Nonlinear Dynamics and Systems Theory, **15** (3) (2015) 313–320

Estimating the Bounds for the General 4-D Continuous-Time Autonomous System

Rezzag Samia *, Zehrour Okba and Aliouche Abdelkrim

Department of Mathematics and Informatics,
University Larbi Ben M'hidi, 04000, Oum-El-Bouaghi, Algeria.

Received: September 6, 2014; Revised: June 24, 2015

Abstract: In the present paper, the general 4-D continuous-time system is considered and the estimate of the upper bound of such a system is investigated, using the multivariable functions analysis. Especially, sufficient conditions for this system to be contained in a four-dimensional ellipsoidal surface are obtained. The results obtained in this investigation generalize all the existing results in the relevant literature concerning the finding of an upper bound for the fourth order dynamical system.

Keywords: *4-D continuous-time system; upper bounds.*

Mathematics Subject Classification (2010): 65P20, 65P30, 65P40.

1 Introduction

Since Lorenz discovered chaos in a simple system of three autonomous ordinary differential equations in order to describe the simplified Rayleigh–Benard problem in 1963 [12], the analysis of dynamics of 3-D chaotic and 4-D hyperchaotic systems has been a focal point of renewed interest for many researchers [2, 3, 5, 6, 8, 13, 15, 17, 19, 21, 22, 26, 27]. Hyperchaos is characterized as a chaotic system with more than one positive exponent, this implies that its dynamics are expended in several different directions simultaneously. Thus, hyperchaotic systems have more complex dynamical behaviors than ordinary chaotic systems. As we know, there are many hyperchaotic systems discovered in the four-dimensional social and economical systems. Typical examples are 4-D hyperchaotic Chua's circuit [1], 4-D hyperchaotic Rôsslor system [18] and 4-D hyperchaotic Lorenz-Haken system [14]. Since hyperchaotic system has the theoretical and practical applications in technological fields, such as secure communications, lasers, nonlinear

* Corresponding author: mailto:rezzag.samia@gmail.com

circuits, neural networks, generation, control, synchronization, it has recently become a central topic in nonlinear sciences research.

The estimate of the bound for a chaotic system is of great importance for chaos control, chaos synchronization, and their applications [4], where the concepts of ultimate bound and attractive set of system serve as excellent tools for analysis of the qualitative behavior of a chaotic system. Such an estimation is quite difficult to achieve technically. Notwithstanding the difficulty, during the past 40 years or so, many good and interesting results on this topic have been obtained for some 3-D continuous-time systems [7, 9, 10, 16, 24].

In recent years, the study of the boundedeness of 4-D dynamical systems have attracted the attention of many engineers, physicists and mathematicians. For example in [11], the ultimate bound and positively invariant set for the 4-D hyperchaotic Lorenz-Haken system were investigated. In [20] the estimation of the bounds for the 4-D hyperchaotic Lorenz-Stenflo system was also obtained. Recently, the boundedness of the generalized 4-D hyperchaotic model containing Lorenz-Stenflo and Lorenz-Haken systems was done in [23] and the boundedness of a kind of hyperchaotic systems that have wide applications in the secure communications was also investigated in [25]. In the present paper, by using the multivariable functions analysis, we generalize all the existing results in the relevant literature concerning the finding of an upper bound for the general 4-D continuous-time system. In particular, we find sufficient conditions for this system to be contained in a four-dimensional ellipsoidal set.

Let us consider the general 4-D continuous-time autonomous system

$$\begin{cases} x^{'} = f\left(x, y, z, w, \delta\right), \\ y' = g\left(x, y, z, w, \delta\right), \\ z' = h\left(x, y, z, w, \delta\right), \\ w' = k\left(x, y, z, w, \delta\right), \end{cases} \tag{1}$$

where f, g, h and k are real functions and $\delta \in \mathbb{R}^m$ is the bifurcation parameter. Assume that system (1) has at least one equilibrium point, so bounded orbits are possible. Without loss of generality we can assume that the origin is an equilibrium point, i.e., $f\left(0, 0, 0, 0, \delta\right) = g\left(0, 0, 0, 0, \delta\right) = h\left(0, 0, , 0, \delta\right) = k\left(0, 0, 0, 0, \delta\right) = 0$.

