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An Inversion of a Fractional Differential Equation and Fixed Points

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Abstract: This is a study of the scalar fractional differential equation of Riemann-Liouville type

$$D^q x(t) = f(t, x(t)), \quad \lim_{t\to 0^+} t^{1-q}x(t) = x^0,$$

where $q \in (0,1)$ and $x^0 \neq 0$. This is first written as a Volterra integral equation

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds.$$

After two existence results for a solution on a short interval $(0, T]$ are presented, it is then transformed in two steps into an integral equation

$$y(t) = F(t) + \int_0^t R(t-s) \left[y(s) + \frac{f(s+T, y(s))}{J} \right] ds,$$

where $y(t) = x(t+T)$. The function R is completely monotone on $(0,\infty)$ and $\int_0^\infty R(t)\, dt = 1$. When f is bounded and continuous for y bounded and continuous on $[0,\infty)$, then the integral maps sets of bounded continuous functions into sets of bounded equicontinuous functions. Moreover, F is uniformly continuous on $[0,\infty)$, $F(t) \to 0$, and $F \in L^1[0,\infty)$, while J is an arbitrary positive constant. A growth condition on f is used to show that all of these equations share solutions.

The point of the work is that an integral equation with two singularities and a kernel having infinite integral is transformed into an equation with a mildly singular kernel and finite integral. That final form is very suitable for a variety of fixed point theorems yielding qualitative properties of solutions of each of the stated equations.

Keywords: *fixed points; fractional differential equations; integral equations; Riemann-Liouville operators; singular kernels.*

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1 Introduction

A myriad of real-world problems can be modeled by the fractional differential equation of Riemann-Liouville type

$$D^q x(t) = f(t, x(t)), \quad \lim_{t\to 0^+} t^{1-q} x(t) = x^0 \quad (0 < q < 1). \tag{1.1}$$

Substantial treatments are found in Diethelm [10], Kilbas et al. [12], Lakshmikantham et al. [14], and Podlubny [19]. An annotated bibliography is found in Oldham and Spanier [17].

Under certain conditions it is known that this initial value problem and the Volterra equation

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds \tag{1.2}$$

share solutions. Equation (1.2) is far more familiar to most analysts than is (1.1), so there is good reason to pursue a study of (1.2) and its relation to (1.1). It can be argued that this last equation has essentially three singularities and a kernel which does not belong to $L^1(0,\infty)$. The singular forcing function immediately feeds back into the function f producing a singularity which can cause us to restrict the values of q for which a solution will exist. These properties offer a strong challenge. Our goal is to transform it into a far more tractable equation.

The conditions with (1.1), and subsequently transferred to (1.2), are of critical importance. Both the literature and the results which we will obtain here dictate very precise properties for solutions contained in this definition.

Definition 1.1 For a given $q \in (0,1)$, a function $\phi\colon (0,T] \to \Re$ is said to be a *solution* of (1.2) if ϕ is continuous, if ϕ satisfies (1.2) on $(0,T]$, and if

$$t^{1-q}\phi(t) \text{ is continuous on } [0,T] \text{ with } \lim_{t\to 0^+} t^{1-q}\phi(t) = x^0.$$

The first task is to obtain some general existence theorems for solutions on a short interval $(0,T]$ which will get us past the singularities and facilitate the transformation. The continuing work does not rely on these particular existence results, but asks only local existence.

Next, we improve the kernel by transforming the Volterra equation into an intermediate equation

$$x(t) = z(t) + \int_0^t R(t-s)\left[x(s) + \frac{f(s,x(s))}{J}\right] ds \tag{1.3}$$

in which J is an arbitrary positive constant, while R is a completely monotone kernel residing in $L^1(0,\infty)$, while

$$z(t) = x^0 t^{q-1} - \int_0^t R(t-s) x^0 s^{q-1} ds \tag{1.4}$$

still contains the singularity in the forcing function. Thus, we make one more transformation mapping that last equation into

$$y(t) = F(t) + \int_0^t R(t-s)\left[y(s) + \frac{f(s+T, y(s))}{J}\right] ds, \tag{1.5}$$

where $y(t) = x(t+T)$. Not only is the kernel nice but now our function F is uniformly continuous on $[0,\infty)$, $F(t) \to 0$ as $t \to \infty$, and $F \in L^1[0,\infty)$.

With this we have achieved our goal. We have transformed the fractional equation into a very standard Volterra equation with a mildly singular kernel. From this the investigator can now move out and apply classical techniques to obtain qualitative properties of solutions of the original fractional differential equation. There is a more complete summary and guide for further work located in the first part of Section 4.

While our goal is the transformation, in the process there emerges a property which seems entirely new. In order to obtain our existence theorem for a solution on $(0,T]$, we ask a growth condition

$$|f(t,x)| \le |f(t,0)| + Kt^{r_1}|x|^{r_2}, \quad \int_0^T |f(t,0)|\,dt < \infty \tag{1.6}$$

for $0 < t \le T$ with $r_1 > -1$ and other technical conditions including

$$r_1 + r_2(q-1) + 1 > 0. \tag{1.7}$$

The local existence follows from these, a contraction mapping, and a nonlinear Lipschitz condition. A similar growth condition is also used (in work to be offered elsewhere because of its length), together with Schauder's theorem, to obtain existence without a Lipschitz condition.

1.1 Two central issues

There are two properties which will play central roles in this paper and we want to alert the reader to them early. The first issue is that existence theory must place restrictions on the values of q in $(0,1)$. As r_1 and r_2 are constants inherently part of $f(t,x)$, (1.7) restricts the values of q to an interval $q_0 < q < 1$ for some $q_0 \ge 0$, a restriction not seen in the aforementioned references. However, Example 2.3 shows that general existence theorems must contain such restrictions. A study of the references reveals that such restrictions were missed since the investigators ask for either a Lipschitz condition, a severe bound on f, or both. See Section 2.5 of [14], [19, p. 127], or [10, p. 77] for example. This brings in the property which to a large extent ties this paper together. Every existence result which we have encountered either in the literature cited just now or in our own work presented here and in preparation has a condition subsumed by

$$|f(t,x)| \le u(t) + K_2 t^{r_1}|x|^{r_2} \tag{1.6*}$$

with mild conditions on $u(t)$ and technical relations between q, r_1, r_2. It is common to find q restricted to an interval smaller than $(0,1)$ [13, p. 1 and Lemma 1] for reasons other than existence theory.

The second issue is encountered almost immediately and continues to be foremost in the considerations. A main sufficient condition to transfer from (1.1) to (1.2) and again to transfer from (1.2) to (1.3) is that a solution on a short interval $(0,T]$ must satisfy

$$\int_0^T [|x(s)| + |f(s,x(s))|]\,ds < \infty. \tag{1.8}$$

Now, the two issues are brought together using Lemma 2.1 and Theorem 2.6. It is shown that if there is a solution and if f does satisfy (1.6) then it will also satisfy (1.8).

Thus, every existence theorem we encounter asks (1.6) and, hence, has as a corollary (1.8). And (1.8) is a main sufficient condition to pass from (1.1) to (1.2) and is also a main sufficient condition to pass from (1.2) to (1.3). The passage from (1.3) to (1.5) is just a translation. Hence, our entire stated problem of passing from (1.1) to (1.5) rests in an essential way on (1.6), and consequently on (1.8), whether we use one of our own existence results or one of the cited works.

2 Existence and Uniqueness

We are concerned with the fractional differential equation and initial condition

$$D^q x(t) = f(t, x(t)), \quad \lim_{t \to 0^+} t^{1-q} x(t) = x^0 \quad (0 < q < 1), \tag{2.1}$$

where $x^0 \in \Re$, $x^0 \neq 0$, $f\colon (0,T] \times \Re \to \Re$ is continuous for some $T > 0$. The symbol D^q denotes the *Riemann-Liouville fractional differential operator of order* q, which for $0 < q < 1$ is defined by

$$D^q x(t) := \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} x(s)\, ds,$$

where $\Gamma\colon (0,\infty) \to \Re$ is Euler's Gamma function:

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt.$$

Our study will focus on the integral equation

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds, \tag{2.2}$$

where $q \in (0,1)$ and $x^0 \in \Re$. However, we exclude $x^0 = 0$ from consideration since this particular value would remove the singularity at $t = 0$, thereby changing (2.2) to a different type of equation.

Notice that this equation contains essentially three singularities. The singular forcing function and kernel are clear. But there is instantaneous feedback of the forcing function into the function f resulting in a complicated singularity in the integrand. This will become more clear as we study existence problems and examine growth properties of f.

The following result given in [4] establishes mild conditions under which (2.1) and (2.2) are equivalent in the sense that they share solutions.

Theorem 2.1 *Let $q \in (0,1)$ and $x^0 \neq 0$. Let $f(t,x)$ be a function that is continuous on the set*

$$\mathcal{B} := \{(t,x) \in \Re^2 : 0 < t \leq T, x \in I\},$$

where $I \subset \Re$ denotes an unbounded interval. Suppose a function $x\colon (0,T] \to I$ is continuous and that both $x(t)$ and $f(t,x(t))$ are absolutely integrable on $(0,T]$. Then $x(t)$ satisfies the initial value problem (2.1) on the interval $(0,T]$ if and only if it satisfies the Volterra integral equation (2.2) on this same interval.

It can give the reader pause to be confronted with the need to show $x(t)$ and $f(t,x(t))$ absolutely integrable. But according to the discussion in the subsection of the introduction, a sufficient condition is that f satisfy (1.6) and that requires no knowledge of the solution.

In order to get past the singularity in the forcing function, t^{q-1}, we will first present two existence results for a short interval $(0,T]$. In the first existence result (cf. Theorem 2.5), we assume there is a positive constant K_2 so that $f\colon [0,T]\times\Re \to \Re$ is continuous and satisfies the Lipschitz condition

$$|f(t,x) - f(t,y)| \le K_2|x-y| \tag{2.3}$$

for $0 \le t \le T$ and all $x, y \in \Re$. Then, because of the continuity of f, there is also a positive constant K_1 such that

$$|f(t,x)| \le |f(t,0)| + K_2|x| \le K_1 + K_2|x| \tag{2.4}$$

for $0 \le t \le T$ and all $x \in \Re$. In the second existence result (cf. Theorem 2.7), f is allowed to have a singularity at $t = 0$ and the Lipschitz condition (2.3) is replaced with a more general condition.

