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Peakons and Soliton Solutions of Newly Developed Benjamin-Bona-Mahony-Like Equations

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Abstract: This paper establishes a set of Benjamin-Bona-Mahony-like equations (BBM-like) equations. By means of an advection dispersion equation, we can develop several BBM-like equations. We show that these established equations share some of the solitary wave solutions of the BBM equation. We also show that these developed equations give paekon solutions, for specific values of the parameter included in these equations, although these equations are not of the Camassa-Holm type of equations. We also derive a variety of solitonic solutions.

Keywords: *BBM-like equation; peakons; solitons.*

Mathematics Subject Classification (2010): 74D10, 74D30, 37G20, 34A45.

1 Introduction

Nonlinear equations have been a subject of intensive study for decades in several areas of mathematics, physics, engineering and other sciences. The study of these nonlinear equations has been the topic of major research projects in nonlinear sciences. Another interesting class of excitations consists of establishing nonlinear equations with significant physical features [1–10].

The KdV equation reads

$$u_t + uu_x + u_{xxx} = 0. \tag{1}$$

This equation models a variety of nonlinear wave phenomena such as shallow water waves, acoustic waves in a harmonic crystal, internal gravity waves in oceans, blood pressure pulses, and ion-acoustic waves in plasmas [1–7]. The KdV equation is completely integrable and admits multiple-soliton solutions and exhibits an infinite number of conserved

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quantities. The Korteweg-de Vries (KdV) equation was derived to describe shallow water waves of long wavelength and small amplitude. The KdV equation admits soliton solutions which have been the subject of intense study for the last few decades. Researchers remain intrigued by the physical properties of the KdV equation, in particular the complete integrability and the possess of an infinite number of conserved quantities.

While the KdV equation has remarkable properties [3], some other aspects of this equation are less favorable. This includes, e.g., an unbounded dispersion relation, that is obviously non-physical [3]. Several noticeable attempts to improve the KdV model were taken over the years. Benjamin-Bona-Mahony introduced the regularized long-wave equation, or the BBM equation that reads

$$u_t + u_x + uu_x - u_{xxt} = 0, \tag{2}$$

replaces the third-order derivative in the KdV equation (1) by a mixed derivative $-uxxt$, which, in turn, results in a bounded dispersion relation [3]. The BBM equation (2) can be used to describe the behavior of an undular bore, in water, which comprises a smooth wavefront followed by a train of solitary waves [6,7]. An undular bore can be interpreted as the dispersive analog of a shock wave in classical non-dispersive, dissipative hydrodynamics [7, 11-20].

Studies on nonlinear evolution equations are growing rapidly because these equations describe real features in science, technology, and engineering fields. In the past decades, a vast variety of powerful methods has been established to determine the exact solutions for these equations and to study the scientific features of these solutions from many points of view. Examples of these methods are the Hirota bilinear method [4], the simplified Hirota's method [6], the Bäcklund transformation method, Darboux transformation, Pfaffian technique, the inverse scattering method [4], the Painlevé analysis, the generalized symmetry method, the subsidiary ordinary differential equation method, and many other methods that can be found in [13–20].

The BBM equation is not integrable and admits one soliton solution given by

$$u(x,t) = -\frac{12k^2}{4k^2-1}\operatorname{sech}^2\left(kx + \frac{k}{4k^2-1}\,t\right), \tag{3}$$

where $k \neq \pm\frac{1}{2}$.

Moreover, the BBM equation has the singular solution

$$u(x,t) = \frac{12k^2}{4k^2-1}\operatorname{csch}^2\left(kx + \frac{k}{4k^2-1}\,t\right). \tag{4}$$

The present paper is aimed at the derivation of entirely new Benjamin-Bona-Mahony-like (BBM-like) equations that will give peakon solutions, i.e peak-shaped soliton solutions, in addition to other travelling wave solutions, although these equations are not of the Camassa-Holm type of equations. The derivation process, as will be seen later, leads to an infinite number of such equations. We will also show that these new forms share the solutions (3)–(4) with the BBM equation (2). To achieve our goals we will use several tools that will be applied in order to extract exact solutions.