2 The Estimate of the Bound for the General 4-D Dynamical System

To study the estimate of the bound for the general system (1), we define the following Lyapunov function

$$V\left(x, y, z, w\right) = \frac{\left(x - \alpha\left(x, y, z, w\right)\right)^2 + \left(y - \beta\left(x, y, z, w\right)\right)^2 + \left(z - \gamma\left(x, y, z, w\right)\right)^2 + \left(w - \theta\left(x, y, z, w\right)\right)^2}{2}, \tag{2}$$

where $\left(\alpha\left(x, y, z, w\right), \beta\left(x, y, z, w\right), \gamma\left(x, y, z, w\right), \theta\left(x, y, z, w\right)\right) \in \mathbb{R}^4$ are real functions, in which the derivative of (2) along the orbits of system (1) is given by

$$\frac{dV}{dt} = \left(x - \alpha\right)\left(x' - \alpha'\right) + \left(y - \beta\right)\left(y' - \beta'\right) + \left(z - \gamma\right)\left(z' - \gamma'\right) + \left(w - \theta\right)\left(w' - \theta'\right), \tag{3}$$

where

$$\begin{cases} \alpha' = \dfrac{\partial \alpha}{\partial x}x' + \dfrac{\partial \alpha}{\partial y}y' + \dfrac{\partial \alpha}{\partial z}z' + \dfrac{\partial \alpha}{\partial w}w' = \psi_1 f + \psi_2 g + \psi_3 h + \psi_4 k, \\ \beta' = \dfrac{\partial \beta}{\partial x}x' + \dfrac{\partial \beta}{\partial y}y' + \dfrac{\partial \beta}{\partial z}z' + \dfrac{\partial \beta}{\partial w}w' = \mu_1 f + \mu_2 g + \mu_3 h + \mu_4 k, \\ \gamma' = \dfrac{\partial \gamma}{\partial x}x' + \dfrac{\partial \gamma}{\partial y}y' + \dfrac{\partial \gamma}{\partial z}z' + \dfrac{\partial \gamma}{\partial w}w' = \xi_1 f + \xi_2 g + \mu\xi_3 h + \xi_4 k, \\ \theta' = \dfrac{\partial \theta}{\partial x}x' + \dfrac{\partial \theta}{\partial y}y' + \dfrac{\partial \theta}{\partial z}z' + \dfrac{\partial \theta}{\partial w}w' = \zeta_1 f + \zeta_2 g + \zeta_3 h + \zeta_4 k. \end{cases} \tag{4}$$

Therefore, we have

$$\frac{dV}{dt} = c_1(x,y,z,w)\,x - \omega x^2 + c_2(x,y,z,w)\,y - \varphi y^2 + c_3(x,y,z,w)\,z - \phi z^2 + c_4(x,y,z,w)\,w - \eta w^2 + c_5(x,y,z,w), \tag{5}$$

where

$$\begin{cases} c_1(x,y,z,w) = f - \psi_1 f - \psi_2 g - \psi_3 h - \psi_4 k + \omega x, \\ c_2(x,y,z,w) = g - \mu_1 f - \mu_2 g - \mu_3 h - \mu_4 k + \varphi y, \\ c_3(x,y,z,w) = h - \xi_1 f - \xi_2 g - \mu\xi_3 h - \xi_4 k + \phi z, \\ c_4(x,y,z,w) = k - \zeta_1 f - \zeta_2 g - \zeta_3 h - \zeta_4 k + \eta w, \\ c_5(x,y,z,w) = c_6(x,y,z,w) + c_7(x,y,z,w), \\ c_6(x,y,z,w) = -\alpha f - \beta g - \gamma h - \theta k + \alpha(\psi_1 f + \psi_2 g + \psi_3 h + \psi_4 k), \\ c_7(x,y,z,w) = \beta(\mu_1 f + \mu_2 g + \mu_3 h + \mu_4 k) + \\ \gamma(\xi_1 f + \xi_2 g + \xi_3 h + \xi_4 k) + \theta(\zeta_1 f + \zeta_2 g + \zeta_3 h + \zeta_4 k). \end{cases} \tag{6}$$