All of our work on existence will be done in a certain weighted space $(X, |\cdot|_g)$, which we define next. The term *g-norm* is what we call $|\cdot|_g$.

Definition 2.1 For a fixed $T > 0$ and for $g(t) := t^{q-1}$, let $(X, |\cdot|_g)$, or simply X, denote the space of continuous functions $\phi\colon (0,T] \to \Re$ for which

$$|\phi|_g := \sup_{0<t\le T} \frac{|\phi(t)|}{g(t)}$$

is finite.

Theorem 2.2 *The space* $(X, |\cdot|_g)$ *is a Banach space.*

Proof. It is a straightforward exercise to show that X is a subspace of the vector space of all continuous functions on $(0,T]$ and to verify that $|\cdot|_g$ is a norm. Thus, $(X, |\cdot|_g)$ is a normed vector space. To show that it is also complete, let $\{x_n\} \subset X$ be a Cauchy sequence. This translates into $\{t^{1-q}x_n(t)\}$ being a uniformly Cauchy sequence of continuous functions on $(0,T]$. By the Cauchy criterion, it converges uniformly on $(0,T]$ to a limit function φ, which is also continuous on $(0,T]$. Finally, $\varphi \in (X, |\cdot|_g)$. In order to see this, choose N large enough so that

$$\left|\frac{\varphi(t)}{t^{q-1}} - \frac{x_N(t)}{t^{q-1}}\right| < 1$$

for all $t \in (0,T]$. Then we have

$$\frac{|\varphi(t)|}{t^{q-1}} \le \left|\frac{\varphi(t)}{t^{q-1}} - \frac{x_N(t)}{t^{q-1}}\right| + \left|\frac{x_N(t)}{t^{q-1}}\right| < 1 + \frac{|x_N(t)|}{t^{q-1}}.$$

Hence

$$\sup_{0<t\le T} \frac{|\varphi(t)|}{t^{q-1}} \le 1 + \sup_{0<t\le T} \frac{|x_N(t)|}{t^{q-1}} < \infty.$$

We now define a mapping P by $\phi \in X$ implies that

$$(P\phi)(t) := x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, \phi(s))\, ds \tag{2.5}$$

and show that $P\colon X \to X$. The next theorem involves the integral

$$\mathcal{H}(t) := \int_0^t (t-s)^{n-1} \phi(s)\, ds. \tag{2.6}$$

It follows from classical theorems for Lebesgue integrals depending on a parameter (e.g. [2, Thm. 10.38]) that if the function ϕ is continuous on a closed interval $[0, T]$, then so is $\mathcal{H}$. Part of the proof of this result depends on ϕ being bounded on $[0, T]$. However, even if $\phi(s)$ has a singularity at $s = 0$, we still have the following lemma.

Lemma 2.1 *Let $n \in \Re^+$. If a function ϕ is continuous and absolutely integrable on $(0, T]$, then the integral $\mathcal{H}$ given by (2.6) defines a function that is also continuous and absolutely integrable on $(0, T]$.*

A proof of this lemma can be found in [4, Lemma 4.6]. It will be used twice in the proof of the following theorem. The transformation of Section 3 will rest heavily on it.

Theorem 2.3 *Let P be the mapping defined by (2.5).*

(i) *If $\phi \in X$, then $\int_0^t (t-s)^{q-1} \phi(s)\, ds \in X$.*

(ii) *If for each $\phi \in X$ a function $\psi_\phi \in X$ exists with*

$$|f(t, \phi(t))| \le \psi_\phi(t) \tag{2.7}$$

for all $0 < t \le T$, then $\int_0^t (t-s)^{q-1} f(s, \phi(s))\, ds \in X$.

(iii) *If (ii) holds and if $\phi \in X$, then $P\phi \in X$.*

Proof. According to the definition of the weighted space, we must show that the integral function in (i) is continuous and that

$$\sup_{0<t\le T} \frac{1}{g(t)} \left| \int_0^t (t-s)^{q-1} \phi(s)\, ds \right| < \infty,$$

where $g(t) = t^{q-1}$.

As for continuity, first notice that as $\phi \in X$ then

$$|\phi(t)| \le |\phi|_g\, t^{q-1}$$

for $0 < t \le T$. Hence, ϕ is absolutely integrable on $(0, T]$ since

$$\int_0^T |\phi(t)|\, dt \le |\phi|_g \int_0^T t^{q-1}\, dt = |\phi|_g \frac{T^q}{q} < \infty.$$

It then follows that $\int_0^t (t-s)^{q-1} \phi(s)\, ds$ is continuous on $(0, T]$ by Lemma 2.1. As for the second part of the proof of (i), we have

$$\begin{aligned}\frac{1}{g(t)}\left|\int_0^t (t-s)^{q-1}\phi(s)\,ds\right| &\leq \frac{1}{t^{q-1}}\int_0^t (t-s)^{q-1}|\phi(s)|\,ds \\ &\leq \frac{1}{t^{q-1}}\int_0^t (t-s)^{q-1}|\phi|_g\, s^{q-1}\,ds = t^{1-q}|\phi|_g \int_0^t (t-s)^{q-1}s^{q-1}\,ds\end{aligned}$$

for $0 < t \leq T$. With the change of variable $s = tv$, the integral becomes

$$\int_0^t (t-s)^{q-1}s^{q-1}\,ds = t^{2q-1}\int_0^1 v^{q-1}(1-v)^{q-1}\,dv.$$

Now it can be expressed in terms of the *Beta function*, namely, the function $B(p,q)$ that is defined by

$$B(p,q) := \int_0^1 v^{p-1}(1-v)^{q-1}\,dv \tag{2.8}$$

and which converges if and only if both p and q are positive. Hence,

$$\int_0^t (t-s)^{q-1}s^{q-1}\,ds = t^{2q-1}B(q,q) < \infty$$

since $B(q,q)$ converges as $q > 0$. Since the Beta function is related to the Gamma function (cf. [11, p. 200] or [18, p. 521]) by the equation

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

we obtain

$$\int_0^t (t-s)^{q-1}s^{q-1}\,ds = t^{2q-1}\frac{\Gamma^2(q)}{\Gamma(2q)}.$$

As a result, we have

$$\frac{1}{g(t)}\left|\int_0^t (t-s)^{q-1}\phi(s)\,ds\right| \leq t^{1-q}\,|\phi|_g\, t^{2q-1}\frac{\Gamma^2(q)}{\Gamma(2q)} \leq \frac{T^q\Gamma^2(q)}{\Gamma(2q)}\,|\phi|_g < \infty$$

for all $t \in (0,T]$. This concludes the proof of (i).

Let $\phi \in X$. Then, as a function $\psi_\phi \in X$ exists satisfying (2.7), we have

$$\int_0^T |f(t,\phi(t)|\,dt \leq \int_0^T \psi_\phi(t)\,dt < \infty.$$

This allows us to invoke Lemma 2.1 again to conclude that the integral function in (ii) is continuous on $(0,T]$. Also, as $\psi_\phi \in X$, it follows from (i) that

$$\begin{aligned}&\frac{1}{g(t)}\left|\int_0^t (t-s)^{q-1}f(s,\phi(s))\,ds\right| \\ &\quad\leq \frac{1}{g(t)}\int_0^t (t-s)^{q-1}|f(s,\phi(s))|\,ds \leq \frac{1}{g(t)}\int_0^t (t-s)^{q-1}\psi_\phi(s)\,ds \\ &\quad\leq \sup_{0<t\leq T}\frac{1}{g(t)}\left|\int_0^t (t-s)^{q-1}\psi_\phi(s)\,ds\right| < \infty\end{aligned}$$

for all $t \in (0,T]$, which completes the proof of (ii).

Finally, it follows from (ii) and $x^0t^{q-1} \in X$ that all terms of P belong to X. Since X is a vector space, $P\phi \in X$.

Theorem 2.4 *Let* $f\colon (0,T] \times \Re \to \Re$ *be continuous. Suppose that a function* $x\colon (0,T_0] \to \Re$ *is a solution of*

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x(s))\, ds \tag{2.2}$$

on $(0,T_0]$ *where* $T_0 \le T$*. Then, for each* $\epsilon \in (0,|x^0|)$*, there is a* $T^* \le T_0$ *so that*

$$(|x^0| - \epsilon)t^{q-1} < |x(t)| < (|x^0| + \epsilon)t^{q-1} < 2|x^0|t^{q-1} \tag{2.9}$$

for $0 < t \le T^*$.

Proof. We have

$$t^{1-q}x(t) = x^0 + t^{1-q}\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x(s))\, ds$$

and $t^{1-q}x(t)$ is continuous on the closed interval $[0,T_0]$ (cf. Def. 1.1). It follows that

$$t^{1-q}\int_0^t (t-s)^{q-1} f(s,x(s))\, ds$$

is continuous on $[0,T_0]$. Now

$$0 = \lim_{t\to 0^+} |t^{1-q}x(t) - x^0| = \frac{1}{\Gamma(q)} \lim_{t\to 0^+} \left| t^{1-q}\int_0^t (t-s)^{q-1} f(s,x(s))\, ds \right|.$$

For a given $\epsilon \in (0,|x^0|)$, there is a $T^* \in (0,T_0]$ such that $0 \le t \le T^*$ implies that

$$\frac{1}{\Gamma(q)}\left| t^{1-q}\int_0^t (t-s)^{q-1} f(s,x(s))\, ds \right| < \epsilon.$$

So, for $0 < t \le T^*$, we have

$$\begin{aligned} &\big| |x(t)| - |x^0|t^{q-1} \big| \\ &\quad \le \big| x(t) - x^0 t^{q-1} \big| = \frac{1}{\Gamma(q)}\left| \int_0^t (t-s)^{q-1} f(s,x(s))\, ds \right| < \epsilon t^{q-1}. \end{aligned}$$

Using the first and last terms, we obtain

$$-\epsilon t^{q-1} < |x(t)| - |x^0|t^{q-1} < \epsilon t^{q-1}$$

so that

$$\big(|x^0| - \epsilon\big)t^{q-1} < |x(t)| < \big(|x^0| + \epsilon\big)t^{q-1} < 2|x^0|t^{q-1},$$

as required.

Corollary 2.1 *For the* T^* *of Theorem 2.4, the solution* $x(t)$ *has the sign of* x^0*. Moreover,* $x(t)$ *is absolutely integrable on* $(0,T^*]$.