2 Formulations of the BBM-Like Equations

In this section, we will establish a class of BBM-like equations with distinct structures. In a manner parallel to that used in [5], we introduce a generalized form of an advection dispersion equation as

$$u_t + Vu_x + \delta u_{xxt} = 0, \tag{5}$$

where δ is an arbitrary dimensionless parameter and $V(u, u_x, u_{xx}, \cdots)$ is an arbitrary function. We assume that the travelling wave

$$u(x,t) = f(x - ct) = f(\xi), \tag{6}$$

solves the BBM equation (2) and also solves the advection dispersion equation (5) for the same speed c. Using $\xi = x - ct$ transforms (2) and (5) to

$$-cf' + (1+f)f' - u_{xxt} = 0, \tag{7}$$

and

$$-cf' + Vf' + \delta u_{xxt} = 0, \tag{8}$$

respectively. Eliminating u_{xxt} from these two equations, and by noting that $f' \neq 0$, we obtain

$$V = (\delta+1)c - \delta(1+f) = (\delta+1)c - \delta(1+u). \tag{9}$$

The advection dispersion equations, or the BBM-like equations can be obtained by using a variety of values of the speed c, that can be obtained by integrating or differentiating (7) as many times as we want and if possible.

We first solve (7) for c where we find

$$c = 1 + u - \frac{u_{xxt}}{u_x}. \tag{10}$$

Substituting (10) into (9) gives

$$V = (\delta+1)(1 + u - \frac{u_{xxt}}{u_x}) - \delta(1+u). \tag{11}$$

Substituting (11) into the generalized advection dispersion equation (5) gives

$$u_t + \left\{(\delta+1)(1 + u - \frac{u_{xxt}}{u_x}) - \delta(1+u)\right\}u_x + \delta u_{xxt} = 0, \tag{12}$$

which gives the standard BBM equation (2) for any value of δ.

We now turn for the derivation of the BBM-like equations. Integrating (7) and solving for c we find

$$c = 1 + \frac{1}{2}u - \frac{u_{xt}}{u}. \tag{13}$$

Substituting for c from (13) into (9) gives

$$V_1 = (\delta+1)\left(1 + \frac{1}{2}u - \frac{u_{xt}}{u}\right) - \delta(1+u). \tag{14}$$

Inserting this result into the advection dispersion equation (5) gives

$$u_t + \left\{(\delta+1)\left(1 + \frac{1}{2}u - \frac{u_{xt}}{u}\right) - \delta(1+u)\right\}u_x + \delta u_{xxt} = 0, \tag{15}$$

that will be termed the first BBM-like equation.

To determine more values for the speed c, we can differentiate (7) as many times as we want. For example, differentiating (7) once and solving for c we find

$$c = 1 + u + \frac{u_x^2 - u_{xxxt}}{u_{xx}}, \tag{16}$$

and by differentiating (7) again and solving for c we obtain

$$c = 1 + u + \frac{3u_x u_{xx} - u_{xxxxt}}{u_{xxx}}. \tag{17}$$

Other values for c can be determined by differentiating (7) as many times as we want. Substituting (16) and (17) into (9) and simplifying one finds

$$V_2 = (\delta + 1)\left(1 + u + \frac{u_x^2 - u_{xxxt}}{u_{xx}}\right) - \delta(1 + u), \tag{18}$$

and

$$V_3 = (\delta + 1)\left(1 + u + \frac{3u_x u_{xx} - u_{xxxxt}}{u_{xxx}}\right) - \delta(1 + u). \tag{19}$$

Notice that V_2 and V_3 involve higher order derivatives than the dispersive term u_{xxt} of the BBM equation. Substituting V_2 and V_3 into (5) gives the following BBM-like equations

$$u_t + \left\{1 + u + (\delta + 1)\left(\frac{u_x^2 - u_{xxxt}}{u_{xx}}\right)\right\} u_x + \delta u_{xxt} = 0, \tag{20}$$

and

$$u_t + \left\{1 + u + (\delta + 1)\left(\frac{3u_x u_{xx} - u_{xxxxt}}{u_{xxx}}\right)\right\} u_x + \delta u_{xxt} = 0, \tag{21}$$

that will be termed the second and the third BBM-like equations respectively.

The first conclusion that we can make here is that the three derived BBM-like equations (15), (20) and (21) share the same soliton and singular solutions (3) and (4) that we derived earlier for the standard BBM equation (2).