Assume that the equation (5) has the form

$$\frac{dV}{dt} = -\omega(x-\alpha_1)^2 - \varphi(y-\beta_1)^2 - \phi(z-\gamma_1)^2 - \eta(w-\theta_1)^2 + r, \tag{7}$$

where ω, φ, ϕ, η and r are strictly positive constants, α_1, β_1, γ_1, θ_1 are unknown constants and it should be determined in which the equation $\frac{dV}{dt} = 0$ determines an ellipsoid in $\mathbb{R}^4$.

Equation (7) is equivalent to

$$\frac{dV}{dt} = -\omega x^2 + 2\omega\alpha_1 x - \varphi y^2 + 2\varphi\beta_1 y - \phi z^2 + 2\phi\gamma_1 z - \eta w^2 + 2\eta\theta_1 w - \omega\alpha_1^2 - \varphi\beta_1^2 - \phi\gamma_1^2 - \eta\theta_1^2 + r. \tag{8}$$

By identification with (5) we get

$$\begin{cases} \alpha_1 = \dfrac{c_1(x,y,z,w)}{2\omega}, \\ \beta_1 = \dfrac{c_2(x,y,z,w)}{2\varphi}, \\ \gamma_1 = \dfrac{c_3(x,y,z,w)}{2\phi}, \\ \theta_1 = \dfrac{c_4(x,y,z,w)}{2\eta}, \\ r = \omega\alpha_1^2 + \varphi\beta_1^2 + \phi\gamma_1^2 + \eta\theta_1^2 + c_5(x,y,z,w). \end{cases} \tag{9}$$

Since α_1, β_1, γ_1, θ_1 and r are real constants, the functions $\{c_i(x,y,z,w), i = 1,2,3,4,5\}$ are also constants, i.e.,

$$\frac{\partial c_i(x,y,z,w)}{\partial x} = \frac{\partial c_i(x,y,z,w)}{\partial y} = \frac{\partial c_i(x,y,z,w)}{\partial z} = \frac{\partial c_i(x,y,z,w)}{\partial w} = 0,\ i = \overline{1,5}. \quad (10)$$

Now, putting

$$H(x,y,z,w) = \frac{(x-\alpha_1)^2}{\frac{r}{\omega}} + \frac{(y-\beta_1)^2}{\frac{r}{\varphi}} + \frac{(z-\gamma_1)^2}{\frac{r}{\phi}} + \frac{(w-\theta_1)^2}{\frac{r}{\eta}} - 1. \quad (11)$$

In order to prove the boundedness of the system (1), we assume that it is bounded and then we will find its bound, i.e., assume that

$$\begin{cases} c_5(x,y,z,w) + \omega\alpha_1^2 + \varphi\beta_1^2 + \phi\gamma_1^2 + \eta\theta_1^2 > 0, \\ \omega > 0,\ \varphi > 0,\ \phi > 0,\ \eta > 0, \end{cases} \quad (12)$$

therefore, the equation $\frac{dV}{dt} = 0$, that means, the surface

$$\Gamma = \left\{(x,y,z,w) \in \mathbb{R}^4 : H(x,y,z,w) = 0,\ \omega,\ \varphi,\ \phi,\ \eta,\ r > 0\ \right\} \quad (13)$$

is an ellipsoid in four-dimensional space. If the system (1) is bounded, then the function (2) can reach its maximum value on Γ. Denote the maximum point as (x_0, y_0, z_0, w_0). In order to find it, we define the function F by

$$F(x,y,z,w) = G(x,y,z,w) + \lambda H(x,y,z,w), \quad (14)$$

where

$$G(x,y,z,w) = x^2 + y^2 + z^2 + w^2 \quad (15)$$

and $\lambda \in \mathbb{R}$ is a finite parameter. It is clear that $\max\limits_{(x,y,z,w)\in\Gamma} G = \max\limits_{(x,y,z,w)\in\Gamma} F$ and let