Proof. For the given $\epsilon \in (0, |x^0|)$ and T^* in the proof of Theorem 2.4, we see that

$$|t^{1-q}x(t) - x^0| < \epsilon$$

or

$$(x^0 - \epsilon)t^{q-1} < x(t) < (x^0 + \epsilon)t^{q-1}$$

for $0 < t \le T^*$. And so if $x^0 > 0$, then $\epsilon < x^0$ and

$$x(t) > \frac{x^0 - \epsilon}{t^{1-q}} > 0$$

for $0 < t \le T^*$. If $x^0 < 0$, then $\epsilon < -x^0$ and

$$x(t) < \frac{x^0 + \epsilon}{t^{1-q}} < 0$$

for $0 < t \le T^*$.

Finally, as $|x(t)| < 2|x^0|t^{q-1}$ for $0 < t \le T^*$,

$$\int_0^{T^*} |x(s)|\, ds \le 2|x^0| \int_0^{T^*} s^{q-1}\, ds = \frac{2|x^0|}{q}(T^*)^q < \infty.$$

Corollary 2.2 *Let $f\colon [0,T] \times \Re \to \Re$ be continuous and satisfy condition (2.4). If $x(t)$ is a solution of (2.2) on the interval $(0, T^*]$ as in Theorem 2.4, then both $x(t)$ and $f(t, x(t))$ are absolutely integrable on $(0, T^*]$.*

Proof. It follows from (2.4) that

$$|f(t, x(t))| \le K_1 + K_2|x(t)|$$

for $0 \le t \le T^*$. We have already shown in Corollary 2.1 that $x(t)$ is absolutely integrable on $(0, T^*]$. Thus,

$$\int_0^{T^*} |f(t, x(t))|\, dt \le K_1T^* + K_2 \int_0^{T^*} |x(t)|\, dt < \infty.$$

Applications.

(a_1) Theorem 2.4 tells us precisely where to look for a function $x(t)$ satisfying the integral equation (2.2). For a sufficiently small $T^* \in (0, T]$, it will lie in the set

$$M := \{\phi \in C(0, T^*] \mid |\phi(t)| \le 2|x^0|t^{q-1}\},$$

where $C(0, T^*]$ denotes the set of all continuous functions on $(0, T^*]$; and it will be sandwiched between two constant multiples of x^0t^{q-1}, as in (2.9).

(a_2) In this paper we mainly consider the growth of $f(t, x)$, not its sign. However in situations where the sign becomes important, then it will be critical to replace M with the following set M^+. Suppose $x^0 > 0$. Then we see from Corollary 2.1 with $\epsilon = x^0/2$ and $T^* \in (0, T]$ sufficiently small that the solution $x(t)$ will reside in the set

$$M^+ := \{\phi \in C(0, T^*] \mid \tfrac{1}{2}x^0 < t^{1-q}\phi(t) < \tfrac{3}{2}x^0\}.$$

There is a parallel statement for $x^0 < 0$.

(b) A suitable space of functions for a fixed point mapping would be the Banach space $(X, |\cdot|_g)$ described in Definition 2.1.

(c) To find a solution of (2.2) we would contrive to define a mapping $P\colon M \to X$ by $\phi \in M$ implies that

$$(P\phi)(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, \phi(s))\, ds$$

and a fixed point of P in M would satisfy (2.2). We would then examine $f(t,x)$ in the light of the sandwich inequality (2.9) to determine the range of values of q for which the remainder of the definition of solution would hold. The sandwich inequality tells us that if there is a solution it will lie very near $x^0 t^{q-1}$. For reasonable functions f, such as polynomials, we will be able to use that sandwich inequality information to tell precisely which values of q will generate a solution.

(d) The absolute integrability of the solution will be used in Theorem 2.1 to show that a solution of (2.2) is a solution of (2.1).

We now prepare to obtain a solution. Let X be the Banach space of continuous functions $\phi\colon (0,T] \to \Re$ satisfying Definition 2.1. Note that because of (2.4) the conditions of part (ii) in Theorem 2.3 are satisfied. As a result, $\phi \in X$ implies

$$\int_0^t (t-s)^{q-1} f(s, \phi(s))\, ds \in X. \tag{2.10}$$

For the given $x^0 \neq 0$ and some $T_0 \in (0,T]$ to be determined, define the set M as before by

$$M := \{\phi \in X\colon |\phi|_g \le 2|x^0|\}. \tag{2.11}$$

Then for each $\phi \in M$,

$$|\phi(t)| \le 2|x^0| t^{q-1}$$

for $0 < t \le T_0$. For the set X, define the natural mapping P by $\phi \in X$ implies that

$$(P\phi)(t) := x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, \phi(s))\, ds \tag{2.12}$$

for $0 < t \le T$.

The following theorem can be proved by showing that P is a *contraction* on the set M: that is to say, $P\colon M \to M$ and a constant $\alpha \in (0,1)$ exists such that

$$\rho(Px, Py) \le \alpha \rho(x,y) \tag{2.13}$$

for all $x, y \in M$, where $\rho(x,y) := |x-y|_g$ is the metric provided by the norm $|\cdot|_g$. Then *Banach's contraction mapping principle* asserts that P has a unique fixed point in M, i.e., a unique $\phi \in M$ such that $P\phi = \phi$. Since this theorem will turn out to be a special case of Theorem 2.7, the proof is omitted.

Theorem 2.5 *Let $f\colon [0,T]\times\Re \to \Re$ be continuous and satisfy the Lipschitz condition (2.3). Then, for each $q \in (0,1)$, there is a $T_0 \in (0,T]$ such that (2.2) has a unique continuous solution ϕ on $(0,T_0]$ with*

$$\lim_{t\to 0^+} t^{1-q} \int_0^t (t-s)^{q-1} f(s,\phi(s))\, ds = 0, \qquad \lim_{t\to 0^+} t^{1-q}\phi(t) = x^0. \tag{2.14}$$

Finally, both $\phi(t)$ and $f(t,\phi(t))$ are absolutely integrable.

Earlier we pointed out that the Volterra equation (2.2) has a singularity at $t = 0$ due to the forcing function, a singularity at the upper limit of integration t due to the kernel, and whatever singularity might arise from f. In the following example, f has an obvious singularity at $t = \pi^2/4$.

Example 2.1 The Volterra equation

$$x(t) = \frac{1}{\sqrt{t}} - \frac{1}{2}\int_0^t \frac{1}{\sqrt{t-s}} \cdot \frac{\mathfrak{J}_1(\sqrt{s})}{\cos(\sqrt{s}\,)}\, x(s)\, ds,$$

where $\mathfrak{J}_1(t)$ denotes the Bessel function of the first kind of order 1, has a unique continuous solution $\phi(t)$ on an interval $(0, T_0]$ for some value of $T_0 \in (0, \pi^2/4)$. It satisfies (2.14), where

$$f(t,x) = -\frac{\sqrt{\pi}\,\mathfrak{J}_1(\sqrt{t})}{2\cos(\sqrt{t}\,)}\, x,$$

and both $\phi(t)$ and $f(t, \phi(t))$ are absolutely integrable on $(0, T_0]$. Furthermore, $\phi(t)$ is also the unique continuous solution of the initial value problem

$$D^{1/2}x(t) = -\frac{\sqrt{\pi}\,\mathfrak{J}_1(\sqrt{t})}{2\cos(\sqrt{t}\,)}\, x(t), \qquad \lim_{t\to 0^+} \sqrt{t}\, x(t) = 1$$

on the interval $(0, T_0]$.

Proof. Comparing the Volterra equation with (2.2), we see that $x^0 = 1$, $q = 1/2$, and the function f is as given above. Since $\mathfrak{J}_1(z)$ is an entire function of z in the complex plane, $\mathfrak{J}_1(\sqrt{t})$ is continuous for all $t \geq 0$. Thus, for any fixed $T \in (0, \pi^2/4)$, the part of f depending only on t is continuous on the closed interval $[0, T]$. This implies there are positive constants K_1, K_2 such that (2.3) and (2.4) hold for $0 \leq t \leq T$ and all $x, y \in \Re$. As a result, all of the conclusions stated in the example, except for the very last one, follow from Theorem 2.5. The last one follows from Theorem 2.1.

Remark 2.1 In fact, the function

$$\phi(t) := \frac{\cos(\sqrt{t})}{\sqrt{t}}$$

is the unique continuous solution of the Volterra equation on all of $(0, \pi^2/4)$. To verify this, use the change of variable $\sqrt{s} = \sqrt{t}\sin\theta$. Then

$$\begin{aligned}\int_0^t \frac{1}{\sqrt{t-s}} \cdot \frac{\mathfrak{J}_1(\sqrt{s})}{\cos(\sqrt{s}\,)}\, \phi(s)\, ds &= \int_0^t \frac{1}{\sqrt{t-s}} \cdot \frac{\mathfrak{J}_1(\sqrt{s})}{\sqrt{s}}\, ds \\ &= 2\int_0^{\pi/2} \mathfrak{J}_1(\sqrt{t}\sin\theta)\, d\theta.\end{aligned}$$

From an integration formula in [20, p. 374], we see that

$$\sqrt{\frac{2z}{\pi}}\int_0^{\pi/2} \mathfrak{J}_1(z\sin\theta)\, d\theta = \mathbb{H}_{1/2}(z),$$

where $\mathbb{H}_{1/2}$ denotes Struve's function of order $\frac{1}{2}$. From [1, (12.1.16)], we have

$$\mathbb{H}_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\,(1-\cos z).$$

Thus,

$$\int_0^{\pi/2} \mathfrak{J}_1(z\sin\theta)\,d\theta = \frac{1-\cos z}{z}.$$

Therefore, letting $x(t)=\phi(t)$, we obtain

$$\begin{aligned}\frac{1}{2}\int_0^t \frac{1}{\sqrt{t-s}}\cdot\frac{\mathfrak{J}_1(\sqrt{s})}{\cos(\sqrt{s}\,)}\,\phi(s)\,ds &= \int_0^{\pi/2}\mathfrak{J}_1(\sqrt{t}\sin\theta)\,d\theta \\ &= \frac{1-\cos\sqrt{t}}{\sqrt{t}} = \frac{1}{\sqrt{t}} - \phi(t)\end{aligned}$$

for $0<t<\pi^2/4$.

