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Delay Independent Stability of Co-operative and Supportive Neural Networks

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Abstract: In this paper a cooperative and supportive neural network proposed recently is considered. Time delays both in transmission of information from subsystems to main system as well as processing of information in subsystem itself are introduced into the network. Criteria on parameters of the system are obtained that establish the stability of the system independent of time delays. Examples are provided for illustration of results.

Keywords: *cooperative and supportive networks; time delays; equilibria; global stability.*

Mathematics Subject Classification (2010): 34D23, 34K20, 92B20, 93D20.

1 Introduction

Neural networks has been a subject of research for decades with growing popularity ([2], [6-11]), for its extensive application in several real world situations ([1], [3], [12-17], [21]). In [20], a new class of networks designated as co-operative and supportive neural network (CSNN, for short) was introduced. The model is suitable for explaining the dynamics of systems exhibiting hierarchy in which the collective capabilities of components involved are utilized for better performance of the system. Such systems find application in industrial information management, financial and economic systems which involve distribution

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and monitoring of various tasks. They are also useful in solving complex network problems [2], classification and clustering problems, in data mining and financial engineering [5]. They are also utilized for parameter estimation of auto regressive signals and to decompose complex classification tasks into simpler subtasks and puzzle them out. In particular, the network considered in the present study is utilized for estimation of key parameters in infectious disease models [18]. The reliability aspects of this network are studied in [14].

The model comprises two neuronal fields say F_x and F_y. Each neuron in F_x is denoted by x_i, $i = 1, 2, ..., n$ and is connected to other neurons x_j, $j = 1, 2, ..., n$ in the same field F_x. Also each x_i is connected to r_i number of neurons in the neuronal field F_y. These are denoted by y_{i_k}, $k = 1, 2, ..., r_i$, $1 \leq r_i \leq n$. These y_{i_k} support x_i in the sense that they coordinate and cooperate with it so that any task assigned to them by x_i will be attended to. The dynamics of the model is described by the following system of equations

$$\begin{aligned} x_i' &= -a_i x_i + \sum_{j=1}^{n} b_{ij} f_j(x_j) + \sum_{k=1}^{r_i} c_{ii_k} g_{i_k}(x_i, y_{i_k}) + I_i, i = 1, 2, ..., n, \\ y_{i_k}' &= -c_{i_k} y_{i_k} + \sum_{l=1}^{r_i} d_{i_l} h_{i_l}(y_{i_l}) + J_{i_k}, \quad k = 1, 2, ..., r_i, \quad 1 \leq r_i \leq n. \end{aligned} \tag{1}$$

In (1), x_i, i=1,2,...,n denotes a typical neuron in F_x and $y_{i_k}, k = 1, 2, ...r_i$ denotes a subgroup of neurons in F_y attached to x_i. $' = \frac{d}{dt}$ denotes the derivative with respect to time variable t. a_i and c_{i_k} are positive constants known as decay rates and b_{ij}, d_{i_l} are the synaptic connection weights for all $i, j = 1, 2, \cdots, n$, $k = 1, 2, ..., r_i$ and are assumed to be real or complex constants. c_{ii_k} is the rate of distribution of information between x_i and y_{i_k}. The functions f_i, g_{i_k} and h_{i_k} are the neuronal output response functions and are more commonly known as the signal functions. I_i, J_{i_k} are exogenous inputs.

We may use the terms main component or task or group element for x_i and sub-component or task or group element for y_{i_k} synonymous with neuron, owing to the application for which system (1) is utilized. For example, (1) may be viewed as a management information system in which x_i are in layer (say managerial or lead group) monitoring the activities of related subgroups of y_{i_k}. Thus, (1) represents both (i) hierarchical systems in which x_i can wait till y_{i_k} complete their task and return to x_i (serial processing) and (ii) coordinating systems where x_i also work along with y_{i_k} to complete their part (parallel processing).

Several modifications of (1) are suggested in [20] that take care of interactions among the neurons as well as time delays. These models are left as open problems for further research. Two types of delays are common in such systems. First one is the time delay in transferring information/completed task from y_{i_k} to x_i, called transmission or propagation delay and the second is the one that occurs while carrying out the job by y_{i_k} themselves, namely, processing delay. Introduction of these two types of time delays into the system modifies (1) as following

$$\begin{aligned} x_i' &= -a_i x_i + \sum_{j=1}^{n} b_{ij} f_j(x_j) + \sum_{k=1}^{r_i} c_{ii_k} g_{i_k}(x_i, y_{i_k}(t - \tau_{i_k})) + I_i, \\ y_{i_k}' &= -c_{i_k} y_{i_k} + \sum_{l=1}^{r_i} d_{i_l} h_{i_l}(y_{i_l}(t - \tau_{i_l})) + J_{i_k}. \end{aligned} \tag{2}$$

Here $i = 1, 2, ..., n$, $k = 1, 2, ..., r_i$ and $1 \leq r_i \leq n$.

In (2) $\tau_{i_k} \geq 0$ denote delays in transmission of data/material from sub-system y_{i_k} to the main system while $\tau_{i_l} \geq 0$ denote processing delays with subcomponents. The present paper studies the qualitative behaviour of the solutions of (2) under the influence of time delays. The present study is important in the context of established influence of time delays on neural network systems and any physical system. (2) is quite general in the sense that it includes several modifications of (1) suggested in [20]. In fact (2) combines the models III and IV of [20].