$$\begin{cases} \frac{\partial F(x,y,z,w)}{\partial x} = 2r^{-1}\left((\omega\lambda + r)x - \omega\lambda\alpha_1\right) = 0, \\ \frac{\partial F(x,y,z,w)}{\partial y} = 2r^{-1}\left((\varphi\lambda + r)y - \varphi\lambda\beta_1\right) = 0, \\ \frac{\partial F(x,y,z,w)}{\partial z} = 2r^{-1}\left((\phi\lambda + r)z - \phi\lambda\gamma_1\right) = 0, \\ \frac{\partial F(x,y,z,w)}{\partial w} = 2r^{-1}\left((\eta\lambda + r)w - \eta\lambda\theta_1\right) = 0. \end{cases} . \quad (16)$$

In the sequel, we can separate some cases to discuss the upper bounds of the system (1).

(i) If $\lambda \neq \frac{-r}{\omega}$, $\lambda \neq \frac{-r}{\varphi}$, $\lambda \neq \frac{-r}{\phi}$ and $\lambda \neq \frac{-r}{\eta}$, we get

$$(x_0, y_0, z_0, w_0) = \left(\frac{\omega\lambda\alpha_1}{r+\omega\lambda}, \frac{\varphi\lambda\beta_1}{r+\varphi\lambda}, \frac{\phi\lambda\gamma_1}{r+\phi\lambda}, \frac{\eta\lambda\theta_1}{r+\eta\lambda}\right) \quad (17)$$

and

$$\max_{(x,y,z,w)\in\Gamma} G = \frac{\omega^2\lambda^2\alpha_1^2}{(r+\omega\lambda)^2} + \frac{\varphi^2\lambda^2\beta_1^2}{(r+\varphi\lambda)^2} + \frac{\phi^2\lambda^2\gamma_1^2}{(r+\phi\lambda)^2} + \frac{\eta^2\lambda^2\theta_1^2}{(r+\eta\lambda)^2}. \quad (18)$$

In this case, there exists parametrized family (in λ) of bounds given by (18) of the system (1).

(ii) If $\lambda = \frac{-r}{\omega}$, $(\omega \neq \varphi,\ \omega \neq \phi,\ \omega \neq \eta)$, $\lambda \neq \frac{-r}{\varphi}$, $\lambda \neq \frac{-r}{\phi}$, $\lambda \neq \frac{-r}{\eta}$, we obtain

$$(x_0, y_0, z_0, w_0) = \left(\pm\sqrt{\frac{r}{\omega}\left(1 - \frac{\xi_1}{\xi_2}\right)} + \alpha_1, \frac{-\varphi\beta_1}{\omega - \varphi}, \frac{-\phi\gamma_1}{\omega - \phi}, \frac{-\eta\theta_1}{\omega - \eta} \right), \tag{19}$$

where

$$\begin{cases} \xi_1 = \omega^2 \left[\varphi\beta_1^2 (\omega - \phi)^2 (\omega - \eta)^2 + \phi\gamma_1^2 (\omega - \varphi)^2 (\omega - \eta)^2 + \eta\theta_1^2 (\omega - \varphi)^2 (\omega - \phi)^2 \right] \\ \xi_2 = r (\omega - \varphi)^2 (\omega - \phi)^2 (\omega - \eta)^2 \\ \xi_2 \geq \xi_1. \end{cases} \tag{20}$$

The last condition of (20) confirms that the value x_0 in (19) is well defined. In this case, we have

$$\max_{(x,y,z,w)\in\Gamma} G = \left(\sqrt{\frac{r}{\omega}\left(1 - \frac{\xi_1}{\xi_2}\right)} + \alpha_1 \right)^2 + \frac{\varphi^2\beta_1^2}{(\omega - \varphi)^2} + \frac{\phi^2\gamma_1^2}{(\omega - \phi)^2} + \frac{\eta^2\theta_1^2}{(\omega - \eta)^2}. \tag{21}$$