Because our main concern of this work is to establish peakon solutions for the derived BBM-like equations, which are not of the CH or DP type, in addition to other solutions, we found that peakon solutions exist only for specific value of δ for each equation. Using selected values of δ for the equations (15), (20) and (21), we obtain the following specific BBM-like equations

$$u_t + \left\{1 - 2\frac{u_{xt}}{u}\right\} u_x + u_{xxt} = 0, \delta = 1, \tag{22}$$

$$u_t + \left\{1 + u - \left(\frac{u_x^2 - u_{xxxt}}{u_{xx}}\right)\right\} u_x - 2u_{xxt} = 0, \delta = -2, \tag{23}$$

and

$$u_t + \left\{1 + u - \frac{1}{3}\left(\frac{3u_x u_{xx} - u_{xxxxt}}{u_{xxx}}\right)\right\} u_x - \frac{4}{3}u_{xxt} = 0, \delta = -\frac{4}{3}. \tag{24}$$

In what follows we will employ distinct tools to derive exact solutions for each of the aforementioned forms (15), (20), and (21), that will be referred to as Form I, Form II, and Form III respectively. Recall that peakon solutions exist only for specific values of the parameter δ, whereas other solutions will be obtained for any selective value of δ.

3 The Nonlinear BBM-Like Equation: Form I

In this section we will study form I of the nonlinear BBM-like equation

$$u_t + \left\{(\delta+1)\left(1+\frac{1}{2}u-\frac{u_{xt}}{u}\right)-\delta(1+u)\right\}u_x+\delta u_{xxt}=0, \tag{25}$$

where we will derive peakon solutions and other travelling wave solutions.

3.1 Peakon solution

As stated before, we found that peakon solution exists for (25) only for $\delta = 1$, where (25) becomes

$$u_t + \left\{1-2\frac{u_{xt}}{u}\right\}u_x+u_{xxt}=0. \tag{26}$$

To determine a peakon solution to (26), we assume the peakon solution is of the form

$$u(x,t) = Re^{-|kx-ct|}. \tag{27}$$

Substituting this assumption into (26) we solve the resulting equation to find that

$$c = -\frac{k}{k^2-1},\ k \neq \pm 1, \tag{28}$$

and R can be any selective real number such as c. Consequently, the peakon solution is given by

$$u(x,t) = Re^{-|kx+\frac{k}{k^2-1}\,t|}. \tag{29}$$

Recall that the standard BBM equation gives soliton solutions but not peakon solutions.

Figure 1 below shows the peakon solution (29).

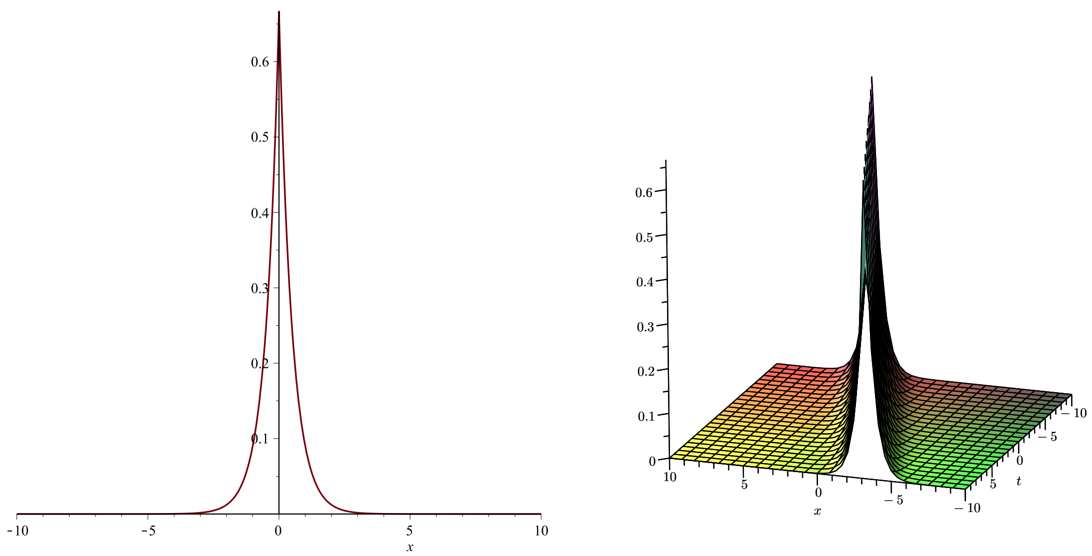

Figure 1: (a) Peakon solution with $R = \frac{2}{3}, k = 2,\ -10 \le x \le 10$,
(b) Peakon solution with $R = \frac{2}{3}, k = 2,\ -10 \le x, t \le 10$.