The paper is organized as follows. For the system (2) we establish the conditions of existence and uniqueness of solutions, equilibria in Section 2. Different Lyapunov functionals are utilized to establish stability of equilibria in Section 3. Examples are provided for an illustration of the results. Finally the paper is concluded with a discussion in Section 4.

2 Existence of Solutions and Equilibria

From the theory of delay differential equations, local Lipschitz condition on the response functions (f_i, g_{i_k} and h_{i_k}) which are at least continuous in their domains of definitions, guarantees the existence of solutions to (2) (see [4,19,20]). However it is useful for researchers to note that conditions weaker than Lipschitz condition on these response functions that guarantee the existence of unique solutions to such systems are also available in literature (e.g., [19]). Thus, we may choose f_i, g_{i_k} and h_{i_k} from a very general class of functions. With this background, we tacitly assume that the system (2) possesses unique solutions that are continuable in their maximal intervals of existence. However, we need the following Lipschitz conditions on these functions to establish the existence of equilibria and their stability in subsequent sections:

$$\begin{aligned} \| g_{i_k}(x_i, y_{i_k}) - g_{i_k}(\overline{x}_i, \overline{y}_{i_k}) \| &\leq M_{1i_k}|y_{i_k} - \overline{y}_{i_k}| + M_{2i_k}|x_i - \overline{x}_i|, && (3)\\ |f_j(x_j) - f_j(\overline{x}_j)| &\leq p_j|x_j - \overline{x}_j|, && (4)\\ |h_{i_k}(y_{i_k}) - h_{i_k}(\overline{y}_{i_k})| &\leq q_{i_k}|y_{i_k} - \overline{y}_{i_k}|, && (5)\end{aligned}$$

where $M_{1i_k},\ M_{2i_k},\ p_j$ *and* q_{i_k} are positive constants.

Since time delays do not disturb the presence of equilibria, as in [20], we have

Theorem 2.1. Let a_i *and* c_{i_k} $(i = 1$ *to* $n,\ k = 1$ *to* $r_i)$ be positive numbers such that

$$\begin{aligned} \sum_{j=1}^{n} |b_{ij}|p_j + \sum_{k=1}^{r_i} |c_{ii_k}|M_{2i_k} &< a_i, i = 1, 2, ..., n,\\ \sum_{l=1}^{r_i} |d_{i_l}|q_{i_l} + \sum_{k=1}^{r_i} |c_{ii_k}|M_{1i_k} &< c_{i_k}, \quad k = 1, 2, ..., r_i, \quad 1 \leq r_i \leq n. && (6)\end{aligned}$$

Then the system (2) possesses a unique positive equilibrium for each $i,\ k$. If we denote this equilibrium solution of (2) by $(x_i^*, y_{i_k}^*)$, then we should have

$$\begin{aligned} a_i x_i^* &= \sum_{j=1}^{n} b_{ij} f_j(x_j^*) + \sum_{k=1}^{r_i} c_{ii_k} g_{i_k}(x_i^*, y_{i_k}^*) + I_i, i = 1, 2, ..., n.\\ c_{i_k} y_{i_k}^* &= \sum_{l=1}^{r_i} d_{i_l} h_{i_l}(y_{i_l}^*) + J_{i_k}, \quad k = 1, 2, ..., r_i, \quad 1 \leq r_i \leq n. && (7)\end{aligned}$$

We shall now proceed to the stability of this unique equilibrium whose existence is ensured by Theorem 2.1.

3 Global Stability Results

In this section we shall establish criteria for the global asymptotic stability of the equilibrium patterns of system (2). The conditions for global stability of (1) are presented in [20]. We shall see how the presence of time delays influences the stability here in the context that time delays have tendency of disturbing the stability by introducing oscillations into the system. We begin with

Case 1. No processing delays within sub components:

We start with a special case of (2) in which we assume that $\tau_{i_l} = 0$ for all i_l. This means that we are considering a state when the sub components finish their part of job without any delay as required by x_i. However the system is characterized by the delays (i.e., $\tau_{i_k} \geq 0$) in transmission of these outcomes to main system.

We need the following inequality for our first result.

For all real numbers u, v and $\eta > 0$ we have

$$uv \leq \frac{1}{4\eta}u^2 + \eta v^2. \tag{8}$$

Theorem 3.1. *Assume that conditions (3)-(5) hold. The equilibrium* $(x_i^*, y_{i_k}^*)$ *of (2) is globally asymptotically stable for any length of time delays* $\tau_{i_k} \geq 0$, *for* $i = 1, 2, ..., n$ *and* $k = 1, 2, ..., r_i$, *provided the parameters satisfy any of the following sets of inequalities:*

$$\begin{aligned} a).\ \sum_{j=1}^{n} |b_{ij}| p_j \frac{1}{4\eta_1} + \sum_{j=1}^{n} |b_{ji}| p_i \eta_1 + \sum_{k=1}^{r_i} |c_{ii_k}|(M_{2i_k} + M_{1i_k}\eta_2) &< a_i, \\ |c_{ii_k}| M_{1i_k} \frac{1}{4\eta_2} + \eta_3 \sum_{k=1}^{r_i} |d_{i_k}| q_{i_k}| + \frac{1}{4\eta_3} \sum_{k=1}^{r_i} |d_{i_k}| q_{i_k}| &< c_{i_k}, \end{aligned}$$