In [4] we also verify directly that the function $\phi(t)$ is a solution of the fractional differential equation and its accompanying initial condition in Example 2.1.

The proof of Theorem 2.5 rests on the Lipschitz condition (2.3). Let us generalize this theorem by replacing the Lipschitz condition with a more general condition (cf. item (iii)) below. In addition, consider the modifications listed below in items (i)–(ii).

(i) For some $T>0$ let $f\colon (0,T]\times\Re\to\Re$ be continuous.

(ii) Let $r_1>-1$. Let $r_2=m/n$, where m,n are positive integers with no common factors and n is odd, and $r_2\geq 1$. (Note then that $x^{r_2}\in\Re$ for all $x\in\Re$.) Furthermore, let r_1, r_2 satisfy the inequality

$$\mu := 1+r_1+(q-1)r_2 > 0. \tag{2.15}$$

(iii) Let the function f satisfy the additional condition that a constant $K>0$ exists such that

$$|f(t,x)-f(t,y)| \leq Kt^{r_1}|x^{r_2}-y^{r_2}| \tag{2.16}$$

for $t\in(0,T]$ and for all $x,y\in\Re$.

Example 2.2 The function $f(t,x)=\sqrt[3]{tx^4}$ satisfies conditions (i), (ii), and (iii) with $K=1$, $r_1=1/3$, $r_2=4/3$ for any fixed $q\in(0,1)$. The function $f(t,x)=t^{-1/2}x^{4/3}$ satisfies conditions (i) and (iii) with $K=1$, $r_1=-1/2$, $r_2=4/3$. As for (ii), $\mu>0$ if $q\in(5/8,1)$.

Before we generalize Theorem 2.5, we present a theorem that will aid in its proof and will be crucial for other results in Sections 3 and 4. The function G in part b) of the theorem is defined by

$$G(t,x) := -\left[x+\frac{f(t,x)}{J}\right],$$

where J is a positive constant.

Theorem 2.6 *Suppose that* $f\colon (0,T]\times\Re\to\Re$ *is continuous where*

$$|f(t,x)| \le |f(t,0)| + Kt^{r_1}|x|^{r_2}$$

with r_1, r_2 *satisfying item (ii) containing (2.15), that* $f(t,0)$ *is absolutely integrable on* $(0,T]$*, and that* $x(t)$ *is a continuous solution of (2.2) on an interval* $(0,T_0]\subset(0,T]$ *satisfying* $|x(t)|\le 2|x^0|t^{q-1}$*. Then:*

a) *There is a constant* $\kappa>0$ *with*

$$\int_0^{T_0}[|x(s)|+|f(s,x(s))|]\,ds=\kappa.$$

b) *For each* $t\in(0,T_0]$*, there is a nonnegative* $\mathfrak{D}(t)\in\Re$ *with*

$$\int_0^t\int_0^s (t-s)^{q-1}(s-u)^{q-1}|G(u,x(u))|\,du\,ds=\mathfrak{D}(t).$$

Proof. We have

$$\begin{aligned}
&\int_0^{T_0}[|x(s)|+|f(s,x(s))|]\,ds\\
&\qquad\le\int_0^{T_0}[2|x^0|s^{q-1}+|f(s,0)|+Ks^{r_1}(2|x^0|s^{q-1})^{r_2}]\,ds\\
&\qquad=2|x^0|\frac{T_0{}^q}{q}+\int_0^{T_0}|f(s,0)|\,ds+K(2|x^0|)^{r_2}\int_0^{T_0}s^{r_1}s^{r_2(q-1)}\,ds
\end{aligned}$$

which is finite because $r_1+r_2(q-1)+1>0$, completing the proof of part a).

The continuity of f and x implies that

$$\phi(s):=|G(s,x(s))|$$

is continuous on $(0,T_0]$ while part a) implies that it is absolutely integrable on this interval. Now, apply Lemma 2.1 to see that

$$\psi(s):=\int_0^s(s-u)^{q-1}\phi(u)\,du$$

is continuous and absolutely integrable. Hence

$$\int_0^t(t-s)^{q-1}\psi(s)\,ds$$

is continuous and absolutely integrable on $(0,T_0]$.

Note. If $f(t,x)=-Jx$ for a given $J>0$, then $G\equiv0$ and $\mathfrak{D}(t)=0$ for $t\in(0,T_0]$. Then (2.2) simplifies to

$$x(t)=x^0t^{q-1}-\frac{J}{\Gamma(q)}\int_0^t(t-s)^{q-1}x(s)\,ds.$$

It is well-established that this linear equation has a unique continuous solution on the entire interval $(0,\infty)$, which can be expressed in terms of the resolvent function. For more details, see (3.3) and (3.4) in Section 3. For nonlinear equations, the focus of this paper, the function G is not identically zero and so $\mathfrak{D}(t)$ is positive.

Theorem 2.7 *Suppose conditions* (i)–(iii) *listed before Example 2.2 hold and that*

$$\lim_{t\to 0^+} t^{1-q}\int_0^t (t-s)^{q-1}|f(s,0)|\,ds = 0. \tag{2.17}$$

Then, for each $q \in (0,1)$ satisfying (2.15), a $T_0 \in (0,T]$ exists such that the integral equation (2.2) has a unique continuous solution ϕ on $(0,T_0]$. Furthermore, both $\phi(t)$ and $f(t,\phi(t))$ are absolutely integrable on $(0,T_0]$. Also, ϕ satisfies (2.14) and is the unique continuous solution of the initial value problem (2.1) on $(0,T_0]$.

Proof. We first show that (2.17) implies that $f(t,0)$ is absolutely integrable on $(0,T]$. Let $\epsilon = 1$. Then there exists a $\delta \in (0,T]$ such that

$$t^{1-q}\int_0^t (t-s)^{q-1}|f(s,0)|\,ds < 1$$

for $t \in (0,\delta)$. And so for $t \in (0,\delta)$, we have

$$\begin{aligned} 0 \le \int_0^t |f(s,0)|ds &= \int_0^t (t-s)^{1-q}(t-s)^{q-1}|f(s,0)|\,ds \\ &\le \int_0^t t^{1-q}(t-s)^{q-1}|f(s,0)|\,ds \le 1. \end{aligned}$$

It follows that $f(s,0)$ is absolutely integrable on $(0,\delta)$. Thus $f(s,0)$ is absolutely integrable on $(0,T]$ because of the continuity of f.

Next consider the set M and the mapping P defined by (2.11) and (2.12), respectively. If $T_0 \in (0,T]$ is sufficiently small, we will show that the "generalized Lipschitz condition" (2.16) implies $P\colon M \to M$. First observe from (2.16) that

$$|f(t,x)| \le |f(t,0)| + Kt^{r_1}|x|^{r_2}$$

for $0 < t \le T$. It follows from this, the integrability of $f(t,0)$, and the proof of Theorem 2.6 a) that for every $\phi \in M$, $f(t,\phi(t))$ is absolutely integrable on $(0,T_0]$. Of course, ϕ being in M is continuous and absolutely integrable on this interval. The absolute integrability and continuity of $f(t,\phi(t))$ imply the integral term of $P\phi$ is continuous on $(0,T_0]$ by Lemma 2.1. Thus $P\phi$ itself is continuous on $(0,T_0]$.

Now we show $P\colon M \to M$. For any $\phi \in M$,

$$\begin{aligned} |(P\phi)(t)| &\le |x^0|t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,\phi(s))|\,ds \\ &\le |x^0|t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left[Ks^{r_1}|\phi(s)|^{r_2} + |f(s,0)|\right]ds \\ &\le |x^0|t^{q-1} + \frac{K}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^{r_1}\left(2|x^0|s^{q-1}\right)^{r_2}ds \\ &\qquad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,0)|\,ds. \end{aligned}$$

Using the assumption that $r_1 + (q-1)r_2 = \mu - 1$ and the Beta function, we have

$$\begin{aligned}|(P\phi)(t)| &\le |x^0|t^{q-1} + \frac{K\big(2|x^0|\big)^{r_2}}{\Gamma(q)}\int_0^t s^{\mu-1}(t-s)^{q-1}\,ds \\ &\qquad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,0)|\,ds \\ &= |x^0|t^{q-1} + \frac{K\big(2|x^0|\big)^{r_2}\Gamma(\mu)}{\Gamma(\mu+q)}\,t^{\mu+q-1} \\ &\qquad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,0)|\,ds.\end{aligned}$$

And so for $\phi \in M$,

$$\begin{aligned}|(P\phi)(t)| \le \bigg\{&|x^0| + \frac{K\Gamma(\mu)\big(2|x^0|\big)^{r_2}}{\Gamma(\mu+q)}\,t^{\mu} \\ &+ \frac{t^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,0)|\,ds\bigg\}t^{q-1}.\end{aligned}$$

Thus, as $\mu > 0$, $1-q > 0$, and because of (2.17),

$$|(P\phi)(t)| \le 2|x^0|t^{q-1}$$

for $0 < t \le T_0$, if T_0 is sufficiently small. For such a T_0, $PM \subset M$.

To prepare the way for showing that P is a contraction mapping in the weighted norm $|\cdot|_g$, first consider the difference $x^{r_2} - y^{r_2}$ for a given pair $x, y \in \Re$ and a given rational number $r_2 > 1$ satisfying the conditions listed in item (ii). It follows from the Mean Value Theorem that there exists a number ξ between x and y such that

$$|x^{r_2} - y^{r_2}| = r_2|\xi|^{r_2-1}|x-y|.$$

Since $r_2 > 1$, the function z^{r_2-1} is increasing on $[0,\infty)$. Consequently, as $|\xi| \le \max\{|x|,|y|\}$,

$$|\xi|^{r_2-1} \le \big(\max\{|x|,|y|\}\big)^{r_2-1}.$$

Thus, for $\phi, \psi \in M$,

$$\begin{aligned}\big|(\phi(t))^{r_2} - (\psi(t))^{r_2}\big| &\le r_2\big(\max\{|\phi(t)|,|\psi(t)|\}\big)^{r_2-1}|\phi(t)-\psi(t)| \\ &\le r_2\left(2|x^0|t^{q-1}\right)^{r_2-1}|\phi(t)-\psi(t)|\end{aligned}$$

for $0 < t \le T_0$.