(iii) If $\lambda = \frac{-r}{\varphi}$, $(\varphi \neq \omega,\ \varphi \neq \phi,\ \varphi \neq \eta)$, $\lambda \neq \frac{-r}{\omega}$, $\lambda \neq \frac{-r}{\phi}$, $\lambda \neq \frac{-r}{\eta}$, we have

$$(x_0, y_0, z_0, w_0) = \left(\frac{-\alpha_1\omega}{\varphi - \omega}, \pm\sqrt{\frac{r}{\varphi}\left(1 - \frac{\xi_3}{\xi_4}\right)} + \beta_1, \frac{-\phi\gamma_1}{\varphi - \phi}, \frac{-\eta\theta_1}{\varphi - \eta} \right), \tag{22}$$

where

$$\begin{cases} \xi_3 = \varphi^2 \left[\omega\alpha_1^2 (\varphi - \phi)^2 (\varphi - \eta)^2 + \phi\gamma_1^2 (\varphi - \omega)^2 (\varphi - \eta)^2 + \eta\theta_1^2 (\varphi - \omega)^2 (\varphi - \phi)^2 \right], \\ \xi_4 = r (\varphi - \omega)^2 (\varphi - \phi)^2 (\varphi - \eta)^2, \\ \xi_4 \geq \xi_3. \end{cases} \tag{23}$$

By the last condition of (23), we can confirm that the value y_0 in (22) is well defined. In this case, we get

$$\max_{(x,y,z,w)\in\Gamma} G = \frac{\alpha_1^2\omega^2}{(\varphi - \omega)^2} + \left(\sqrt{\frac{r}{\varphi}\left(1 - \frac{\xi_3}{\xi_4}\right)} + \beta_1 \right)^2 + \frac{\phi^2\gamma_1^2}{(\varphi - \phi)^2} + \frac{\eta^2\theta_1^2}{(\varphi - \eta)^2}. \tag{24}$$

(iv) If $\lambda = \frac{-r}{\phi}$, $(\phi \neq \omega,\ \phi \neq \varphi,\ \phi \neq \eta)$, $\lambda \neq \frac{-r}{\omega}$, $\lambda \neq \frac{-r}{\varphi}$, $\lambda \neq \frac{-r}{\eta}$, we obtain

$$(x_0, y_0, z_0, w_0) = \left(\frac{-\alpha_1\omega}{\phi - \omega}, \frac{-\varphi\beta_1}{\phi - \varphi}, \pm\sqrt{\frac{r}{\phi}\left(1 - \frac{\xi_5}{\xi_6}\right)} + \gamma_1, \frac{-\eta\theta_1}{\phi - \eta} \right), \tag{25}$$

where

$$\begin{cases} \xi_5 = \phi^2 \left[\omega\alpha_1^2 (\phi - \varphi)^2 (\phi - \eta)^2 + \varphi\beta_1^2 (\phi - \omega)^2 (\phi - \eta)^2 + \eta\theta_1^2 (\phi - \omega)^2 (\phi - \varphi)^2 \right], \\ \xi_6 = r (\phi - \omega)^2 (\phi - \varphi)^2 (\phi - \eta)^2, \\ \xi_6 \geq \xi_5. \end{cases} \tag{26}$$

Also, the last condition of (26) confirms that the value z_0 in (25) is well defined. In this case, we have

$$\max_{(x,y,z,w)\in\Gamma} G = \frac{\alpha_1^2\omega^2}{(\phi-\omega)^2} + \frac{\varphi^2\beta_1^2}{(\phi-\varphi)^2} + \left(\sqrt{\frac{r}{\phi}\left(1-\frac{\xi_5}{\xi_6}\right)} + \gamma_1\right)^2 + \frac{\eta^2\theta_1^2}{(\phi-\eta)^2}. \tag{27}$$