$$\begin{aligned} b).\ \sum_{j=1}^{n} |b_{ij}| p_j^2 \frac{1}{4\eta_1} + \sum_{j=1}^{n} |b_{ji}| \eta_1 + \sum_{k=1}^{r_i} |c_{ii_k}|(M_{2i_k} + M_{1i_k}\eta_2) &< a_i, \\ |c_{ii_k}| M_{1i_k} \frac{1}{4\eta_2} + \eta_3 \sum_{k=1}^{r_i} |d_{i_k}| q_{i_k}| + \frac{1}{4\eta_3} \sum_{k=1}^{r_i} |d_{i_k}| q_{i_k}| &< c_{i_k}, \end{aligned}$$

$$\begin{aligned} c).\ \sum_{j=1}^{n} |b_{ij}| \frac{1}{4\eta_1} + \sum_{j=1}^{n} |b_{ji}| p_i^2 \eta_1 + \sum_{k=1}^{r_i} |c_{ii_k}|(M_{2i_k} + M_{1i_k}\eta_2) &< a_i, \\ |c_{ii_k}| M_{1i_k} \frac{1}{4\eta_2} + \eta_3 \sum_{k=1}^{r_i} |d_{i_k}| q_{i_k}| + \frac{1}{4\eta_3} \sum_{k=1}^{r_i} |d_{i_k}| q_{i_k}| &< c_{i_k}, \end{aligned}$$

where η_1, η_2 *and* η_3 *are positive parameters chosen appropriately.*

Proof. We construct a Lyapunov functional suitable for our purpose here. We first consider

$$V_1(t) = \sum_{i=1}^{n} \left\{ \frac{(x_i(t) - x_i^*)^2}{2} \right\}.$$

The derivative of V_1 along the solutions of (2), using (7), is given by

$$
\begin{aligned}
V_1'(t) &= \sum_{i=1}^{n}\Big\{(x_i(t)-x_i^*)(x_i'(t)-x_i^{*'})\Big\} \\
&= \sum_{i=1}^{n}\Big\{(x_i(t)-x_i^*)\Big\{-a_i(x_i(t)-x_i^*)+\sum_{j=1}^{n} b_{ij}(f_j(x_j)-f_j(x_j^*)) \\
&\quad +\sum_{k=1}^{r_i} c_{ii_k}(g_{i_k}(x_i,y_{i_k}(t-\tau_{i_k}))-g_{i_k}(x_i^*,y_{i_k}^*))\Big\}\Big\} \\
&\le \sum_{i=1}^{n}\Big\{\Big\{-a_i(x_i(t)-x_i^*)^2+|x_i(t)-x_i^*|\sum_{j=1}^{n}|b_{ij}|p_j|x_j(t)-x_j^*| \\
&\quad +|x_i(t)-x_i^*|\sum_{k=1}^{r_i}|c_{ii_k}|\Big[M_{2i_k}|x_i-x_i^*|+M_{1i_k}|y_{i_k}(t-\tau_{i_k}))-y_{i_k}^*|\Big]\Big\}\Big\},
\end{aligned}
$$

utilizing (4),(3) for the last two terms respectively.

We utilize the inequality (8) for $\eta=\eta_1$ and $\eta=\eta_2$ in the second and fourth terms of the above inequality to get

$$
\begin{aligned}
p_j|x_i(t)-x_i^*||x_j-x_j^*| &\le p_j\Big[\frac{1}{4\eta_1}(x_i(t)-x_i^*)^2+\eta_1(x_j-x_j^*)^2\Big], \\
|y_{i_k}(t-\tau_{i_k}))-y_{i_k}^*||x_i-x_i^*| &\le \Big[\frac{1}{4\eta_2}(y_{i_k}(t-\tau_{i_k})-y_{i_k}^*)^2+\eta_2(x_i-x_i^*)^2\Big]. \qquad (9)
\end{aligned}
$$

Then we have

$$
\begin{aligned}
V_1'(t) &\le \sum_{i=1}^{n}\Big[-a_i(x_i(t)-x_i^*)^2+\sum_{j=1}^{n}|b_{ij}|p_j\Big[\frac{1}{4\eta_1}(x_i(t)-x_i^*)^2+\eta_1(x_j-x_j^*)^2\Big] \\
&\quad +\sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k}(x_i-x_i^*)^2 \\
&\quad +\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\Big[\frac{1}{4\eta_2}(y_{i_k}(t-\tau_{i_k})-y_{i_k}^*)^2+\eta_2(x_i-x_i^*)^2\Big]\Big] \\
&= -\sum_{i=1}^{n}\Big[a_i-\sum_{j=1}^{n}|b_{ij}|p_j\frac{1}{4\eta_1}-\sum_{j=1}^{n}|b_{ji}|p_i\eta_1-\sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k} \\
&\quad -\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\eta_2\Big](x_i-x_i^*)^2+\sum_{i=1}^{n}\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\frac{1}{4\eta_2}(y_{i_k}(t-\tau_{i_k})-y_{i_k}^*)^2. \qquad (10)
\end{aligned}
$$

