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Mathematical Contributions to the Dynamics of the Josephson Junctions: State of the Art and Open Problems

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Abstract: Mathematical models related to some Josephson junctions are pointed out and attention is drawn to the solutions of certain initial boundary problems and to some of their estimates. In addition, results of rigorous analysis of the behaviour of these solutions when $t \to \infty$ and when the small parameter ε tends to zero are cited. These analyses lead us to mention some of the open problems.

Keywords: *third order parabolic operator; fundamental solution; superconductivity; Josephson junction.*

Mathematics Subject Classification (2010): 82D55, 74K30, 35K35, 35E05.

1 Introduction

Our purpose is to:

i) furnish a short review of the mathematical contributions to the dynamics of the Josephson junctions,

ii) introduce some possible open problems.

From the mathematical point of view, many descriptions of superconductivity phenomena have been developed and an important contribution has been given by Brian David Josephson. He predicted in 1962 the tunnelling of superconducting Cooper pairs through an insulating barrier to pass from one superconductor to another (Josephson effect). He also predicted the exact form of the current and voltage relations for the junction (Josephson junction) [1]. (Experimental work proved that his theory was right, and Josephson was awarded the 1973 Nobel Prize in Physics.)

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The flux-dynamics of a Josephson junction, i.e., two layers of superconductors separated by a very thin layer of insulating material, can be described by means of Sine–Gordon equation (SGE):

$$u_{xx} - u_{tt} = \sin u, \tag{1}$$

where x denotes the direction of propagation, t is time and the variable $u = u(x,t)$ represents the difference between the phases of the wave functions of the two superconductors.

However, in dealing with real junctions it seems necessary to take into account other effects such as losses and bias. Therefore, many authors prefer to consider the so-called perturbed Sine–Gordon equation (PSGE):

$$\varepsilon u_{xxt} + u_{xx} - u_{tt} - a u_t = \sin u - \gamma. \tag{2}$$

In this case, terms εu_{xxt} and $a u_t$ represent respectively the dissipative normal electron current flow along and across the junction, (longitudinal and shunt losses) while γ is the normalized current bias [2]. The value's range for a and ε depends on the real junction. Indeed, there are cases with $0 < a, \varepsilon < 1$ and, when the shunt resistance of the junction is low, the case a large with respect to 1 arises [2–4].

In some cases, extra terms must be considered. For example in a semiannular or in a S-shaped Josephson junction, when an applied magnetic field b parallel to the plane of the dielectric barrier is considered, the dynamic equation is:

$$\varepsilon u_{xxt} + u_{xx} - u_{tt} - a u_t = \sin u - \gamma - b\cos(kx), \tag{3}$$

where the last term evaluates a transient force on the trapped fluxons and locates these ones at the center of the junction [2,5,6]. Moreover, if an annular junction, also provided with a microshort, is considered, the vortex dynamics in a static magnetic field is modelled with the general perturbed sine–Gordon equation (see, f.i. [7]):

$$\varepsilon u_{xxt} + u_{xx} - u_{tt} - a u_t = [1 - \delta(x)\mu]\sin u - \gamma - b\cos(kx), \tag{4}$$

where μ is the current density associated with the microshort.

Nowadays, in addition to rectangular or annular junctions, many other geometries for Josephson junctions have been proposed. For instance, window Josephson junctions (WJJ) ([8] and reference therein) or exponentially shaped Josephson junctions (ESJJ) [9–12]. This type of junction is only a particular case of a structure covering a region

$$0 \le x \le L, \qquad g_2(x) \le y \le g_1(x). \tag{5}$$

Denoting by

$$0 < w(x) = g_1(x) - g_2(x) \ll 1, \tag{6}$$

the evolution of the phase inside the junction is given by:

$$\varepsilon u_{xxt} + u_{xx} - u_{tt} - a u_t = \sin u - \Gamma(x) - \frac{\dot w(x)}{w(x)}(u_x + \varepsilon u_{xt}) + \eta_y \frac{\dot w(x)}{w(x)}, \tag{7}$$

where $\Gamma(x) = \frac{\eta_x|_{g_2} - \eta_x|_{g_1}}{w(x)}$ and η_x η_y is the normalized magnetic field respectively in the x and y directions [10]. When one assumes $g_1(x) = -g_2(x) = w_o\, e^{-\lambda x}$, where λ is a constant that, generally, is less than one, an ESJJ is obtained. Moreover, assuming that there is no bias current so that $\Gamma(x) = 0$ and $\eta_y = 0$, the equation achieved is the following:

$$\varepsilon u_{xxt} + u_{xx} - u_{tt} - \varepsilon\lambda u_{xt} - \lambda u_x - a u_t = \sin u. \tag{8}$$