(v) If $\lambda = \frac{-r}{\eta}$, $(\eta \neq \omega,\ \eta \neq \varphi,\ \eta \neq \phi)$, $\lambda \neq \frac{-r}{\omega}$, $\lambda \neq \frac{-r}{\varphi}$, $\lambda \neq \frac{-r}{\phi}$, we get

$$(x_0, y_0, z_0, w_0) = \left(\frac{-\alpha_1\omega}{\eta-\omega}, \frac{-\varphi\beta_1}{\eta-\varphi}, \frac{-\phi\gamma_1}{\eta-\phi}, \pm\sqrt{\frac{r}{\eta}\left(1-\frac{\xi_7}{\xi_8}\right)} + \theta_1\right), \tag{28}$$

where

$$\begin{cases} \xi_7 = \eta^2\left[\omega\alpha_1^2(\eta-\varphi)^2(\eta-\phi)^2 + \varphi\beta_1^2(\eta-\omega)^2(\eta-\phi)^2 + \phi\gamma_1^2(\eta-\omega)^2(\eta-\varphi)^2\right], \\ \xi_8 = r(\eta-\omega)^2(\eta-\varphi)^2(\eta-\phi)^2, \\ \xi_8 \geq \xi_7. \end{cases} \tag{29}$$

The last condition of (29) confirms that the value w_0 in (28) is well defined. In this case, we obtain

$$\max_{(x,y,z,w)\in\Gamma} G = \frac{\alpha_1^2\omega^2}{(\eta-\omega)^2} + \frac{\varphi^2\beta_1^2}{(\eta-\varphi)^2} + \frac{\phi^2\gamma_1^2}{(\eta-\phi)^2} + \left(\sqrt{\frac{r}{\eta}\left(1-\frac{\xi_7}{\xi_8}\right)} + \theta_1\right)^2. \tag{30}$$

Finally, the other possible cases can be treated using the same technique.

Theorem 2.1 *Assume that conditions (9), (10) and (12) hold, then the general 4-D continuous-time autonomous system (1) is bounded, i.e., it is contained in the 4-D ellipsoid (13).*

Also, similar results can be found using the cases discussed above.

3 Example

We consider the Lorenz-Stenflo system studied in [20] and given by

$$\begin{cases} x' = ay - ax + dw, \\ y' = cx - xz - y, \\ z' = xy - bz, \\ w' = -x - aw. \end{cases} \tag{31}$$

We choose the Lyapunov function $V(x,y,z,w) = \lambda x^2 + y^2 + (z - \lambda a - c)^2 + \lambda d w^2$ as in [20]. Suppose that λ and d are strictly positive and denote $\sqrt{\lambda}x = \widetilde{x}$, $\sqrt{\lambda d}w = \widetilde{w}$, thus we get $V(\widetilde{x}, y, z, \widetilde{w}) = \widetilde{x}^2 + y^2 + (z-\lambda a - c)^2 + \widetilde{w}^2$ i.e., $\alpha = \beta = \theta = 0$, $\gamma = \lambda a + c$ and the system (31) becomes

$$\begin{cases} \widetilde{x}' = -a\widetilde{x} + \sqrt{\lambda}ay + \sqrt{d}\widetilde{w}, \\ y' = \frac{c}{\sqrt{\lambda}}\widetilde{x} - y - \frac{1}{\sqrt{\lambda}}\widetilde{x}z, \\ z' = -bz + \frac{1}{\sqrt{\lambda}}\widetilde{x}y, \\ \widetilde{w}' = -\sqrt{d}\widetilde{x} - a\widetilde{w}, \end{cases} \tag{32}$$

i.e., $f(\widetilde{x},y,z,\widetilde{w}) = -a\widetilde{x} + \sqrt{\lambda}ay + \sqrt{d}\widetilde{w}$, $g(\widetilde{x},y,z,\widetilde{w}) = \frac{c}{\sqrt{\lambda}}\widetilde{x} - y - \frac{1}{\sqrt{\lambda}}\widetilde{x}z$, $h(\widetilde{x},y,z,\widetilde{w}) = -bz + \frac{1}{\sqrt{\lambda}}\widetilde{x}y$, $k(\widetilde{x},y,z,\widetilde{w}) = -\sqrt{d}\widetilde{x} - a\widetilde{w}$. Thus, we have $\omega = a > 0$, $\varphi = 1 > 0$, $\phi = b > 0$, $\eta = a > 0$, $\alpha_1 = \beta_1 = \theta_1 = 0$, $\gamma_1 = \dfrac{\lambda a + c}{2}$ and $r = b\left(\dfrac{\lambda a + c}{2}\right)^2$. Then, we get $\dfrac{1}{2}\dfrac{dV}{dt} = -a\widetilde{x}^2 - y^2 - b\left(z - \frac{\lambda a + c}{2}\right)^2 - a\widetilde{w}^2 + b\left(\frac{\lambda a+c}{2}\right)^2$, i.e., $\dfrac{1}{2}\dfrac{dV}{dt} = -a\lambda x^2 - y^2 - bz^2 - a\lambda d w^2 + (\lambda a + c)\, bz$ which is the same as in [20]. Also, it is easy to verify that all conditions of Theorem 2.1 hold for this case. The 4-D ellipsoid Γ is given by