Now define

$$V_2(t)=\sum_{i=1}^{n}\sum_{k=1}^{r_i}\frac{(y_{i_k}(t)-y_{i_k}^*)^2}{2}.$$

Then along the solutions of (2) we have

$$\begin{aligned}
V_2'(t) &= \sum_{i=1}^{n}\sum_{k=1}^{r_i}(y_{i_k}(t)-y_{i_k}^*)(y_{i_k}'(t)-y_{i_k}^{*'}) \\
&= \sum_{i=1}^{n}\sum_{k=1}^{r_i}(y_{i_k}(t)-y_{i_k}^*)\Big[-c_{i_k}(y_{i_k}(t)-y_{i_k}^*)+\sum_{l=1}^{r_i}d_{i_l}[h_{i_l}(y_{i_l})-h_{i_l}(y_{i_l}^*)]\Big] \\
&\le \sum_{i=1}^{n}\sum_{k=1}^{r_i}\Big[-c_{i_k}(y_{i_k}(t)-y_{i_k}^*)^2+|y_{i_k}(t)-y_{i_k}^*|\sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}|y_{i_l}-y_{i_l}^*|\Big] \\
&\le -\sum_{i=1}^{n}\sum_{k=1}^{r_i}\Big[c_{i_k}-\frac{1}{4\eta_3}\sum_{k=1}^{r_i}|d_{i_k}|q_{i_k}-\eta_3\sum_{k=1}^{r_i}|d_{i_k}|q_{i_k}\Big](y_{i_k}-y_{i_k}^*)^2, \qquad (11)
\end{aligned}$$

again utilizing the inequality (8) for $\eta=\eta_3>0$. Now consider

$$V_3(t) = \sum_{i=1}^{n}\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\frac{1}{4\eta_2}\int_{t-\tau_{i_k}}^{t}(y_{i_k}(z)-y_{i_k}^*)^2dz.$$

Then we have

$$V_3'(t)=\sum_{i=1}^{n}\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\frac{1}{4\eta_2}(y_{i_k}(t)-y_{i_k}^*)^2-\sum_{i=1}^{n}\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\frac{1}{4\eta_2}(y_{i_k}(t-\tau_{i_k})-y_{i_k}^*)^2. \qquad (12)$$

We now define our Lyapunov functional by $V(t)=V_1(t)+V_2(t)+V_3(t)$. Then along the solutions of (2) utilizing (10),(11) and (12), we get

$$\begin{aligned}
V'(t) &= V_1'(t)+V_2'(t)+V_3'(t) \\
&\le -\sum_{i=1}^{n}\Big[a_i-\sum_{j=1}^{n}|b_{ij}|p_j\frac{1}{4\eta_1}-\sum_{j=1}^{n}|b_{ji}|p_i\eta_1-\sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k} \\
&\quad -\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\eta_2\Big](x_i-x_i^*)^2+\sum_{i=1}^{n}\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\frac{1}{4\eta_2}(y_{i_k}(t-\tau_{i_k})-y_{i_k}^*)^2 \\
&\quad -\sum_{i=1}^{n}\sum_{k=1}^{r_i}\Big[c_{i_k}-\eta_3\sum_{k=1}^{r_i}|d_{i_k}|q_{i_k}-\frac{1}{4\eta_3}\sum_{k=1}^{r_i}|d_{i_k}|q_{i_k}\Big](y_{i_k}(t)-y_{i_k}^*)^2 \\
&\quad +\sum_{i=1}^{n}\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\frac{1}{4\eta_2}(y_{i_k}(t)-y_{i_k}^*)^2 \\
&\quad -\sum_{i=1}^{n}\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\frac{1}{4\eta_2}(y_{i_k}(t-\tau_{i_k})-y_{i_k}^*)^2 \\
&= -\sum_{i=1}^{n}\Big[[a_i-\sum_{j=1}^{n}|b_{ij}|p_j\frac{1}{4\eta_1}-\sum_{i=1}^{n}|b_{ji}|p_i\eta_1-\sum_{k=1}^{r_i}|c_{ii_k}|(M_{2i_k}+M_{1i_k}\eta_2)\Big](x_i-x_i^*)^2 \\
&\quad -\sum_{i=1}^{n}\sum_{k=1}^{r_i}\Big[c_{i_k}-\eta_3\sum_{k=1}^{r_i}|d_{i_k}|q_{i_k}-\frac{1}{4\eta_3}\sum_{k=1}^{r_i}|d_{i_k}|q_{i_k}-|c_{ii_k}|M_{1i_k}\frac{1}{4\eta_2}\Big](y_{i_k}(t)-y_{i_k}^*)^2
\end{aligned}$$

If we choose

$$\begin{aligned} a_i &> \sum_{j=1}^{n}|b_{ij}|p_j\frac{1}{4\eta_1}+\sum_{j=1}^{n}|b_{ji}|p_i\eta_1+\sum_{k=1}^{r_i}|c_{ii_k}|(M_{2i_k}+M_{1i_k}\eta_2), \\ c_{i_k} &> |c_{ii_k}|M_{1i_k}\frac{1}{4\eta_2}+\eta_3\sum_{k=1}^{r_i}|d_{i_k}|q_{i_k}+\frac{1}{4\eta_3}\sum_{k=1}^{r_i}|d_{i_k}|q_{i_k}, \end{aligned}$$

as in assumption (a), then we have

$$V'(t)<0.$$

Clearly V has all the properties of a Lyapunov functional to serve our purpose here. Rest of argument may be followed as in [4] or [19]. Hence $(x_i(t),\ y_{i_k}(t))$ converges to $(x_i^*,\ y_{i_k}^*)$ as $t\to\infty$.