It follows from the previous inequality and (2.16) that

$$\begin{aligned}\frac{|(P\phi)(t) - (P\psi)(t)|}{t^{q-1}} &\le \frac{t^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,\phi(s)) - f(s,\psi(s))|\,ds \\ &\le \frac{t^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1}Ks^{r_1}\big|(\phi(s))^{r_2} - (\psi(s))^{r_2}\big|\,ds \\ &\le \frac{Kt^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^{r_1}r_2\left(2|x^0|s^{q-1}\right)^{r_2-1}|\phi(s)-\psi(s)|\,ds.\end{aligned}$$

Hence, because of the definition of the g-norm and (2.15),

$$\begin{aligned}
&\frac{|(P\phi)(t)-(P\psi)(t)|}{t^{q-1}} \\
&\le \frac{r_2Kt^{1-q}}{\Gamma(q)}\big(2|x^0|\big)^{r_2-1}\int_0^t (t-s)^{q-1}s^{r_1+(q-1)(r_2-1)}|\phi(s)-\psi(s)|\,ds \\
&\le \frac{r_2Kt^{1-q}}{\Gamma(q)}\big(2|x^0|\big)^{r_2-1}|\phi-\psi|_g\int_0^t (t-s)^{q-1}s^{r_1+(q-1)r_2}\,ds \\
&\le \frac{r_2Kt^{1-q}}{\Gamma(q)}\big(2|x^0|\big)^{r_2-1}|\phi-\psi|_g\int_0^t s^{\mu-1}(t-s)^{q-1}\,ds.
\end{aligned}$$

Evaluating the integral with the Beta function, we obtain

$$\begin{aligned}
\frac{|(P\phi)(t)-(P\psi)(t)|}{t^{q-1}} &\le \frac{r_2Kt^{1-q}}{\Gamma(q)}\big(2|x^0|\big)^{r_2-1}|\phi-\psi|_g\cdot t^{\mu+q-1}\frac{\Gamma(\mu)\Gamma(q)}{\Gamma(\mu+q)} \\
&= \left[\frac{r_2K\Gamma(\mu)}{\Gamma(\mu+q)}\big(2|x^0|\big)^{r_2-1}t^{\mu}\right]|\phi-\psi|_g.
\end{aligned}$$

Although this was derived for $r_2 > 1$, it is also true for $r_2 = 1$. Since $\mu > 0$, the bracketed quantity is less than 1 for $t \in (0, T_0]$ if T_0 is small enough. We conclude a $T_0 \in (0, T]$ exists such that $P\colon M \to M$ and P is a contraction on M. Therefore, by Banach's contraction mapping principle, there is a unique $\phi \in M$ such that $P\phi = \phi$.

Both the fixed point $\phi(t)$ and the function $f(t, \phi(t))$ are absolutely integrable on $(0, T_0]$ since this is true of all functions in M, as we saw earlier in the proof.

It follows from the bound we obtained for $P\phi$ that

$$\begin{aligned}
&\frac{t^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,\phi(s))|\,ds \\
&\qquad \le \frac{K\big(2|x^0|\big)^{r_2}\Gamma(\mu)}{\Gamma(\mu+q)}\,t^{\mu} + \frac{t^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,0)|\,ds,
\end{aligned}$$

where $\mu > 0$. This along with (2.17) implies the first limit in (2.14). This in turn implies the second limit in (2.14) as

$$\lim_{t\to 0^+} t^{1-q}\phi(t) = \lim_{t\to 0^+} t^{1-q}(P\phi)(t) = x^0.$$

Finally, the fixed point ϕ fulfills all of the conditions of Theorem 2.1. Therefore it is also the unique continuous solution of (2.1) on $(0, T_0]$.

Remark 2.2 Two items worth noticing are:

(i) Consider the function $f(t, x)$ when $x = 0$. The continuity of f in Theorem 2.5 implies that $f(t, 0)$ is bounded on the closed interval $[0, T]$. Contrast this with Theorem 2.7 which no longer requires it to be defined at $t = 0$ nor bounded as long as it satisfies (2.17).

(ii) Theorem 2.7 generalizes Theorem 2.5.

Item (ii) follows from observing that if condition (2.3) holds on a closed interval $[0, T]$, then condition (2.16) certainly holds on the half-open interval $(0, T]$ with $r_1 = 0$ and $r_2 = 1$. Also, condition (2.15) holds as

$$\mu = 1 + r_1 + (q-1)r_2 = 1 + 0 + (q-1)(1) = q > 0.$$

Furthermore, as $f(t, 0)$ is bounded on $[0, T]$, condition (2.17) holds.

Example 2.3 Part 1: the range of q. We now show that existence must take q into account. We will examine an assumed solution of (2.2) taking $f(t, x) = x^{2n+1}$ with n a positive integer and $x^0 > 0$. The work takes place in the context of Theorem 2.4 and (a_2) in the applications located just after Corollary 2.2. Thus, any solution $x\colon (0, T] \to \Re$ will be continuous, while $t^{1-q}x(t)$ will be continuous on the closed interval $[0, T]$ and for T small enough will satisfy

$$(1/2)x^0 t^{q-1} \le x(t) \le (3/2)x^0 t^{q-1}.$$

Here is the important part and it can be used to test many functions in the same way to determine permissible values of q. The function $f(t, x)$ is increasing in $x > 0$ so it preserves inequalities: for $x^0 > 0$ and for s small we have

$$\Big[(1/2)x^0 s^{q-1}\Big]^{2n+1} \le \Big[x(s)\Big]^{2n+1} \le \Big[(3/2)x^0 s^{q-1}\Big]^{2n+1}$$

which we write for convenience as

$$A(s) \le B(s) \le C(s).$$

Moreover,

$$Q(t) := t^{1-q} \int_0^t (t-s)^{q-1} (x(s))^{2n+1} ds$$

must be continuous on $[0, T]$ for some sufficiently small positive T; in particular, the limit as $t \downarrow 0$ of $Q(t)$ must exist. But notice that

$$t^{1-q} \int_0^t (t-s)^{q-1} A(s)\, ds \le t^{1-q} \int_0^t (t-s)^{q-1} B(s)\, ds \le t^{1-q} \int_0^t (t-s)^{q-1} C(s)\, ds.$$

However the end terms differ only by a multiplicative constant so if we can prove that the end terms both have limit zero as $t \downarrow 0$, then the middle term will have the same limit of zero. We will see that the end terms have a limit if and only if

$$q > \frac{2n}{2n+1} \tag{2.18}$$

and that limit is zero. If that fails to hold, then both of the end terms are unbounded. This means that if (2.18) fails then the middle term can not have a limit, while if (2.18) holds then the middle term has the same limit of zero.

Using the Beta function to compute the integral, we obtain

$$t^{1-q} \int_0^t (t-s)^{q-1} s^{q(2n+1)-2n-1}\, ds = K t^{q(2n+1)-2n}$$

for some $K > 0$. This gives the required convergence if and only if (2.18) holds.

Now let us return to Theorem 2.7 and condition (2.15). We have

$$\mu = 1 + 0 + (q-1)(2n+1) > 0$$

which is the same as (2.18). Note that $f(t,x) = x^{2n+1}$ trivially satisfies all of the other conditions of Theorem 2.7. We conclude a continuous solution of (2.2) exists on some interval $(0, T]$ if and only if

$$\frac{2n}{2n+1} < q < 1.$$

Moreover, one of the statements of Theorem 2.7 tells us that the solution $x(t)$ as well as $f(t, x(t))$ are absolutely integrable on $(0, T]$. As a result, we also conclude from Theorem 2.1 that $x(t)$ is also a continuous solution of (2.1) on $(0, T]$.

Part 2: the third singularity.

We readily see that (2.2) has a singularity in the forcing function and one in the kernel. But both of them are mild in a technical sense. However they coalesce as $t \downarrow 0$. From (2.9) we see that as $\epsilon \downarrow 0$ then $x(s)$ in the integrand of (2.2) gets as close to $x^0 s^{q-1}$ as we please on a sufficiently short interval $(0, T]$. For instance, with $f(t,x) = x^{2n+1}$, (2.2) is approximated arbitrarily well for very small t by

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} (x^0 s^{q-1})^{2n+1}\, ds.$$

As $t \downarrow 0$ both terms in the integrand tend to infinity producing a product of singularities which is no longer mild in any technical sense.

3 A Transformation

Equation (2.2) and its solution on some short interval $[0, T)$ that is ensured by Theorem 2.7 hold many challenges if we wish to continue that solution beyond T. The forcing function is singular at $t = 0$ so the solution is singular there too and that introduces another singularity in the integrand besides the one already at $s = t$. In Example 2.3, Part 2 we discussed how this added singularity will coalesce with the singularity in the kernel producing a singularity of a radically different type than either that in the forcing function or in the kernel. And this added singularity cannot be avoided because it occurs as $t \downarrow 0$. This is also the situation in Example 2.1. Note however its integrand has even more singularities: those located at the zeroes of $\cos(\sqrt{t})$, which were avoided in that example by simply restricting the interval under consideration.

To make matters worse, consider the integral in the mapping P, calling it H for now, and suppose for the moment that f satisfies a global Lipschitz condition with constant $\alpha < 1$. If the functions $\phi_i : [0, \infty) \to \Re$ $(i = 1, 2)$ are bounded and continuous with the supremum norm $\|\cdot\|$, then

$$H(t, x) := \int_0^t (t-s)^{q-1} f(s, x(s))\, ds$$

satisfies

$$\begin{aligned}|H(t,\phi_1(t)) - H(t,\phi_2(t)| &\le \int_0^t (t-s)^{q-1}\alpha|\phi_1(s) - \phi_2(s)|\,ds \\ &\le \alpha\|\phi_1 - \phi_2\| \int_0^t s^{q-1}\,ds \\ &= \alpha\|\phi_1 - \phi_2\| t^q/q.\end{aligned}$$

So for this situation H, as well as P, is not a contraction for $t \ge (q/\alpha)^{1/q}$. Moreover, if $f(t,x)$ contains a bounded additive function $u(t)$, then it transforms into $\int_0^t (t-s)^{q-1}u(s)\,ds$ passing from a bounded u to a possibly unbounded integral.

There is a simple way out of all these difficulties. In [5] we introduced a transformation for a fractional differential equation of Caputo type which has turned out to be very useful in the construction of fixed point mappings. The first part of it will now be given. It will take a second step to make it work for fractional differential equations of Riemann-Liouville type because of the singular forcing function.