$$\Gamma = \left\{ \begin{array}{c} (x,y,z,w) \in \mathbb{R}^4 : \dfrac{\widetilde{x}^2}{\frac{b}{a}\left(\frac{\lambda a+c}{2}\right)^2} + \dfrac{y^2}{b\left(\frac{\lambda a+c}{2}\right)^2} + \dfrac{\left(z - \frac{\lambda a+c}{2}\right)^2}{\left(\frac{\lambda a+c}{2}\right)^2} + \dfrac{\widetilde{w}^2}{\frac{b}{a}\left(\frac{\lambda a+c}{2}\right)^2} = 1, \\ a > 0,\ b > 0,\ c > 0,\ d > 0,\ \lambda > 0 \end{array} \right\} \tag{33}$$

i.e.,

$$\Gamma = \left\{ \begin{array}{c} (x,y,z,w) \in \mathbb{R}^4 : \lambda a x^2 + y^2 + b\left(z - \dfrac{\lambda a + c}{2}\right)^2 + \lambda a d w^2 = \dfrac{b(\lambda a + c)^2}{4}, \\ a > 0,\ b > 0,\ c > 0,\ d > 0,\ \lambda > 0 \end{array} \right\} \tag{34}$$

which is also the same as in [20]. Finally, we have the result shown in [20] that confirms that if $a > 0$, $b > 0$, $c > 0$, $d > 0$, $\lambda > 0$, then the Lorenz-Stenflo system is contained in the following set

$$\Omega_\lambda = \left\{ (x,y,z,w) \in \mathbb{R}^4 : \lambda x^2 + y^2 + (z - \lambda a - c)^2 + \lambda d w^2 \le R^2 \right\}, \tag{35}$$

where

$$R^2 = \begin{cases} \dfrac{(\lambda a + c)^2 b^2}{4(b-1)}, & \text{if } a \ge 1,\ b \ge 2, \\ (\lambda a + c)^2, & \text{if } a > \frac{b}{2},\ b < 2, \\ \dfrac{(\lambda a + c)^2 b^2}{4a(b-a)}, & \text{if } 0 < a < 1,\ b \ge 2. \end{cases} \tag{36}$$

4 Conclusion

In this paper, based on the multivariable functions analysis, a generalization of all the existing results in the relevant literature for the upper bound of the general 4-D continuous-time system is investigated. Especially, sufficient conditions for this system to be contained in a four-dimensional ellipsoidal surface are determined.

The strategy presented in this work is sufficiently general, so it would be possible to apply the present method to consider other systems with high order and more complicated nonlinearity, which will be the topic for further papers.

References

[1] Cafagna, D. and Grassi, G. New 3D-scroll attractors in hyperchaotic Chua's circuits forming a ring. *Int. J. of Bifur. & Chaos* **13** (10) (2003) 2889–2903.

[2] Celikovsky, S. and Chen, G. On the generalized Lorenz canonical form of chaotic systems. *Int. J. of Bifur. & Chaos* **12** (2002) 1789–1812.

[3] Cenys, A., Tamaservicius, A., Baziliauskas, A., Krivickas, R. and Lindberg, E. Hyperchaos in coupled Colpitts oscillators. *Chaos, Solitons & Fractals* **17** (2-3) (2003) 349–353.

[4] Chen, G. Controlling Chaos and Bifurcations in Engineering Systems. CRC Press, Boca Raton, FL, United Stated, 1999.

[5] Chen, G. and Ueta, T. Yet another chaotic attractor. *Int. J. of Bifur. & Chaos* **9** (1999) 1465–1466.