3.2 Soliton solutions

In this section, we will derive soliton solutions that satisfy the generalized BBM-like equation (25) for specific values of the parameter δ. For this reason, we assume that the solution for (25) has the form

$$u(x,t) = R + \operatorname{sech}^2(kx - ct). \tag{30}$$

Substituting this assumption into the nonlinear BBM-like equation (25), and solving the resulting equation for R and c, we find two sets of solutions given by

$$\begin{aligned} c &= \tfrac{k}{4k^2+1}, \\ R &= -\tfrac{2}{3}, \end{aligned} \tag{31}$$

valid for $\delta = 1$, and

$$\begin{aligned} c &= \tfrac{1}{12k}, \\ R &= \tfrac{1-16k^2}{12k^2}, \end{aligned} \tag{32}$$

valid for $\delta = -1$.

This in turn gives the soliton solutions

$$u(x,t) = -\frac{2}{3} + \operatorname{sech}^2(kx - \frac{k}{4k^2+1}t), \delta = 1, \tag{33}$$

and

$$u(x,t) = \frac{1-16k^2}{12k^2} + \operatorname{sech}^2(kx - \frac{1}{12k}t), \delta = -1. \tag{34}$$

We point out that the first solution justifies also the BBM equation, whereas the second solution satisfies only the BBM-like equation (25).

In a similar manner, we can derive the singular soliton solutions

$$u(x,t) = \frac{2}{3} + \operatorname{csch}^2(kx - \frac{k}{4k^2+1}t), \delta = 1. \tag{35}$$

and

$$u(x,t) = -\frac{1+8k^2}{12k^2} + \operatorname{csch}^2(kx + \frac{1}{12k}t), \delta = -1. \tag{36}$$

Unlike the previous results of the soliton solutions, the first singular soliton solution (35) satisfies the BBM-like equation (25), whereas the second solution (36) satisfies the BBM and the BBM-like equations.

3.3 Travelling waves solutions

In this section, we will derive more exact solutions that satisfy the generalized BBM-like equation (25), for specific values of the parameter δ. In what follows, we will present the approaches that will be used to derive these new solutions.

3.3.1 Solutions in the $\sec^2$ or $\csc^2$ form

We assume that the solution for (25) has the form

$$u(x,t) = R + \sec^2(kx - ct). \tag{37}$$

Substituting this assumption into the nonlinear BBM-like equation (25), and solving the resulting equation for R and c, we find two sets of solutions given by

$$\begin{aligned} c &= -\tfrac{k}{4k^2-1}, \\ R &= -\tfrac{2}{3}. \end{aligned} \tag{38}$$

valid for $\delta = 1$, and

$$\begin{aligned} c &= -\tfrac{1}{12k}, \\ R &= -\tfrac{1+16k^2}{12k^2}. \end{aligned} \tag{39}$$

valid for $\delta = -1$. This gives the exact solutions

$$u(x,t) = -\frac{2}{3} + \sec^2(kx + \frac{k}{4k^2-1}\,t), \delta = 1, \tag{40}$$

and

$$u(x,t) = -\frac{1+16k^2}{12k^2} + \sec^2(kx + \frac{1}{12k}\,t), \delta = -1, \tag{41}$$

3.3.2 Solutions in the $\sin^2$ or $\cos^2$ form

We assume that the solution for (25) has the form

$$u(x,t) = R + \sin^2(kx - ct). \tag{42}$$

Substituting this assumption into the nonlinear BBM-like equation (25), and solving the resulting equation for R and c, we find only one set of solutions given by

$$\begin{aligned} c &= -\tfrac{k}{4k^2+1}, \\ R &= -\tfrac{1}{2}. \end{aligned} \tag{43}$$

valid for $\delta = 1$ This gives the exact solution

$$u(x,t) = -\frac{1}{2} + \sec^2(kx + \frac{k}{4k^2+1}\,t), \delta = 1. \tag{44}$$

