
&+ 4M^2b^2N_2(1+||x||_{H_2}^2) + 4M^2L_\sigma K_2 b(1+||x||_{H_2}^2) \\
&\leq 4M^2[2\mathbb{E}||x_0||^2 + 2M_g(1+||x||_{H_2}^2)] + 4M^2M_B^2b^2\frac{M_u}{\alpha^2}(1+||x||_{H_2}^2) \\
&\quad +4M^2b^2N_2(1+||x||_{H_2}^2) + 4M^2L_\sigma K_2 b(1+||x||_{H_2}^2) \\
&\leq 4M^2[2\mathbb{E}||x_0||^2 + 2M_g(1+q)] + 4M^2M_B^2b^2\frac{M_u}{\alpha^2}(1+q) \\
&\quad +4M^2b^2N_2(1+q) + 4M^2L_\sigma K_2 b(1+q) \\
&\leq \bigg(8M^2\mathbb{E}||x_0||^2 + 8M^2M_g + \frac{4M^2M_B^2b^2M_u}{\alpha^2} \\
&+ 4M^2b^2N_2 + 4M^2L_\sigma K_2 b\bigg) \\
&+ \bigg(8M^2M_g + \frac{4M^2M_B^2b^2M_u}{\alpha^2} + 4M^2b^2N_2 + 4M^2L_\sigma K_2 b\bigg)q.
\end{aligned}$$

Dividing both sides by q and taking the limit as $q \to \infty$, we get

$$8M^2M_g + 4M^2\bigg(\frac{M_B^2b^2M_u}{\alpha^2} + b^2N_2 + L_\sigma K_2 b\bigg) > 1.$$

This contradicts condition (5). Hence for some positive number q, $P_\alpha B_q \subseteq B_q$.

Step 3. Now, we define operators P_{α_1} and P_{α_2} as

$$\begin{aligned}
(P_{\alpha_1}x)(t) &= S(t)[x_0+g(x)], \\
(P_{\alpha_2}x)(t) &= \int_0^t S(t-s)[Bu^\alpha(s,x)+f(s,x(s))]ds + \int_0^t S(t-s)\sigma(s,x(s))d\omega(s)
\end{aligned}$$

for $t \in J$. Here, we will prove that P_{α_1} is completely continuous, while P_{α_2} is a contraction operator.

By assumption (iii), it is clear that P_{α_1} is a completely continuous operator. Next we show that P_{α_2} is the contraction operator. For this, let $x, y \in B_q$, then for each $t \in J$, we have from assumptions (ii),(iii)

$$\begin{aligned}
\mathbb{E}||(P_{\alpha_2}x)(t)-(P_{\alpha_2}y)(t)||_H^2 &\leq 3\mathbb{E}\bigg|\bigg|\int_0^t S(t-s)B[u^\alpha(s,x)-u^\alpha(s,y)]ds\bigg|\bigg|_H^2\\
&\quad +3\mathbb{E}\bigg|\bigg|\int_0^t S(t-s)[f(s,x(s))-f(s,y(s))]ds\bigg|\bigg|_H^2\\
&\quad +3\mathbb{E}\bigg|\bigg|\int_0^t S(t-s)[\sigma(s,x(s))-\sigma(s,y(s))]d\omega(s)\bigg|\bigg|_H^2\\
&\leq 3M^2M_B^2\int_0^t \mathbb{E}||u^\alpha(s,x)-u^\alpha(s,y)||_H^2 ds\\
&+ 3M^2\int_0^t \mathbb{E}||f(s,x(s))-f(s,y(s))||^2 ds\\
&\quad +3M^2\int_0^t \mathbb{E}||\sigma(s,x(s))-\sigma(s,y(s))||^2 dw(s)\\
&\leq 3M^2M_B^2 b\frac{M_u}{\alpha^2}||x-y||_{H_2}^2+3M^2bN_1||x-y||_{H_2}^2\\
&+ 3M^2L_G||x-y||_{H_2}^2\\
&\leq \left(\frac{3M^2M_B^2bM_u}{\alpha^2}+3M^2bN_1+3M^2L_\sigma\right)||x-y||_{H_2}^2
\end{aligned}$$

therefore $||(P_{\alpha_2}x)-(P_{\alpha_2}y)||_{H_2}^2 \leq L_0||x-y||_{H_2}^2$, where

$$L_0=\left(\frac{3M^2M_B^2bM_u}{\alpha^2}+3M^2bN_1+3M^2L_G\right)<1.$$

Thus P_{α_2} is a contraction mapping.