The other two cases (b) and (c) may be proved on similar lines using the inequalities

$$\begin{aligned} p_j|x_i(t)-x_i^*||x_j-x_j^*| &\le \Big[\frac{p_j^2}{4\eta_1}(x_i(t)-x_i^*)^2+\eta_1(x_j(t)-x_j^*)^2\Big], \\ p_j|x_i(t)-x_i^*||x_j-x_j^*| &\le \Big[\frac{1}{4\eta_1}(x_i(t)-x_i^*)^2+\eta_1 p_j^2(x_j(t)-x_j^*)^2\Big], \end{aligned}$$

respectively in place of (9). The proof is complete.

The two-delay system:

We shall now consider the general case of (2) in which we assume delays both in transmission of information from and processing of information within subcomponents. The following result establishes sufficient conditions for the global asymptotic stability of equilibrium solution for this case.

Theorem 3.2. *Assume that the parameters of the system (2) satisfy the following conditions:*

$$a_i>\sum_{j=1}^{n}|b_{ji}|p_i+\sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k},\quad c_{i_k}>\sum_{k=1}^{r_i}|d_{i_k}|q_{i_k}+\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k},$$

for all $k=1,2,...,r_i,\quad 1\le r_i\le n,\ i=1,2,...,n$. *Then the equilibrium* $(x_i^*,y_{i_k}^*)$ *is globally asymptotically stable independent of delays in the sense that all solutions of (2) satisfy the convergence requirement*

$$\lim_{t\to\infty}y_{i_k}\to y_{i_k}^*,\quad \lim_{t\to\infty}x_i\to x_i^*.$$

Proof. Utilizing (7) in (2), we rewrite (2) as

$$\begin{aligned} (x_i-x_i^*)' &= -a_i(x_i-x_i^*)+\sum_{j=1}^{n}b_{ij}[f_j(x_j)-f_j(x_j^*)] \\ &\quad +\sum_{k=1}^{r_i}c_{ii_k}[g_{i_k}(x_i,y_{i_k}(t-\tau_{i_k}))-g_{i_k}(x_i^*,y_{i_k}^*)], \\ (y_{i_k}-y_{i_k}^*)' &= -c_{i_k}(y_{i_k}-y_{i_k}^*)+\sum_{l=1}^{r_i}d_{i_l}[h_{i_l}(y_{i_l}(t-\tau_{i_l}))-h_{i_l}(y_{i_l}^*)]. \end{aligned}$$

We employ the functional

$$\begin{aligned} V(t) &= \sum_{i=1}^{n}\Big[|x_i - x_i^*| + |y_{i_k} - y_{i_k}^*| + \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\int_{t-\tau_{i_k}}^{t}|y_{i_k}(s) - y_{i_k}^*|ds \\ &\quad + \sum_{l=1}^{r_i}|d_{i_l}|\int_{t-\tau_{i_l}}^{t}|h_{i_l}(y_{i_l}(s) - h_{i_l}(y^*{}_{i_l})|ds\Big], \end{aligned} \tag{13}$$

$$\begin{aligned} D^+V(t) &\leq \sum_{i=1}^{n}\Big[-a_i|x_i - x_i^*| + \sum_{j=1}^{n}|b_{ij}||f_j(x_j) - f_j(x_j^*)| \\ &\quad + \sum_{k=1}^{r_i}|c_{ii_k}||g_{i_k}(x_i, y_{i_k}(t-\tau_{i_k})) - g_{i_k}(x_i, y_{i_k}^*) + g_{i_k}(x_i, y_{i_k}^*) - g_{i_k}(x_i^*, y_{i_k}^*)| \\ &\quad + \Big[-c_{i_k}|y_{i_k} - y_{i_k}^*| + \sum_{l=1}^{r_i}|d_{i_l}|h_{i_l}(y_{i_l}(t-\tau_{i_l})) - h_{i_l}(y_{i_l}^*)|\Big] \\ &\quad + \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}|y_{i_k} - y_{i_k}^*| - \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}|y_{i_k}(t-\tau_{i_k}) - y_{i_k}^*| \\ &\quad + \sum_{l=1}^{r_i}|d_{i_l}|h_{i_l}(y_{i_l}(t)) - h_{i_l}(y_{i_l}^*)| - \sum_{l=1}^{r_i}|d_{i_l}|h_{i_l}(y_{i_l}(t-\tau_{i_l})) - h_{i_l}(y_{i_l}^*)|\Big], \end{aligned}$$