In a similar manner, we can derive the solution

$$u(x,t) = -\frac{1}{2} + \cos^2(kx + \frac{k}{4k^2+1}\,t), \delta = 1. \tag{45}$$

4 The Nonlinear BBM-Like Equation: Form II

In this section we will study form II of the nonlinear BBM-like equation

$$u_t + \left\{1 + u + (\delta+1)\frac{u_x^2 - u_{xxxt}}{u_{xx}}\right\} u_x + \delta u_{xxt} = 0, \tag{46}$$

where we will derive peakon solutions and other travelling wave solutions.

4.1 Peakon solution

As stated before, we found that peakon solution exists for (46) only for $\delta = -2$, where (46) becomes

$$u_t + \left\{1 + u - \frac{u_x^2 - u_{xxxt}}{u_{xx}}\right\} u_x - 2u_{xxt} = 0, \delta = -2, \tag{47}$$

To determine a peakon solution to (47), we assume the peakon solution is of the form

$$u(x,t) = Re^{-|kx-ct|}. \tag{48}$$

Substituting this assumption into (47) we solve the resulting equation to find that

$$c = -\frac{k}{k^2-1},\ k \neq \pm 1, \tag{49}$$

and R can be any selective real number such as c. Consequently, the peakon solution is given by

$$u(x,t) = Re^{-|kx+\frac{k}{k^2-1}\,t|}. \tag{50}$$

Recall that the standard BBM equation gives soliton solutions but not peakon solutions. Moreover, the obtained peakon solution (50) is identical to the peakon solution obtained earlier for the first form.

4.2 Soliton solutions

In this section, we will derive soliton solutions that satisfy the generalized BBM-like equation (46). For this reason, we assume that the solution for (46) has the form

$$u(x,t) = R + \operatorname{sech}^2(kx - ct). \tag{51}$$

Substituting this assumption into the nonlinear BBM-like equation (46), and solving the resulting equation for R and c, we find one set of solutions given by

$$\begin{array}{rcl} c &=& \frac{1}{12k}, \\ R &=& \frac{1-16k^2}{12k^2}, \end{array} \tag{52}$$

valid for any real value of δ.

This in turn gives the soliton solutions

$$u(x,t) = \frac{1-16k^2}{12k^2} + \operatorname{sech}^2(kx - \frac{1}{12k}\,t), \tag{53}$$

which also satisfies the BBM equation.

In a similar manner, we can derive the singular soliton solutions

$$u(x,t) = -\frac{1+8k^2}{12k^2} + \operatorname{csch}^2(kx + \frac{1}{12k}\,t), \tag{54}$$

which also satisfies the BBM equation.

4.3 Travelling waves solutions

In this section, we will derive more exact solutions that satisfy the generalized BBM-like equation (46). In what follows, we will present the approaches that will be used to derive these new solutions.

4.3.1 Solutions in the $\sec^2$ or $\csc^2$ form

We assume that the solution for (46) has the form

$$u(x,t) = R + \sec^2(kx - ct). \tag{55}$$

Substituting this assumption into the nonlinear BBM-like equation (46), and solving the resulting equation for R and c, we find one set of solutions given by

$$\begin{aligned} c &= -\tfrac{1}{12k}, \\ R &= -\tfrac{1+16k^2}{12k^2}. \end{aligned} \tag{56}$$

valid for any real value of δ. This gives the exact solutions

$$u(x,t) = -\frac{1+16k^2}{12k^2} + \sec^2(kx + \frac{1}{12k}\,t). \tag{57}$$

In a like manner, we can derive another exact solution of the form

$$u(x,t) = -\frac{1+16k^2}{12k^2} + \csc^2(kx + \frac{1}{12k}\,t). \tag{58}$$

5 The Nonlinear BBM-Like Equation: Form III

In this section we will study form III of the nonlinear BBM-like equation

$$u_t + \left\{1 + u + (\delta+1)\frac{3u_x u_{xx} - u_{xxxxt}}{u_{xxx}}\right\} u_x + \delta u_{xxt} = 0, \tag{59}$$

where we will derive peakon solutions and other travelling wave solutions.