The current due to the tapering is represented by terms λu_x and $\lambda \varepsilon u_{xt}$. In particular λu_x characterizes the geometrical force driving the fluxons from the wide edge to the narrow edge. These junctions assure many advantages compared to rectangular ones, such as a voltage which is not chaotic anymore, but rather periodic excluding, in this way, some among the possible causes of large spectral width. It is also proved that the problem of trapped flux can be avoided (see f.i. [10]).

There exist numerous applications of Josephson junctions especially as superconducting quantum interference device (SQUID), which consists of a loop of superconductor with one or more Josephson junctions. These devices are one of the most important applications of superconductivity. They are basically extremely sensitive sensors of magnetic flux. This peculiarity allows to diagnose heart and/or blood circuit problems using magnetocardiograms and even to evaluate magnetic fields generated by electric currents in the brain using magnetoencephalography -MEG- [2]. SQUIDs are also used in nondestructive testing as a convenient alternative to ultra sound or x-ray methods (see [2] and reference therein). In geophysics, instead, they are used as gradiometers [3] or as gravitational wave detectors (see [4] and reference therein). SQUIDs play an important role in the study of the potential virtues of superconducting digital electronics, too [13].

2 Mathematical Models and Equivalences

All equations previously considered have something in common. More precisely, if one denotes by $\mathcal{L}$ the following linear third order parabolic operator:

$$\mathcal{L} = \varepsilon\partial_{xxt} - \partial_{tt} + \partial_{xx} - \alpha\partial_t, \tag{9}$$

(1)-(4) and (8) can be expressed by means of the unique equations:

$$\mathcal{L}u = f(x, t, u). \tag{10}$$

According to the meaning of f, numerous other examples of dissipative phenomena can be considered. For example, equation (10) arises in the motion of viscoelastic fluids or solids (see [14–17] and references therein) and in the study of viscoelastic plates with memory, when the relaxation function is given by an exponential function [18]. It can also be employed in the analysis of phase-change problems for an extended heat conduction model [19,20]. In addition, equation (10) arises also in heat conduction at low temperature [15, 21] and in the propagation of localized magnetohydrodinamic models in plasma physics [22]. Still, it is possible to find others in [23–26].

Then, an equivalence between the third order equation (10), typical of Josephson junctions, and biological phenomena has been pointed out in [27]. Indeed, let us consider the FitzHugh-Nagumo system (FHN) [28,29]:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon \dfrac{\partial^2 u}{\partial x^2} - v - a u + u^2 (a + 1 - u) \qquad (0 < a < 1), \\ \dfrac{\partial v}{\partial t} = b u - \beta v, \end{cases} \tag{11}$$

where $u(x, t)$ represents a membrane potential of a nerve axon at distance x and time t, and $v(x, t)$ is a recovery variable that models the transmembrane current.

This reaction-diffusion model characterizes the theory of the propagation of nerve impulses, and the connection between a third order equation like (10) and the (FHN) system can be realized changing the first one into the second one under continuous parameter variations [27].

An equation that is able to model all these physical problems has been introduced in [30] and it is represented by the following parabolic integro-differential equation:

$$\mathcal{L}_R\, u \equiv u_t - \varepsilon u_{xx} + au + b \int_0^t e^{-\beta(t-\tau)}\, u(x,\tau)\, d\tau \;=\; F(x,t,u). \tag{12}$$

Indeed, it has been proved that (12) characterizes both reaction diffusion models like the FitzHugh-Nagumo system and superconductive models [30–34].