$$\begin{aligned} D^+V(t) &\leq \sum_{i=1}^{n}\Big[-a_i|x_i - x_i^*| + \sum_{j=1}^{n}|b_{ij}|p_j|x_j - x_j^*| \\ &\quad + \sum_{k=1}^{r_i}|c_{ii_k}||g_{i_k}(x_i, y_{i_k}(t-\tau_{i_k})) - g_{i_k}(x_i, y_{i_k}^*)| \\ &\quad + \sum_{k=1}^{r_i}|c_{ii_k}||g_{i_k}(x_i, y_{i_k}^*) - g_{i_k}(x_i^*, y_{i_k}^*)| \\ &\quad - c_{i_k}|y_{i_k} - y_{i_k}^*| + \sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}|y_{i_l} - y_{i_l}^*| \\ &\quad + \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}|y_{i_k} - y_{i_k}^*| - \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}|y_{i_k}(t-\tau_{i_k}) - y_{i_k}^*|\Big], \end{aligned}$$

$$\begin{aligned} D^+V(t) &\leq \sum_{i=1}^{n}\Big[-a_i|x_i - x_i^*| + \sum_{j=1}^{n}|b_{ij}|p_j|x_j - x_j^*| \\ &\quad + \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}|y_{i_k}(t-\tau_{i_k}) - y_{i_k}^*)| + \sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k}|x_i - x_i^*| \\ &\quad - c_{i_k}|y_{i_k} - y_{i_k}^*| + \sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}|y_{i_l} - y_{i_l}^*| \\ &\quad + \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}|y_{i_k} - y_{i_k}^*| - \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}|y_{i_k}(t-\tau_{i_k}) - y_{i_k}^*|\Big], \end{aligned}$$

$$
\begin{aligned}
D^+V(t) &\leq -\sum_{i=1}^{n}\Big[[a_i - \sum_{j=1}^{n}|b_{ji}|p_i - \sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k}]|x_i - x_i^*| \\
&\quad +[c_{i_k} - \sum_{k=1}^{r_i}|d_{i_k}|q_{i_k} - \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}]|y_{i_k} - y_{i_k}^*|\Big] \\
&\leq -\widetilde{A}\sum_{i=1}^{n}\Big[|x_i - x_i^*| + |y_{i_k} - y_{i_k}^*|\Big] \\
&< 0,
\end{aligned}
$$

where $\widetilde{A} = \min\{\overline{A}, \overline{B}\} > 0$, in which

$$
\overline{A} = \min_{1\leq i\leq n}\Big\{a_i - \sum_{j=1}^{n}|b_{ji}|p_i - \sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k}\Big\} > 0,
$$

$$
\overline{B} = min_{1\leq i\leq n}\Big\{c_{i_k} - \sum_{k=1}^{r_i}|d_{i_k}|q_{i_k} - \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\Big\} > 0.
$$

It is clear that V is the required Lyapunov functional and rest of the proof may be completed employing standard arguments (see e.g., [4,20]).

Remark 3.3. Stability of system (2) may be studied in two ways. Firstly, the subsystem $\{y_{i_k}\}$ may converge first and x_i converge then. In this case, the x_i wait for information for any length of time from their subsystems and finish the task only after y_{i_k} come up with their contribution. In the second case, x_i work along with subsystem y_{i_k} simultaneously to finish the job. That means, x_i and y_{i_k} converge together. The first approach was taken in [20] well. The present study is along second approach and Theorems 3.1 and 3.2 are in this direction.

For a delay free system (1) conditions for stability of equilibrium when x_i wait for y_{i_k} to converge first are given by (Theorem 4.1, [20])

$$
a_i > \sum_{j=1}^{n}|b_{ji}|p_i + \sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k}, i = 1, 2, ..., n; \tag{14}
$$

$$
c_{i_k} > \sum_{k=1}^{r_i}|d_{i_k}|q_{i_k}, \quad 1 \leq r_i \leq n. \tag{15}
$$

A straightforward comparison of parametric conditions of Theorems 3.1 and 3.2 of this paper with those of (14) and (15), shows that parameters are more strained here. However, this is tolerable when the system can not wait a long time for convergence of subsystems and have to compete the task all at a time, working in parallel with subsystem. This distinguishes the study here from earlier work ([20]). Further, since Theorems 3.1 and 3.2 are valid for $\tau_{i_k} = 0 = \tau_{i_l}$ also, these two results provide independent sets of sufficient conditions for global asymptotic stability of equilibrium solution of (1) also.

A close look at the parametric conditions of Theorems 3.1 and 3.2 for the choice of $\eta_1 = \eta_2 = \eta_3 = \frac{1}{2}$ reveals that a part of strain on parameters c_{i_k} represented by $\eta_2\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}$ is taken by a_i. Thus, we remark that the x_i are actually sharing the burden of monitoring y_{i_k} and simultaneously converge with them. □

A more general case:

One may notice that primary units x_i are supported by y_{i_k}. But there is no information (input) nor instructions from x_i directly to y_{i_k}. Nor there is any check or supervision by x_i as far as dynamics in second equation of (2) are considered. What ever information provided by subsystem is taken up by x_i. That is, flow of information is uni-directional. This raises a doubt on the relevance of information/contrbution from y_{i_k}. To overcome this lapse in model, it was proposed in [20] that the inputs to y_{i_k} are from x_i but not mere constants, J_{i_k}. This may be more realistic in the sense that, y_{i_k} are chosen to aid x_i and hence, are motivated by x_i rather than some other inputs. Moreover x_i are also variables and thus, this choice reflects the presence of variable input which always influences the dynamics of y_{i_k}. To realize this, it was assumed that $J_{i_k} = J_{i_k}(x_i)$ for each i_k. In the present paper, to further enhance the quality of performance of y_{i_k}, we assume that the present task of y_{i_k} depends on some previous information/instructions from x_i. To be more specific, we admit time delays in these inputs also. That is, we consider, $J_{i_k} = J_{i_k}(x_i(t-\tau_i))$. This allows us to modify (2) as