Let J be an arbitrary positive constant and write (2.2) as

$$\begin{aligned}x(t) &= x^0 t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[-Jx(s) + Jx(s) + f(s,x(s))]\,ds \\ &= x^0 t^{q-1} - \frac{J}{\Gamma(q)}\int_0^t (t-s)^{q-1}x(s)\,ds \\ &\quad + \frac{J}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left[x(s) + \frac{f(s,x(s))}{J}\right]ds,\end{aligned}$$

which we then rewrite as

$$x(t) = x^0 t^{q-1} - \int_0^t C(t-s)[x(s) + G(s,x(s))]\,ds, \tag{3.1}$$

where

$$C(t) := \frac{Jt^{q-1}}{\Gamma(q)} \tag{3.2a}$$

and

$$G(s,x(s)) := -\left[x(s) + \frac{f(s,x(s))}{J}\right]. \tag{3.2b}$$

Now view the linear equation

$$z(t) = x^0 t^{q-1} - \int_0^t C(t-s)z(s)\,ds \tag{3.3}$$

as the linear part of the nonlinear equation (3.1). Closely allied to both (3.1) and (3.3) is the *resolvent equation*

$$R(t) = C(t) - \int_0^t C(t-s)R(s)\,ds. \tag{3.4}$$

It is well-established that (3.4) has a unique continuous solution on $(0,\infty)$, which is known as the *resolvent* (cf. [3, Thm. 4.2]). Because of this uniqueness, it follows from multiplying both sides of (3.4) by $x^0\Gamma(q)/J$ that (3.3) also has a unique continuous solution on $(0,\infty)$, namely

$$z(t) = \frac{x^0\Gamma(q)}{J}R(t).$$

Substituting this for $z(s)$ in the integrand of (3.3) and using (3.2a), we obtain

$$\begin{aligned} z(t) &= x^0t^{q-1} - \int_0^t C(t-s)\frac{x^0\Gamma(q)}{J}R(s)\,ds \\ &= x^0t^{q-1} - \int_0^t \frac{J}{\Gamma(q)}(t-s)^{q-1}\frac{x^0\Gamma(q)}{J}R(s)\,ds \\ &= x^0t^{q-1} - \int_0^t (t-s)^{q-1}x^0R(s)\,ds. \end{aligned}$$

With an obvious change of variable, we can also write this as

$$z(t) = x^0t^{q-1} - \int_0^t R(t-s)x^0s^{q-1}\,ds. \tag{3.5}$$

Important properties of $R(t)$ (cf. [15, p. 212 f.]) that we rely on are

$$0 < R(t) \le C(t), \quad \int_0^\infty R(s)\,ds = 1 \tag{3.6}$$

and the fact that $R(t)$ is completely monotone on $(0,\infty)$ ([15, p. 224]).

Suppose the conditions of an existence theorem, such as Theorem 2.5 or 2.7, are satisfied so that a solution $x(t)$ of (2.2), equivalently of (3.1), is known to exist on an interval $(0,T_0]$. In that case, a variation of parameters formula found in Miller [15, (1.4), p. 192] states that $x(t)$ will also satisfy the equation

$$x(t) = z(t) - \int_0^t R(t-s)G(s,x(s))\,ds \tag{3.7}$$

provided

$$\begin{aligned} &\int_0^t\int_0^s R(t-s)C(s-u)G(u,x(u))\,du\,ds \\ &\qquad = \int_0^t\int_u^t R(t-s)C(s-u)G(u,x(u))\,ds\,du \end{aligned} \tag{3.8}$$

for $0 < t \le T_0$. Note that this interchange in the order of integration is valid if the conditions of either Theorem 2.5 or Theorem 2.7 are satisfied since those conditions imply part b) of Theorem 2.6, which in turn implies (3.8) by the Hobson-Tonelli test (cf. [16, p. 93]). We further note a function satisfying (3.7) and (3.8) will also satisfy (2.2) since, as Miller [15, p. 192] points out, the steps from (3.1) to (3.7) are reversible.

The solution z of (3.5) will play a major role in the subsequent work and the following result offers its properties.

Lemma 3.1 *For each $\epsilon > 0$, the function z defined by (3.5) is bounded on $[\epsilon,\infty)$ and tends to zero as $t\to\infty$. Furthermore*

$$|z(t)| \le |x^0|t^{q-1}\left[1 - \int_0^t R(s)\,ds\right] \tag{3.9}$$

for all $t > 0$. The bounds on both z and $\int_0^t R(t-s)s^{q-1}\,ds$ are independent of the positive constant J. Moreover,

$$z(t) = \frac{x^0\Gamma(q)}{J}R(t).$$

Proof. From (3.4) we see that

$$\begin{aligned} R(t) &= \frac{J}{\Gamma(q)}t^{q-1} - \frac{J}{\Gamma(q)}\int_0^t (t-s)^{q-1}R(s)\,ds \\ &= \frac{J}{\Gamma(q)}\left[t^{q-1} - \int_0^t R(t-s)s^{q-1}\,ds\right] \\ &= \frac{J}{x^0\Gamma(q)}z(t). \end{aligned}$$

Consequently, for $t > 0$,

$$0 \le \int_0^t R(t-s)s^{q-1}\,ds \le t^{q-1} \tag{3.10}$$

as $R(t) > 0$ for $t > 0$ [15, p. 222]. Note this is independent of J. Hence (3.5) has the following limit:

$$\lim_{t\to\infty} z(t) = x^0\left[\lim_{t\to\infty} t^{q-1} - \lim_{t\to\infty}\int_0^t R(t-s)s^{q-1}ds\right] = 0.$$

This limit, along with the continuity of $z(t)$ on $(0,\infty)$, implies that $z(t)$ is bounded on $[\epsilon,\infty)$ for each $\epsilon > 0$.

By the previous inequality,

$$|z(t)| = |x^0|\left|t^{q-1} - \int_0^t R(t-s)s^{q-1}\,ds\right| = |x^0|\left(t^{q-1} - \int_0^t R(t-s)s^{q-1}\,ds\right).$$

As t^{q-1} is decreasing,

$$\int_0^t R(t-s)s^{q-1}\,ds \ge t^{q-1}\int_0^t R(t-s)\,ds.$$

Therefore,

$$|z(t)| \le |x^0|\left(t^{q-1} - t^{q-1}\int_0^t R(t-s)\,ds\right) = |x^0|t^{q-1}\left[1 - \int_0^t R(u)\,du\right]. \tag{3.11}$$

4 A Translation

We have one more step to take and it is a large one. Fix x^0 and let x be a solution of (2.2). Redefine the interval of definition and say that it is a solution on the interval $(0, 2T]$ satisfying Definition 1.1 as well as (3.7) and (3.8). In particular, we have

$$|x(t)| \le 2|x^0|t^{q-1}$$

on that interval. However, we must keep in mind that z still has a singularity. This section is devoted to showing that a translation with $y(t) = x(t+T)$ will transform (3.7) into

$$y(t) = F(t) + \int_0^t R(t-s)\left[y(s) + \frac{f(s+T, y(s))}{J}\right] ds,$$

where F is bounded, continuous, in $L^1[0,\infty)$, and converges to zero as $t \to \infty$. But most of all we want to remember (3.6).

The value of x^0 determines $z(t)$ and we know from Lemma 3.1 that $z(t)$ is bounded and continuous for $t \geq T$ and that $z(t) \to 0$ as $t \to \infty$. Now translate (3.7) as follows:

$$\begin{aligned} x(t+T) &= z(t+T) - \int_0^{t+T} R(t+T-s)G(s,x(s))\,ds \\ &= z(t+T) + \int_0^T R(t+T-s)\left[x(s) + \frac{f(s,x(s))}{J}\right] ds \\ &\quad + \int_T^{t+T} R(t+T-s)\left[x(s) + \frac{f(s,x(s))}{J}\right] ds \\ &= F(t) + \int_0^t R(t-s)\left[x(s+T) + \frac{f(s+T, x(s+T))}{J}\right] ds, \end{aligned} \tag{4.1}$$

where

$$F(t) := z(t+T) + \int_0^T R(t+T-s)\left[x(s) + \frac{f(s,x(s))}{J}\right] ds. \tag{4.2}$$

Next, let

$$y(t) := x(t+T)$$

and rewrite the translated equation (4.1) as

$$y(t) = F(t) + \int_0^t R(t-s)\left[y(s) + \frac{f(s+T, y(s))}{J}\right] ds, \tag{4.3}$$

where $y(0) = x(T)$. From this we see how to define an appropriate mapping for establishing solutions in the Banach space of bounded continuous functions on $[0,\infty)$ with the sup norm, which we denote by $(BC, \|\cdot\|)$. For a specified subset Q of this space, define the mapping $P\colon Q \to BC$ by $\phi \in Q$ implies

$$(P\phi)(t) := F(t) + \int_0^t R(t-s)\left[\phi(s) + \frac{f(s+T, \phi(s))}{J}\right] ds. \tag{4.4}$$

The last line of the following theorem need not be disquieting. If we ask that f satisfy (1.6), then we invoke Theorem 2.6 and find that $x(s)$ and $f(s,x(s))$ are absolutely integrable on $(0,T]$ so that (3.8) is assured by the Hobson-Tonelli theorem.