In particular, perturbed Sine-Gordon equation (2) can be obtained by (12) as soon as one assumes

$$a \;=\; \alpha - \frac{1}{\varepsilon}, \qquad b = -\,\frac{a}{\varepsilon}, \qquad \beta \;=\frac{1}{\varepsilon} \tag{13}$$

and F is such that

$$F(x,t,u) \;=\; -\int_0^t e^{-\,\frac{1}{\varepsilon}\,(t-\tau\,)}\;[\;sen\,u(x,\tau)\;-\gamma\;]\;d\tau. \tag{14}$$

Furthermore, the integro-differential equation (12) is able to describe the evolution inside an exponentially shaped Josephson junction, too. Indeed, as it has already been underlined in [12], assuming

$$\beta\;=\frac{1}{\varepsilon}, \qquad b=\,\beta^2\,(1-\alpha\,\varepsilon), \qquad a\,\beta=\;\frac{\lambda^2}{4}\;-\,b, \tag{15}$$

$$F\;=\;-\int_0^t\; e^{-\,\frac{1}{\varepsilon}(t-\tau)} f_1(x,\tau,u)\,d\tau,$$

with

$$f_1 = \,e^{-\frac{\lambda}{2}\,x}\,[\,\sin\,(e^{\,x\,\lambda/2}\;u)\;-\gamma], \tag{16}$$

from the integro-differential equation (12) it follows:

$$\varepsilon u_{xxt}\;-\;u_{tt}\;+\;u_{xx} - \,(\alpha\;+\varepsilon\,\frac{\lambda^2}{4})u_t\;\;-\,\frac{\lambda^2}{4}\,u =\;f_1. \tag{17}$$

Therefore, assuming $e^{\frac{\lambda}{2}\,x}\;u = \bar{u},$ (17) turns into equation (8).

Remark: In (12) the kernel $e^{-\beta(t-\tau)}\,u(x,\tau)$ can be modified as physical situations demand and in this way many other physical phenomena could be described (see, f.i. [35–38] and references therein). The particular choice made here is due to describe the superconductive and biological models considered.

3 Mathematical Results

There exist many significant analytic results concerning the qualitative analysis of equations related to Josephson junctions and many initial-boundary problems have been discussed in a lot of papers (see [15, 39–43] and references therein).

A first analysis, where the fundamental solution is determined, concerns operator $\mathcal{L}$ in case $\alpha = 0$ [14,44]. Later, in [45,46], the fundamental solution of the whole operator $\mathcal{L}$ of (9) is explicitly determined and various properties are analyzed. Estimates and properties of continuous dependence for the solution of initial value problem are determined, too.

Moreover, in [47], in order to deduce an exhaustive asymptotic analysis, the Green function of the linear operator $\mathcal{L}$ of (9) has been determined by Fourier series and by means of its properties, an exponential decrease of solution related to the Dirichlet problem is deduced. And still by means of Fourier series, existence and uniqueness for Dirichlet, Neumann and pseudoperiodic initial-boundary conditions are achieved, too [42,43].

The Dirichlet problem is still considered with respect to equation (8) and in [11] the problem is reduced to an integral equation with kernel G endowed with rapid convergence and exponentially vanishing as t tends to infinity. Indeed, let

$$\gamma_n = \frac{n\pi}{l}, \qquad b_n = (\gamma_n^2 + \lambda^2/4\,), \quad g_n = \frac{1}{2}(\alpha + \varepsilon\, b_n\,), \quad \omega_n = \sqrt{g_n^2 - b_n} \tag{18}$$

and

$$G_n(t) = \frac{1}{\omega_n}\, e^{-g_n\, t}\, sinh(\omega_n t), \tag{19}$$

the Green function is given by

$$G(x,t,\xi) = \frac{2}{l}\, e^{\frac{\lambda}{2}\, x} \sum_{n=1}^{\infty} G_n(t) \;\; sin\gamma_n\xi \;\; sin\gamma_n x. \tag{20}$$

The initial boundary problem with Dirichlet conditions is analyzed and an appropriate analysis implies results on the existence and uniqueness of the solution.

