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An Inversion of a Fractional Differential Equation and Fixed Points

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Abstract: This is a study of the scalar fractional differential equation of Riemann-Liouville type

$$D^q x(t) = f(t, x(t)), \quad \lim_{t\to 0^+} t^{1-q} x(t) = x^0,$$

where $q \in (0,1)$ and $x^0 \neq 0$. This is first written as a Volterra integral equation

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds.$$

After two existence results for a solution on a short interval $(0, T]$ are presented, it is then transformed in two steps into an integral equation

$$y(t) = F(t) + \int_0^t R(t-s) \left[y(s) + \frac{f(s+T, y(s))}{J} \right] ds,$$

where $y(t) = x(t+T)$. The function R is completely monotone on $(0, \infty)$ and $\int_0^\infty R(t)\, dt = 1$. When f is bounded and continuous for y bounded and continuous on $[0, \infty)$, then the integral maps sets of bounded continuous functions into sets of bounded equicontinuous functions. Moreover, F is uniformly continuous on $[0, \infty)$, $F(t) \to 0$, and $F \in L^1[0, \infty)$, while J is an arbitrary positive constant. A growth condition on f is used to show that all of these equations share solutions.

The point of the work is that an integral equation with two singularities and a kernel having infinite integral is transformed into an equation with a mildly singular kernel and finite integral. That final form is very suitable for a variety of fixed point theorems yielding qualitative properties of solutions of each of the stated equations.

Keywords: *fixed points; fractional differential equations; integral equations; Riemann-Liouville operators; singular kernels.*

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