[6] Ehadj, Z. Analysis of a new three-dimensional quadratic chaotic system. *Radio Engineering* **17** (1) (2008) 9–13.

[7] Elhadj, Z. and Sprott, J. C. About the boundedness of 3D continuous time quadratic systems. *Nonlinear Oscillations* **13** (2-3) (2010) 515–521.

[8] Korkmaz, E. and C. Tunc, C. On the convergence of solutions of some nonlinear differential equations of fourth order. *Nonlinear Dynamics and Systems Theory* **14** (3) (2014) 313–322.

[9] Leonov, G., Bunin, A. and Koksch, N. Attractor localisation of the Lorenz system. *Zeitschrift fur Angewandte Mathematik und Mechanik* **67** (1987) 649–656.

[10] Li, D., Lu, J.A., Wu, X. and Chen, G. Estimating the bounds for the Lorenz family of chaotic systems. *Chaos, Solitons & Fractal*s **23** (2005) 529–534.

[11] Li, D., Wu, X. and Lu, J. Estimating the ultimating bound and positively invariant set for the hyperchaotic Lorenz-Haken system. *Chaos, Solitons & Fractals* **39** (2009) 1290–1296.

[12] Lorenz, E. N. Deterministic non-periodic flow. *J. Atmos. Sci.* **20** (1963) 130–141.

[13] Lu, J. H., Chen, G. and Zhou, T. A unified Lorenz-type system and its canonical form. *Int. J. of Bifur. & Chaos* **16** (10) (2006) 2855–2871.

[14] Ning, C. Z. and Haken, H. Detuned lasers and the complex Lorenz equation: subcritical and subcritical Hopf bifurcations. *Phys Rev A* **41** (7) (1999) 326.

[15] Ouanass, A. Chaos synchronization approach based on new criterion of stability. *Nonlinear Dynamics and Systems Theory* **14** (4) (2014) 395–401.

[16] Pogromsky, A. Y., Santoboni, G. and Nijmeijer, H. An ultimate bound on the trajectories of the Lorenz systems and its applications. *Non-linearity* **16** (2003) 1597–1605.

[17] Rôsslor, O. E. An equation for continuous chaos. *Phys Lett A* **57** (1976) 397–398.

[18] Rôsslor, O. E. An equation for the hyperchaos. *Phys Lett A* **71** (2-3) (1979) 155–157.

[19] Stenflo, L. Generalized Lorenz equations for acoustic-gravity waves in the atmosphere. *Phys Scr* **53** (1996) 83–84.

[20] Wang, P., Li, D. and Hu, Q. Bounds of the hyper-chotic Lorenz-Stenflo system. *Commun Nonlinear Sci Numer Simulat* **15** (2010) 2514–2520.

[21] Xavier, J. C. and Rech, P. C. Regular and chaotic dynamics of the Lorenz-Stenflo system. *Int. J. of Bifur. & Chaos* **20** (2010) 145–152.

[22] Yu, M. Y. Some chaotic aspects of the Lorenz-Stenflo equations. *Phys Scr* **82** (1999) 10–11.

[23] Zehrour, O. and Elhadj, Z. Boundedness of the generalized 4-D hyper-chaotic model containing Lorenz-Stenflo and Lorenz-Haken systems. *Nonlinear studies* **19** (4) (2012) 1–7.

[24] Zehrour, O. and Elhadj, Z. Ellipsoidal chaos. *Nonlinear studies* **19** (1) (2012) 71-77.

[25] Zhang, F., Li, Y. and Mu, C. Bounds of Solutions of a Kind of Hyper-Chaotic Systems and Application. *Journal of Mathematical Research with Applications* **33** (3) (2013) 345–352.

[26] Zhou, T. and Chen, G. Classification of chaos in 3-D autonomous quadratic systems-I. Basic framework and methods. *Int. J. of Bifur. & Chaos* **16** (9) (2006) 2459–2479.

[27] Zhou, T., Chen, G. and Yang, Q. Constructing a new chaotic system based on the Si'lnikov criterion. *Chaos, Solitons & Fractals* **19** (2004) 985–993.