That is, indicating by

$$\Omega_T \equiv \{\, (x,t) : \, 0 \le x \le L \,;\, 0 < t \le T\,\},$$

the following initial boundary problem

$$\begin{cases} (\partial_{xx} - \lambda\,\partial_x\,)\;(\varepsilon u_t + u) - \partial_t(u_t + \alpha\, u) = F(x,t,u), \quad (x,t) \in \Omega_T, \\ u(x,0) = h_0(x), \quad u_t(x,0) = h_1(x), \quad x \in [0,L], \\ u(0,t) = g_1(t), \quad u(l,t) = g_2(t), \quad 0 < t \le T. \end{cases} \tag{21}$$

for $g_1 = g_2 = 0$ admits the following integral equation:

$$u(x,t) = (\partial_t + \alpha + \varepsilon\,\lambda\partial_x - \varepsilon\partial_{xx}) \int_0^L h_0(\xi) e^{-\frac{\lambda\xi}{2}} G(x,\xi,t) d\xi \tag{22}$$

$$+ \int_0^L h_1(\xi) e^{-\frac{\lambda\xi}{2}} G(x,\xi,t) d\xi + \int_0^t d\tau \int_0^L G(x,\xi,t-\tau) e^{-\frac{\lambda\xi}{2}} F(\xi,\tau,u(\xi,\tau)) d\xi.$$

So, a priori estimates, continuous dependence and asymptotic behaviour of the solution, are deduced, too.

When boundary data are non null, in order to achieve explicit estimates of boundary contributions related to the Dirichlet problem, equivalence between the equation describing the evolution inside an (ESJJ) and the integro-differential equation (12) has been considered. Indeed, operator $\mathcal{L}_R$ of (12) has already been extensively examined in [30] and the fundamental solution K with many of its properties have been determined.

More in detail, if a, b, ε, β are positive constants, $r = |x|/\sqrt{\varepsilon}$ and $J_n(z)$ denotes the Bessel function of first kind and order n, let us consider the function

$$K(r,t) = \frac{e^{-\frac{r^2}{4t}}}{2\sqrt{\pi\varepsilon t}} e^{-at} - \frac{1}{2}\sqrt{\frac{b}{\pi\varepsilon}} \int_0^t \frac{e^{-\frac{r^2}{4y}-ay}}{\sqrt{t-y}} e^{-\beta(t-y)} J_1(2, \sqrt{by(t-y)})dy. \quad (23)$$

The following theorem has been proved:

Theorem 3.1 *The function* K *has the same basic properties of the fundamental solution of the heat equation, that is:* $K(x,t) \in C^\infty$ *for* $t > 0$, $x \in \Re$.

For fixed $t > 0$, K *and its derivatives are exponentially vanishing as fast as* $|x|$ *tends to infinity.*

For any fixed $\delta > 0$, *uniformly for all* $|x| \geq \delta$, *it results:*

$$\lim_{t\downarrow 0} K(x,t) = 0. \quad (24)$$

For $t > 0$, *it is* $\mathcal{L}_R K = 0$.

Moreover, it results

$$|K(x,t)| \leq \frac{e^{-\frac{x^2}{4\varepsilon t}}}{2\sqrt{\pi\varepsilon t}} \left[e^{-at} + bt \, \frac{e^{-at} - e^{-\beta t}}{\beta - a} \right]. \quad (25)$$

Previous estimates show, as well, that K exponentially decays to zero as t increases. These and other properties also allowed to prove in [12] numerous properties of the following function which is similar to *theta functions*:

$$\theta(x,t) = K(x,t) + \sum_{n=1}^{\infty} [K(x+2nL,t) + K(x-2nL,t)] = \sum_{n=-\infty}^{\infty} K(x+2nL,t). \quad (26)$$

So that, as for problem (21), denoting by

$$G(x,\xi,t) = \theta(|x-\xi|, t) - \theta(x+\xi, t)$$

and

$$F(x,t,u) = e^{-\frac{\lambda}{2}x} \left[\int_0^t e^{-\frac{1}{\varepsilon}(t-\tau)} [\sin(e^{x\lambda/2} u) - \gamma] \, d\tau - h_1(x) \, e^{-\frac{t}{\varepsilon}} \right],$$

it has been proved that the problem admits the following integral equation:

$$u(x,t) = \int_0^L G(x,\xi,t) e^{-\frac{\lambda}{2}x} h_0(\xi) d\xi + \int_0^t d\tau \int_0^L G(x,\xi,t) F(\xi,\tau,u(x,\tau)) d\xi \quad (27)$$

$$-2\varepsilon \int_0^t \theta_x(x, t-\tau) g_1(\tau) d\tau + 2\varepsilon \int_0^t \theta_x(x-L, t-\tau) e^{-\frac{\lambda L}{2}} g_2(\tau) d\tau.$$

Besides, a priori estimates and asymptotic properties have proved that when t tends to infinity, the effect due to the initial disturbances (h_0, h_1) is vanishing, while the effect of the non linear source is bounded for all t. Furthermore, for large t, the effects due to boundary disturbances g_1, g_2 are null or at least everywhere bounded.