Theorem 4.1 *Let $q \in (0,1)$, $f\colon (0,\infty)\times\Re \to \Re$ be continuous, and $x^0 \in \Re$ with $x^0 \neq 0$. Let $x(t)$ be a solution of*

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x(s))\,ds \tag{2.2}$$

on an interval $(0,T]$*. For a given constant* $J>0$*, let* $z(t)$ *denote the unique continuous solution of (3.3) on* $(0,\infty)$ *and let a function* $F\colon[0,\infty)\to\Re$ *be defined by (4.2). Lastly, let* $y(t)$ *be a solution of (4.3) on an interval* $[0,\tau]$ *for some* $\tau>0$*.*

If the piecewise-defined function

$$x_c(t) := \begin{cases} x(t), & \text{if } \ 0<t\le T, \\ y(t-T), & \text{if } \ T<t\le T+\tau, \end{cases} \tag{4.5}$$

satisfies (3.8) at each $t\in(0,T+\tau]$*, then it is a solution of (2.2) on* $(0,T+\tau]$*.*

Proof. Suppose $x_c(t)$ satisfies (3.8) at each $t\in(0,T+\tau]$. Then the solution $x(t)$ must satisfy (3.8) at each $t\in(0,T]$. Hence, by the variation of parameters formula $x(t)$ is also a solution of (3.7). Thus,

$$x_c(t) = z(t) - \int_0^t R(t-s)G(s,x_c(s))\,ds \qquad (0<t\le T), \tag{4.6}$$

where from (3.2b)

$$G(s,x_c(s)) = -\left[x_c(s) + \frac{f(s,x_c(s))}{J}\right].$$

Since $y(t)$ is a solution of (4.3) for $0\le t\le\tau$, $y(0)=F(0)$. Setting $t=0$ in (4.2) and replacing $x(t)$ with $x_c(t)$, we get

$$y(0) = z(T) - \int_0^T R(T-s)G(s,x_c(s))\,ds.$$

Comparing this with (4.6) when $t=T$, we see $y(0)=x_c(T)$. And so

$$y(0) = x(T).$$

Thus, as x and y are continuous functions on their respective domains, the piecewise-defined function x_c is continuous on the interval $(0,T+\tau]$.

Since the function $y(t)$ satisfies (4.3) on the interval $[0,\tau]$, we have

$$\begin{aligned} y(t) &= F(t) - \int_0^t R(t-s)G(s+T,y(s))\,ds \\ &= z(t+T) - \int_0^T R(t+T-s)G(s,x(s))\,ds \\ &\qquad - \int_0^t R(t-s)G(s+T,y(s))\,ds. \end{aligned}$$

With the change of variable $u=s+T$, this becomes

$$\begin{aligned} y(t) = z(t+T) &- \int_0^T R(t+T-s)G(s,x(s))\,ds \\ &- \int_T^{t+T} R(t+T-u)G(u,y(u-T))\,du. \end{aligned}$$

Rewriting the right-hand side in terms of the function x_c, we have

$$y(t) = z(t+T) - \int_0^T R(t+T-s)G(s, x_c(s))\, ds \\ - \int_T^{t+T} R(t+T-u)G(u, x_c(u))\, du.$$

And so

$$y(t) = z(t+T) - \int_0^{t+T} R(t+T-s)G(s, x_c(s))\, ds$$

for $0 \leq t \leq \tau$. Or,

$$y(t-T) = z(t) - \int_0^t R(t-s)G(s, x_c(s))\, ds$$

for $T \leq t \leq T+\tau$. That is,

$$x_c(t) = z(t) - \int_0^t R(t-s)G(s, x_c(s))\, ds. \qquad (T \leq t \leq T+\tau)$$

This and (4.6) implies that the function $x_c(t)$ is a solution of the intermediate equation (3.7) on the interval $(0, T+\tau]$.

Finally since $x_c(t)$ satisfies (3.8) for $0 < t \leq T+\tau$, we can invoke the variation of parameters result to conclude $x_c(t)$ is also a solution of the integral equation (2.2) on $(0, T+\tau]$.

Summary

Our stated goal was to transform (2.2) into a standard Volterra integral equation with a singularity only in the kernel which was to be completely monotone and have integral equal to one. That final equation is (4.3). In the next subsection we will develop the properties of the function F because F did not appear in (2.2). The solution of (2.2) will be that original solution on the short interval $(0, T]$ and then continued with the solution y of (4.3). Here are details which should guide the investigator.

Suppose that some existence theorem yields a solution of (2.2) on an interval $(0, T]$. That solution resides in

$$\text{A)} \quad M = \{\phi \in X : |\phi(t)| \leq 2|x^0|t^{q-1}\}$$

provided that T is sufficiently small.

Regardless of which existence theorem we might use, suppose that we have assumed $f : (0, \infty) \times \Re \to \Re$ is continuous and satisfies

$$\text{B)} \quad |f(t,x)| \leq |f(t,0)| + Kt^{r_1}|x|^{r_2}$$

and with the assumption that $f(t, 0)$ is absolutely integrable on $(0, T]$. We need only note from A) and B) above that

$$\int_0^T s^{r_1}(s^{q-1})^{r_2}\, ds = \int_0^T s^{r_1+r_2(q-1)}\, ds = kt^{r_1+r_2(q-1)+1}$$

for some $k > 0$ and our basic requirement for integrability is

$$\text{C)} \quad r_1 + r_2(q-1) + 1 > 0.$$

By Theorem 2.6 if A), B), and C) hold then

D) $|x| + |f(t,x)|$ is integrable on $(0,T]$

so Theorem 2.1 holds, as does (3.8) making (2.2), (3.7), and (4.3) equivalent in the sense of Theorem 4.1 as long as the solution extending $x(t)$ from $(0,T]$ to $(0,T+\tau]$ is continuous.

In conclusion, after verifying B), C), and any existence result, the investigator may go directly to (4.3) and begin the task of extracting properties of continuous solutions. Those properties are inherited by both (2.1) and (2.2).

4.1 Properties of the forcing function

Properties of the function F defined by (4.2) that will govern solutions of (4.3) are stated in the next theorem. We noted earlier that a solution of (2.2) lies in the set M defined in (2.11) so it is absolutely integrable. We gave conditions in Theorem 2.6 to ensure that a solution x will have $|x| + |f(t,x)|$ integrable. Moreover, when that holds so does conclusion b) of that theorem which is a sufficient condition for (3.8) to hold and, indeed, to assure us by Theorem 2.1 that the solution satisfies (2.1). That, in turn, was used together with a solution of (2.2) on a short interval to pass from (2.2) to (3.7) and then on to our final equation (4.3). This paragraph then is describing the fundamental position of item (iii) in the next theorem. Items (i) and (ii) are reminding us of Definition 1.1.

Theorem 4.2 *Let $f\colon (0,T_1] \times \Re \to \Re$ be continuous. Suppose there exists a $T \in (0, T_1/2]$ and a continuous function $x\colon (0,2T] \to \Re$ that is absolutely integrable and satisfies the equation*

$$\begin{aligned} x(t) &= z(t) - \int_0^t R(t-s)G(s,x(s))\,ds \qquad (3.7)\\ &= z(t) + \int_0^t R(t-s)\left[x(s) + \frac{f(s,x(s))}{J}\right] ds \end{aligned}$$

on $(0,2T]$. Suppose further that

(i) $t^{1-q}x(t)$ is continuous on $[0,2T]$,

(ii) $\lim_{t\to 0^+} t^{1-q}x(t) = x^0$,

(iii) $f(t,x(t))$ is absolutely integrable on $(0,2T]$.

Then the function $F\colon [0,\infty) \to \Re$ defined by (4.2), where J denotes an arbitrary positive constant, is uniformly continuous on $[0,\infty)$, tends to zero as $t \to \infty$, and is in $L^1[0,\infty)$. Moreover, a bound for F exists that is independent of the value of J.

The proof of this theorem is a consequence of the following three lemmas, namely Lemmas 4.1–4.3.

Lemma 4.1 *Under the conditions of Theorem 4.2, for any given constant $J > 0$, the function*

$$\begin{aligned} \mathsf{G}(t) &:= -\int_0^T R(t+T-s)G(s,x(s))\,ds \qquad (4.7)\\ &= \int_0^T R(t+T-s)\left[x(s) + \frac{f(s,x(s))}{J}\right] ds \end{aligned}$$

is uniformly continuous on $[\eta,\infty)$ *for any* $\eta > 0$.

Proof. Fix an arbitrary $J > 0$. Let $\epsilon > 0$. We see from (3.6) that $R(t)$ is continuous on $[\eta,\infty)$ and $R(t) \to 0$ as $t \to \infty$, which implies it is uniformly continuous on this interval. For a given $\lambda > 0$, there is a $\gamma > 0$ such that distinct $t_1, t_2 \in [\eta,\infty)$ and

$$|t_1 + T - s - t_2 - T + s| = |t_1 - t_2| < \gamma$$

and $T - s \geq 0$ imply that

$$|R(t_1 + T - s) - R(t_2 + T - s)| < \lambda.$$

Hence for these t_i, we have

$$\begin{aligned} |\mathsf{G}(t_1) - \mathsf{G}(t_2)| &\leq \int_0^T \big|R(t_1 + T - s) - R(t_2 + T - s)\big|\big|G(s, x(s))\big|\, ds \\ &\leq \lambda \int_0^T \big|G(s, x(s))\big|\, ds \\ &\leq \lambda \int_0^T \left[|x(s)| + \frac{|f(s, x(s))|}{J}\right] ds =: \lambda H_J, \end{aligned}$$

where H_J denotes the constant defined by the last integral. So choose $\lambda < \epsilon/H_J$. Therefore, for the given $\epsilon > 0$, there is a $\gamma > 0$ such that $|t_1 - t_2| < \gamma$ implies $|G(t_1) - G(t_2)| < \epsilon$.

The function F in the following two lemmas refers to the function defined by (4.2).

Lemma 4.2 *Under the conditions of Theorem 4.2, for any given constant* $J > 0$*, the function* $F(t)$ *is right-continuous at* $t = 0$*. Furthermore, it is uniformly continuous on* $[0,\infty)$.

Proof. By hypothesis, a continuous function $x(t)$ exists satisfying (3.7) on $(0, 2T]$. Recall earlier we defined y by

$$y(t) := x(t + T) \tag{4.8}$$

with the purpose of continuing the solution of (3.7) beyond $2T$. This then yielded equation (4.3), which we find convenient here to write as

$$y(t) = F(t) + L(t), \tag{4.9}$$

where

$$L(t) := \int_0^t R(t - s)\left[y(s) + \frac{f(s + T, y(s))}{J}\right] ds. \tag{4.10}$$

This suggests defining $L(0) = 0$. In so doing, we have from (4.9), (4.2), and (3.7) that

$$y(0) = F(0) = z(T) - \int_0^T R(T - s)G(s, x(s))\, ds = x(T).$$

Note this is consistent with (4.8). Thus let $L(0) := 0$.