Now we have $P_\alpha = P_{\alpha_1}+P_{\alpha_2}$ is a condensing map on B_q, so Sadovskii's fixed point theorem is satisfied. Hence we conclude that there exists a fixed point $x(.)$ for P_α on B_q, which is the mild solution of (1).

Theorem 3.2 *Assume assumptions* $(i)-(iv)$ *are satisfied and if* f *and* σ *are uniformly bounded, then the system* (1) *is approximately controllable on* $[0,b]$.

Proof. Let x_α be a fixed point of $\mathbf{P}_\alpha$ in H_2. By using the stochastic Fubini theorem, it is easy to see that

$$\begin{aligned}
x_\alpha(b) &= x_b-\alpha(\alpha I+\Gamma_0^b)^{-1}\Big(\mathbb{E}x_b-S(b)(x_0+g(x))\Big)\\
&\quad +\alpha\int_0^b(\alpha I+\Gamma_s^b)^{-1}S(b-s)f(s,x_\alpha(s))ds\\
&\quad +\alpha\int_0^b(\alpha I+\Gamma_s^b)^{-1}[S(b-s)\sigma(s,x_\alpha(s))-\phi(s)]d\omega(s).
\end{aligned}$$

By the assumption that f and σ are uniformly bounded, there exists $D>0$ such that

$$||f(s,x_\alpha(s))||^2+||\sigma(s,x_\alpha(s))||^2\leq D$$

in $[0, b] \times \Omega$. Then there is a subsequence denoted by $\{f(s, x_\alpha(s)), \sigma(s, x_\alpha(s))\}$ weakly converging to say $\{f(s, \omega), \sigma(s, \omega)\}$ in $H \times L_2^0$. Now, the compactness of $S(t)$ implies that $S(b-s)f(s, x_\alpha(s)) \to S(b-s)f(s)$ and $S(b-s)\sigma(s, x_\alpha(s)) \to S(b-s)\sigma(s)$ in $J \times \Omega$.

Now, from the above equation, we get

$$\begin{aligned}
\mathbb{E}||x_\alpha(b) - x_b||^2 \quad \le \quad & 6\left|\left|\alpha(\alpha I + \Gamma_0^b)^{-1}\left[\mathbb{E}x_b - S(b)[x_0 + g(x))]\right]\right|\right|^2 \\
& +6\mathbb{E}\left(\int_0^b ||\alpha(\alpha I + \Gamma_s^b)^{-1}\phi(s)||_{L_2^0}^2 ds\right) \\
& +6\mathbb{E}\left(\int_0^b ||\alpha(\alpha I + \Gamma_s^b)^{-1}||\ ||S(b-s)[f(s, x_\alpha(s)) - f(s)]||ds\right)^2 \\
& +6\mathbb{E}\left(\int_0^b ||\alpha(\alpha I + \Gamma_s^b)^{-1}S(b-s)f(s)||ds\right)^2 \\
& +6\mathbb{E}\left(\int_0^b ||\alpha(\alpha I + \Gamma_s^b)^{-1}||\ ||S(b-s)[\sigma(s, x_\alpha(s)) - \sigma(s)]||_{L_2^0}^2 ds\right) \\
& +6\mathbb{E}\left(\int_0^b ||\alpha(\alpha I + \Gamma_s^b)^{-1}S(b-s)\sigma(s)||_{L_2^0}^2 ds\right)\Bigg].
\end{aligned}$$

Since by assumption (iv), for all $0 \le s < b$ the operator $\alpha(\alpha I + \Gamma_s^b)^{-1} \to 0$ strongly as $\alpha \to 0^+$ and moreover $||\alpha(\alpha I + \Gamma_s^b)^{-1}|| \le 1$. Thus by the Lebesgue dominated convergence theorem, we obtain $\mathbb{E}||x_\alpha(b) - x_b||^2 \to 0^+$. This gives the approximate controllability.