5.1 Peakon solution

As stated before, we found that peakon solution exists for (59) only for $\delta = -\frac{4}{3}$, where (59) becomes

$$u_t + \left\{1 + u - \frac{1}{3}(\frac{3u_x u_{xx} - u_{xxxxt}}{u_{xxx}})\right\} u_x - \frac{4}{3}u_{xxt} = 0. \tag{60}$$

To determine a peakon solution to (60), we assume the peakon solution is of the form

$$u(x,t) = Re^{-|kx-ct|}. \tag{61}$$

Substituting this assumption into (60) we solve the resulting equation to find that

$$c = -\frac{k}{k^2-1},\ k \neq \pm 1, \tag{62}$$

and R can be any selective real number such as c. Consequently, the peakon solution is given by

$$u(x,t) = Re^{-|kx+\frac{k}{k^2-1}\,t|}. \tag{63}$$

It is obvious that although the three forms of the BBM-like equations differ in its structures, but all three models gave the same peakon solution.

5.2 Soliton solutions

In this section, we will derive soliton solutions that satisfy the generalized BBM-like equation (59) for specific values of the parameter δ. For this reason, we assume that the solution for (59) has the form

$$u(x,t) = R + \operatorname{sech}^2(kx - ct). \tag{64}$$

Substituting this assumption into the nonlinear BBM-like equation (59), and solving the resulting equation for R and c, we find two sets of solutions given by

$$\begin{aligned} c &= \tfrac{1}{12k}, \\ R &= \tfrac{1-16k^2}{12k^2}, \end{aligned} \tag{65}$$

valid for any real value of δ.

This in turn gives the soliton solutions

$$u(x,t) = \frac{1-16k^2}{12k^2} + \operatorname{sech}^2(kx - \frac{1}{12k}\,t), \delta = -1. \tag{66}$$

In a similar manner, we can derive the singular soliton solutions

$$u(x,t) = -\frac{1+8k^2}{12k^2} + \operatorname{csch}^2(kx + \frac{1}{12k}\,t), \delta = -1. \tag{67}$$

5.3 Travelling waves solutions

In this section, we will derive more exact solutions that satisfy the generalized BBM-like equation (59), for specific values of the parameter δ. In what follows, we will present the approaches that will be used to derive these new solutions.

5.3.1 Solutions in the $\sec^2$ or $\csc^2$ form

We assume that the solution for (59) has the form

$$u(x,t) = R + \sec^2(kx - ct). \tag{68}$$

Substituting this assumption into the nonlinear BBM-like equation (59), gives the same solution obtained before for form II, namely

$$u(x,t) = -\frac{1+16k^2}{12k^2} + \sec^2(kx + \frac{1}{12k}\,t), \tag{69}$$

and

$$u(x,t) = -\frac{1+16k^2}{12k^2} + \csc^2(kx + \frac{1}{12k}\,t). \tag{70}$$

5.3.2 Solutions in the $\sin^2$ or $\cos^2$ form

We assume that the solution for (59) has the form

$$u(x,t) = R + \sin^2(kx - ct). \tag{71}$$

Substituting this assumption into the nonlinear BBM-like equation (59), and solving the resulting equation for R and c, we find only one set of solutions given by

$$\begin{aligned} \delta &= -\tfrac{4}{3}, \\ c &= \tfrac{k(3+2R}{2(4k^2+1)}, \end{aligned} \tag{72}$$

where R is left as a free parameter. This gives the exact solution

$$u(x,t) = R + \sin^2(kx + \frac{k}{4k^2+1}\,t), \tag{73}$$

In a similar manner, we can derive the solution

$$u(x,t) = R + \cos^2(kx + \frac{k}{4k^2+1}\,t). \tag{74}$$

6 Conclusion

In this work we established three (BBM-like) equations that share some of the solitary wave solutions with the standard BBM equation. We showed that these forms, although are not of the same type as the CH or DP list of equations, but give peakon solutions provided that each form has specific value of the parameter δ included in the equation. This shows that the developed BBM-like equations can model solitary wave solutions and peaked solitary waves solutions. In addition, the developed equations contain terms with higher derivatives than the third-order term u_{xxt}.

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