Since x is continuous on $(0, 2T]$, we see from (4.8) that y is continuous on $(-T, T]$. So it follows from (4.9) that if we can show that $L(t)$ is right-continuous at $t = 0$, then the same will be true of F. It follows from the hypothesis that the function f is continuous on $(0, 2T] \times \Re$. Thus, as y is continuous on $[0, T]$, there is a constant C_J such that

$$\left| y(s) + \frac{f(s+T, y(s))}{J} \right| \leq C_J$$

on $[0, T]$. This along with (3.6) implies

$$|L(t)| \leq C_J \int_0^t R(t-s)\, ds \leq \frac{C_J J}{\Gamma(q)} \int_0^t s^{q-1}\, ds \leq \frac{C_J J t^q}{q\Gamma(q)}. \tag{4.11}$$

Hence $L(t) \to L(0)$ as $t \to 0^+$. As a result, F is right-continuous at $t = 0$.

As for uniform continuity, first observe from (4.2) and (4.7) that

$$F(t) = z(t+T) + \mathsf{G}(t). \tag{4.12}$$

By default, G is right-continuous at $t = 0$ since $z(t+T)$ is continuous on $[0, \infty)$. Because of this and Lemma 4.1, we see that G is continuous on $[0, \infty)$. This, together with the uniform continuity of G on $[\eta, \infty)$ for every $\eta > 0$, implies that G is uniformly continuous on $[0, \infty)$. It follows from Lemma 3.1 that $z(t+T) \to 0$ as $t \to \infty$. This and the continuity of $z(t+T)$ on $[0, \infty)$ imply that $z(t+T)$ is also uniformly continuous on $[0, \infty)$. Since the sum of uniformly continuous functions is uniformly continuous, we conclude F is uniformly continuous on $[0, \infty)$.

Finally note that the foregoing argument is valid for any given $J > 0$. This concludes the proof.

Lemma 4.3 *Under the conditions of Theorem 4.2, $F \in L^1[0, \infty)$, $F(t) \to 0$ as $t \to \infty$, and F is bounded on $[0, \infty)$. Moreover, there is a bound for F on $[0, \infty)$ that is independent of J.*

Proof. Let us start with the first term of $F(t)$, namely $z(t+T)$. We have already determined that $z(t+T) \to 0$ as $t \to \infty$. Now consider $\mathsf{G}(t)$, the other term of $F(t)$. Recall that R is completely monotone, so it is decreasing on $(0, \infty)$. Consequently,

$$\begin{aligned} |\mathsf{G(t)}| &\leq \int_0^T R(t+T-s) \big| G(s, x(s)) \big|\, ds \\ &\leq R(t) \int_0^T \left| x(s) + \frac{f(s, x(s))}{J} \right| ds \\ &\leq R(t) \int_0^T \left[|x(s)| + \frac{|f(s, x(s))|}{J} \right] ds = KR(t), \end{aligned} \tag{4.13}$$

where K denotes the last integral, which has a finite value since both $x(s)$ and $f(s, x(s))$ are absolutely integrable on $(0, T]$. As $t \to \infty$, $\mathsf{G}(t) \to 0$ since $R(t) \to 0$. Because both terms of $F(t)$ tend to zero, so does $F(t)$.

It also follows from (4.13) that $\mathsf{G} \in L^1[0, \infty)$ because $R \in L^1[0, \infty)$. Moreover, $z(t+T) \in L^1[0, \infty)$ since $z(t)$ is proportional to $R(t)$ (cf. Lemma 3.1). Hence, $F \in L^1[0, \infty)$.

Recall from Lemma 4.2 that F is uniformly continuous on $[0, \infty)$. This together with $F(t) \to 0$ as $t \to \infty$ implies that F is bounded on $[0, \infty)$.

We now come to the final part of the proof, which is to show that a bound for F exists independent of J. Consider $z(t+T)$, the first term of $F(t)$. From (3.9) we see that

$$|z(t+T)| \leq |x^0|(t+T)^{q-1} \leq |x^0|T^{q-1} \tag{4.14}$$

for $t \geq 0$, yielding a bound for $z(t+T)$ on $[0,\infty)$ that depends on T but not on J. So what remains is to prove that the integral term $\mathsf{G}(t)$ in (4.12) also has a bound on $[0,\infty)$ independent of J.

Condition (i) of Theorem 4.2 implies the existence of a constant k such that $|x(t)| \leq kt^{q-1}$ for all $t \in (0, 2T]$. Then in view of (3.6), (3.2a), and the monotonicity of R, we have

$$\begin{aligned}
|\mathsf{G(t)}| &\leq \int_0^T R(t+T-s)\left[|x(s)| + \frac{|f(s,x(s))|}{J}\right] ds \\
&\leq \int_0^T R(t+T-s)ks^{q-1}\, ds \\
&\qquad + \int_0^T \frac{J}{\Gamma(q)}(t+T-s)^{q-1}\frac{|f(s,x(s))|}{J}\, ds \\
&\leq k\int_0^T R(T-s)s^{q-1}\, ds + \frac{1}{\Gamma(q)}\int_0^T (t+T-s)^{q-1}|f(s,x(s))|\, ds.
\end{aligned}$$

Applying (3.10), we obtain the bound

$$|\mathsf{G(t)}| \leq kT^{q-1} + \frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}|f(s,x(s))|\, ds. \tag{4.15}$$

By hypothesis, $f(s,x(s))$ is absolutely integrable on $(0,T]$. So the integral in (4.15) converges according to Lemma 2.1. Because of this, (4.12), and (4.14), we have

$$|F(t)| \leq \big(|x^0| + k\big)T^{q-1} + \frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}|f(s,x(s))|\, ds$$

for $t \geq 0$. Thus the right-hand side serves as a bound for F. Since it does not depend on J, the proof is complete.

The completion of the proof of this last lemma also completes the proof of Theorem 4.2.

5 Future Work

The objective was to reduce the fractional differential equation to a very common Volterra integral equation. Equation (4.3) now has three properties which make it ideal for fixed point theory.

First, the kernel $R(t-s)$ has two properties used extensively in fixed point theory. If Q is a convex set in the Banach space of bounded continuous functions with the supremum norm and if $f(s+T, y(s))$ is bounded for $y \in Q$, then

$$\int_0^t R(t-s)\left[y(s) + \frac{f(s+T,y(s))}{J}\right] ds$$

maps Q into an equicontinuous set [6, Thm. 5.1] all ready for numerous fixed point theorems of the Schauder type. Further work will actually give compactness of the mapping provided that Q is a ball in the Banach space [9]. On the other hand, if the function in large brackets defines a contraction, it will be preserved by that same integral.

Next, we have said nothing of J, but it serves a prime function, together with that extra $y(s)$ in the integrand. These work together to secure a self mapping set parallel to that seen in [6] concerning Caputo problems.

There are many directions we can take from here. Our next project involves offering an existence theorem based on the growth condition given here, but without any kind of contraction assumption. We then pick up (4.3) and obtain results on bounded solutions, solutions in $L^1[T,\infty)$, and solutions which are asymptotically periodic. While such results are of interest in themselves, they put us in a position to compare and contrast the behavior of solutions of Caputo equations having a Volterra representation parallel to (2.2) of the form

$$x(t) = x^0 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,x(s))\,ds.$$

The difference in initial conditions is clear and that is a prime reason for considering them. The present work leading up to (4.3) reveals new differences which an investigator would like to take into account. For example, the function $z(t)$ discussed in Lemma 3.1 is in $L^1[T,\infty)$, but it is seen in [7] the corresponding $z(t)$ for the Caputo equation is in $L^p[0,\infty)$ if and only if $p > 1/q$. Other differences appear in the study of asymptotically periodic solutions in (4.3) compared to those for the Caputo equation as shown in [8].

References

[1] Abramowitz, M. and Stegun, I. A. (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 2nd printing. National Bureau of Standards, Applied Mathematical Series 55, 1964.

[2] Apostol, T. M., *Mathematical Analysis*, Second ed. Addison-Wesley, Reading MA, 1974.

[3] Becker, Leigh C., Resolvents for weakly singular kernels and fractional differential equations. *Nonlinear Anal.: TMA* **75** (2012) 4839–4861.

[4] Becker, L. C., Burton, T. A., and Purnaras, I. K., Complementary equations: A fractional differential equation and a Volterra integral equation. *Electron. J. Qual. Theory Differ. Equ.* (12) (2015) 1–24.

[5] Burton, T. A., Fractional differential equations and Lyapunov functionals. *Nonlinear Anal.: TMA* **74** (2011) 5648–5662.

[6] Burton, T. A. and Zhang, Bo, Fixed points and fractional differential equations: Examples, *Fixed Point Theory* **14** (2) (2013) 313–326.

[7] Burton, T. A. and Zhang, Bo, L^p-solutions of fractional differential equations. *Nonlinear Studies* **19** (2) (2012) 161–177.

[8] Burton, T. A. and Zhang, Bo, Asymptotically periodic solutions of fractional differential equations. *Dynamics of Continuous, Discrete and Impulsive Systems, Series A: Mathematical Analysis* **2** (2013) 1–21.

[9] Burton, T. A. and Zhang, Bo, A Schauder-type fixed point theorem. *J. Math. Anal. Appl.* **417** (2014) 552–558.

[10] Diethelm, Kai, *The Analysis of Fractional Differential Equations*. Springer, Heidelberg, 2010.

[11] Jones, Frank, *Lebesgue Integration on Euclidean Spaces*, revised ed., Jones and Bartlett, Boston, MA, 2001.

[12] Kilbas, A., Srivastava, H., and Trujillo, J., *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematical Studies **204**. Elsevier, 2006.

[13] Kumar, S. and Sukavanam, N., On the approximate controlability of fractional order control systems with a delay. *Nonlinear Dynamics and Systems Theory* **13** (1) (2013) 69–78.

[14] Lakshmikantham, V., Leela, S. and Devi, J. Vasundhara, *Theory of Fractional Dynamic Systems*. Cambridge Scientific Publishers, Cottenham, Cambridge, 2009.

[15] Miller, R. K., *Nonlinear Volterra Integral Equations*. Benjamin, Menlo Park, CA, 1971.

[16] Natanson, I. P., *Theory of Functions of a Real Variable*, Vol. II, Frederick Ungar Publishing Co., New York, 1961.

[17] Oldham, Keith B. and Spanier, Jerome, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*. Dover, Mineola, NY, 2006.

[18] Olmsted, John M. H., *Real Variables*. Appleton-Century-Crofts, New York, 1959.

[19] Podlubny, I., *Fractional Differential Equations*. Academic Press, San Diego, 1999.

[20] Watson, G. N., *A Treatise on the Theory of Bessel Functions*, 2nd ed. Cambridge University Press, Cambridge, 1966.