$$\begin{aligned} x_i' &= -a_i x_i + \sum_{j=1}^{n} b_{ij} f_j(x_j) + \sum_{k=1}^{r_i} c_{ii_k} g_{i_k}(x_i, y_{i_k}(t-\tau_{i_k})) + I_i, i = 1, 2, ..., n; \\ y_{i_k}' &= -c_{i_k} y_{i_k} + \sum_{l=1}^{r_i} d_{i_l} h_{i_l}(y_{i_l}(t-\tau_{i_l})) + \sum_{k=1}^{r_i} J_{i_k}(x_i(t-\tau_i)), \quad 1 \le r_i \le n. \end{aligned} \tag{16}$$

An equilibrium solution, say $(x_i^*,\ y_{i_k}^*)$, for this system should satisfy the equations

$$\begin{aligned} a_i x_i^* &= \sum_{j=1}^{n} b_{ij} f_j(x_i^*) + \sum_{k=1}^{r_i} c_{ii_k} g_{i_k}(x_i^*, y_{i_k}^*) + I_i, i = 1, 2, ..., n; \\ c_{i_k} y_{i_k}^* &= \sum_{l=1}^{r_i} d_{i_l} h_{i_l}(y_{i_l}^*) + \sum_{k=1}^{r_i} J_{i_k}(x_i^*). \end{aligned} \tag{17}$$

We assume that the function J_{i_k} satisfies $|J_{i_k}(x_i(t)) - J_{i_k}(x_i^*)| \le \alpha_{i_k}|x_i - x_i^*|$, where $\alpha_{i_k} > 0$.

Assuming that the algebraic system (17) yields a unique solution (i.e., system (16) has a unique equilibrium pattern), we directly proceed to the global asymptotic stability of the equilibrium pattern of system (16). Using (17) in (16), we get

$$\begin{aligned} (x_i - x_i^*)' &= -a_i(x_i - x_i^*) + \sum_{j=1}^{n} b_{ij}[f_j(x_j) - f_j(x_j^*)] \\ &\quad + \sum_{k=1}^{r_i} c_{ii_k}[g_{i_k}(x_i, y_{i_k}(t-\tau_{i_k})) - g_{i_k}(x_i^*, y_{i_k}^*)], \\ (y_{i_k} - y_{i_k}^{*'})' &= -c_{i_k}(y_{i_k} - y_{i_k}^*) + \sum_{l=1}^{r_i} d_{i_l}[h_{i_l}(y_{i_l}(t-\tau_{i_l})) - h_{i_l}(y_{i_l}^*)] \\ &\quad + \sum_{k=1}^{r_i}[J_{i_k}(x_i(t-\tau_i)) - J(x_i^*)]. \end{aligned} \tag{18}$$

We employ the following Lyapunov functional for our purpose here

$$\begin{aligned} V(t) &= \sum_{i=1}^{n}\Big[|x_i - x_i^*| + |y_{i_k} - y_{i_k}^*| + \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\int_{t-\tau_{i_k}}^{t}|y_{i_k}(s) - y_{i_k}^*|ds \\ &\quad + \sum_{l=1}^{r_i}|d_{i_l}|\int_{t-\tau_{i_l}}^{t}|h_{i_l}(y_{i_l}(s)) - h_{i_l}(y_{i_l}^*)|ds \\ &\quad + \sum_{k=1}^{r_i}\int_{t-\tau_i}^{t}|J_{i_k}(x_i(s) - J_{i_k}(x_i^*)|ds\Big]. \end{aligned} \tag{19}$$

The upper right derivative of V along the solutions of (16) employing (18) may be given by

$$\begin{aligned} D^+V(t) &\leq \sum_{i=1}^{n}\Big[-a_i|x_i - x_i^*| + \sum_{j=1}^{n}|b_{ij}||f_j(x_j) - f_j(x_j^*)| \\ &\quad + \sum_{k=1}^{r_i}|c_{ii_k}||g_{i_k}(x_i, y_{i_k}(t-\tau_{i_k})) - g_{i_k}(x_i^*, y_{i_k}^*)| \\ &\quad - c_{i_k}|y_{i_k} - y_{i_k}^*| + \sum_{l=1}^{r_i}|d_{i_l}||h_{i_l}(y_{i_l}(t-\tau_{i_l})) - h_{i_l}(y_{i_l}^*)| \\ &\quad + \sum_{k=1}^{r_i}|J_{i_k}(x_i(t-\tau_i)) - J_{i_k}(x_i^*)| \\ &\quad + \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}|y_{i_k} - y_{i_k}^*| - \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}|y_{i_k}(t-\tau_{i_k})) - y_{i_k}^*| \\ &\quad + \sum_{l=1}^{r_i}|d_{i_l}||h_{i_l}(y_{i_l}(t)) - h_{i_l}(y_{i_l}^*)| - \sum_{l=1}^{r_i}|d_{i_l}||h_{i_l}(y_{i_l}(t-\tau_{i_l})) - h_{i_l}(y_{i_l}^*)| \\ &\quad + \sum_{k=1}^{r_i}|J_{i_k}(x_i(t) - J_{i_k}(x_i^*)| - \sum_{k=1}^{r_i}|J_{i_k}(x_i(t-\tau_i)) - J_{i_k}(x_i^*)|\Big]. \end{aligned}$$