Indeed, if $h_0 = h_1 = 0$ and $F = 0$, the following theorem holds:

Theorem 3.2 *When t tends to infinity and data g_i $(i = 1, 2)$ are two continuous functions convergent for large t, one has:*

$$u = g_{1,\infty} \frac{\sinh \sigma_0 \ (L - x)}{\sinh \ \sigma_0 L} + g_{2,\infty} \frac{\sinh \sigma_0 x}{\sinh \ \sigma_0 L}, \tag{28}$$

where $\sigma_0 = \frac{\lambda}{2}$ and $g_{i,\infty} = \lim_{t\to\infty} g_i$, $(i = 1, 2)$. Otherwise, when $\dot{g}_i \in L_1[0, \infty]$ $(i = 1, 2)$ too, the effects determined by boundary disturbance vanish.

Another aspect frequently highlighted in many papers is that the linear third order operator $\mathcal{L}$ is an example of wave operator perturbed by higher order viscous terms. The behaviour of solution of (10) when $\alpha = 0$, has been analyzed in various applications of artificial viscosity method [48, 49]. Moreover, in [50], when ε is vanishing, the interaction between diffusion effects and pure waves has been evaluated by means of slow time εt and fast times t/ε. These aspects are also analyzed in [16] referring to the strip problem for equation (10) with a linear source term f, while in the non-linear case, the Neumann boundary problem has been discussed in [51].

Also equation (8) can be considered as a semilinear hyperbolic equation perturbed by viscous terms described by higher-order derivatives with small diffusion coefficients ε. In [52], the influence of the dissipative terms has been estimated proving that they are both bounded when ε tends to zero and when time tends to infinity, giving a mathematical proof of what has been observed in [9].

As for explicit solutions, an extensive literature exists, and more recently, various classes of solutions for (SGE) have been determined (see, f.i., [53, 54]). Furthermore, when $\varepsilon = 0$, some travelling-wave solutions for (2) have been obtained both for $|\gamma|$ not larger than 1 and for $|\gamma| > 1$ [55, 56]. Still when $\varepsilon = 0$, some classes of explicit solutions have been determined for equation (8), too [52].

4 Open Problems

In light of what has been stated until now, many open problems can be highlighted.

It would be interesting, for example, to study equation (2) when interface conditions for the phase (and its normal gradient) are added, connecting, in this way, with the problems of window Josephson junctions (WJJ) when the influence of an external magnetic field must be considered [57]. Indeed, letting $\varepsilon = 0$, (2) exactly recalls one of the equations usually considered for (WJJ).

When, on the other hand, ε is not vanishing, a viscous term, represented by the third order term, appears. So that, it would be interesting to give an estimate of the diffusive effects due to the ε-term, too.

Moreover, according to the analogy between superconductor equations and reaction-diffusion models, the Robin boundary problem would be considered in order to achieve results for many biological phenomena, too [58, 59].

Besides, as for analysis on asymptotic effects due to the boundary perturbations related to equation (8), as it has been pointed out, the Dirichlet boundary problem has already been considered in [12]. So, the evaluation could be extended to other boundary problems, such as, for instance, Neumann and mixed ones.

Of course, in order to achieve estimates for other more significant physical problems, this analysis and many other estimates could be carried out for solution of equation (3) and for equations like (4) where the presence of a gap in the vacuum chamber is considered, too [41].

The analysis conducted so far required that in (12) constants a, b, ε, β were all positive. This can be valid if we look for an analogy with an (ESJJ), but excludes application of (12) to some other junctions. Therefore it would be interesting to extend the analysis of operator $\mathcal{L}_R$ for any value of a, b, ε, β.

Finally a qualitative analysis of operators should be made in case $\varepsilon, \alpha, \lambda$ were not constant.

5 Conclusion

The state of the art proves that many significant analytic results concerning the qualitative analysis of equations related to Josephson junctions have been obtained and many initial-boundary problems have been discussed. However other many important open problems may be considered and solved.

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