4 Example

Consider the stochastic control system:

$$\left.\begin{array}{l}
d_t z(t, \theta) = [z_{\theta\theta} + Bu(t, \theta) + p(t, z(t))]dt + k(t, z(t))d\omega(t), \\
z(t, 0) = z(t, \pi) = 0, \quad 0 \le t \le T, \quad 0 < \theta < \pi, \\
z(0, \theta) + \displaystyle\sum_{i=1}^{n} \alpha_i z(t_i, \theta) = z_0(\theta) \quad t \in J,
\end{array}\right\} \tag{6}$$

where B is a bounded linear operator from a Hilbert space U into X; $p : J \times X \to X$, $k : J \times X \to L_2^0$ are all continuous and uniformly bounded, $u(t)$ is a feedback control and w is a Q-Wiener process.

Let $X = L_2[0, \pi]$, and let $A : D(A) \subset X \to X$ be an operator defined by

$$Az = z_{\theta\theta}$$

with domain

$$D(A) = \{z(.) \in X | z, z_\theta \text{ are absolutely continuous }, z_{\theta\theta} \in X, z(0) = z(\pi) = 0\}.$$

Furthermore, A has discrete spectrum, the eigenvalues are $-n^2, n = 1, 2, \cdots$ with the corresponding normalized characteristic vectors $e_n(s) = (2/\pi)^{1/2} \sin ns$, then

$$Az = \sum_{n=1}^{\infty} -n^2 < z, e_n > e_n, \quad z \in X.$$

It is known that A generates a compact semigroup $S(t), t > 0$ in X and is given by

$$S(t)z = \sum_{n=1}^{\infty} e^{-n^2 t} < z, e_n > e_n(\theta), \quad z \in X.$$

Let $f : J \times X \to X$ be defined by

$$f(t, x(t))(\theta) = p(t, x(t))(\theta)), \quad (t, x_t) \in J \times X, \theta \in [0, \pi].$$

Let $\sigma : J \times X \to L_2^0$ be defined by

$$\sigma(t, x(t))(\theta) = k(t, x(t))(\theta)), \quad (t, x_t) \in J \times X, \theta \in [0, \pi].$$

The function $g : C(J, X) \to X$ is defined as

$$g(z)(\theta) = \sum_{i=1}^{n} \alpha_i z(t_i, \theta)$$

for $0 < t_i < T$ and $\theta \in [0, \pi]$.

With this choice of A, B, f, σ and g, (1) is the abstract formulation of (6) such that the conditions in (i) and (ii) are satisfied.

Now define an infinite-dimensional space

$$U = \left\{ u : u = \sum_{n=2}^{\infty} u_n e_n(\theta) \ \ | \sum_{n=2}^{\infty} u_n^2 < \infty \right\}$$

with the norm defined by

$$||u||_U = \left(\sum_{n=2}^{\infty} u_n^2 \right)^{1/2}$$

and a linear continuous mapping B from $U \to X$ as follows:

$$Bu = 2u_2 e_1(\theta) + \sum_{n=2}^{\infty} u_n(t) e_n(\theta).$$

It is obvious that for $u(t, \theta, \omega) = \sum_{n=2}^{\infty} u_n(t, \omega) e_n(\theta) \in L_2^{\Im}(J, U)$

$$Bu(t) = 2u_2(t) e_1(\theta) + \sum_{n=2}^{\infty} u_n(t) e_n(\theta) \in L_2^{\Im}(J, X).$$

Moreover,

$$B^* v = (2v_1 + v_2) e_2(\theta) + \sum_{n=3}^{\infty} v_n e_n(\theta),$$

$$B^* S^*(t) z = (2z_1 e^{-t} + z_2 e^{-4t}) e_2(\theta) + \sum_{n=3}^{\infty} z_n e^{-n^2 t} e_n(\theta),$$

for $v = \sum_{n=1}^{\infty} v_n e_n(\theta)$ and $z = \sum_{n=1}^{\infty} z_n e_n(\theta)$.

Let $||B^*S^*(t)z|| = 0, \quad t \in [0,T]$, it follows that

$$||2z_1e^{-t} + z_2e^{-4t}||^2 + \sum_{n=3}^{\infty} ||z_n e^{-n^2t}||^2 = 0, \quad t \in [0,T]$$

$\Rightarrow z_n = 0, \quad n = 1, 2, \cdots \Rightarrow z = 0.$

Thus by Theorem 4.1.7 of [23], the deterministic linear system corresponding to (6) is approximate controllable on $[0,T]$. Therefore the system (6) is approximate controllable provided that f, σ and g satisfy the assumptions (i) and (ii).

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