This, on further simplification, as done in earlier results, gives

$$\begin{aligned} D^+V(t) &\leq -\sum_{i=1}^{n}\Big[[a_i - \sum_{j=1}^{n}|b_{ji}|p_i - \sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k} - \sum_{k=1}^{r_i}\alpha_{i_k}]|x_i - x_i^*| \\ &\quad + \sum_{k=1}^{r_i}[c_{i_k} - \sum_{k=1}^{r_i}|d_{i_k}|q_{i_k} - \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}]|y_{i_k} - y_{i_k}^*|\Big] \\ &\leq -\widetilde{A}\Big[\sum_{i=1}^{n}\Big\{|x_i - x_i^*| + \sum_{k=1}^{r_i}|y_{i_k} - y_{i_k}^*|\Big\}\Big] \\ &< 0, \end{aligned} \tag{20}$$

provided

$$\widetilde{A} \equiv \min_{1\leq i\leq n}\Big\{a_i - \sum_{j=1}^{n}|b_{ij}|p_j - \sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k} - \sum_{k=1}^{r_i}\alpha_{i_k},\ c_{i_k} - \sum_{k=1}^{r_i}|d_{i_k}|q_{i_k} - \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\Big\} > 0,$$

$k = 1, 2, ..., r_i$, holds. We are now in a position to state

Theorem 3.4. *The equilibrium solution of (12) is globally asymptotically stable provided the parameters satisfy*

$$a_i - \sum_{j=1}^{n} |b_{ij}|p_j - \sum_{k=1}^{r_i} |c_{ii_k}|M_{2i_k} - \sum_{k=1}^{r_i} \alpha_{i_k} > 0,$$
$$c_{i_k} - \sum_{k=1}^{r_i} |d_{i_k}|q_{i_k} - \sum_{k=1}^{r_i} |c_{ii_k}|M_{1i_k} > 0,$$

for all $i = 1, 2, ..., n$ and $\alpha_{i_k} > 0$ is such that $|J_{i_k}(x_i(t)) - J_{i_k}(x_i^*)| \leq \alpha_{i_k}|x_i - x_i^*|$.

Proof. The proof is obvious from standard arguments noticing that $V(t)$ defined by (19) and (20) is the required Lyapunov functional.

We shall illustrate the above results by means of numerical examples.

Example 3.5. Consider the following system having two neurons in X supported by two neurons in Y involving time delays as given by

$$\begin{aligned}\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} &= -\begin{pmatrix} 6x_1 \\ 8x_2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} f_1(x_1) \\ f_2(x_2) \end{pmatrix} \\ &+ \begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} g_{1_1}(x_1, y_{1_1}(t-\tau_{1_1}) & g_{2_1}(x_2, y_{2_1}(t-\tau_{1_2})) \\ g_{1_2}(x_1, y_{1_2}(t-\tau_{2_1})) & g_{2_2}(x_2, y_{2_2}(t-\tau_{2_2})) \end{pmatrix} + \begin{pmatrix} I_1 \\ I_2 \end{pmatrix},\end{aligned}$$

$$\begin{pmatrix} y_{1_1}' \\ y_{1_2}' \end{pmatrix} = -\begin{pmatrix} 4.5y_{1_1} \\ 8y_{1_2} \end{pmatrix} + \begin{pmatrix} \sqrt{2} & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} h_{1_1}(y_{1_1}(t-\tau_{1_1})) \\ h_{1_2}(y_{1_2}(t-\tau_{1_2})) \end{pmatrix} + \begin{pmatrix} J_{1_1} \\ J_{1_2} \end{pmatrix},$$

$$\begin{pmatrix} y_{2_1}' \\ y_{2_2}' \end{pmatrix} = -\begin{pmatrix} 6.5y_{2_1} \\ 7.5y_{2_2} \end{pmatrix} + \begin{pmatrix} 2 & -2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} h_{2_1}(y_{2_1}(t-\tau_{2_1})) \\ h_{2_2}(y_{2_2}(t-\tau_{2_2})) \end{pmatrix} + \begin{pmatrix} J_{2_1} \\ J_{2_2} \end{pmatrix}.$$

Choose $f_i(x_i) = tanh(x_i), h_{i_k} = tanh(y_{i_k})$ and $g_{i_k}(x_i, y_{i_k}) = x_i + y_{i_k}$. Then $p_j = q_{i_k} = M_{1i_k} = M_{2i_k} = 1, i = 1, 2, k = 1, 2.$. Let us choose $\eta_1 = \frac{1}{2}$, $\eta_3 = \frac{1}{2}$. It is easy to see that for the above parametric values of the system, all the conditions of Theorem 3.1 are satisfied for the range of values of $\frac{3}{8} < \eta_2 < \frac{3}{7}$. Hence the equilibrium of the above system is globally asymptotically stable by virtue of Theorem 3.1 for $\tau_{i_l} = 0$.

Example 3.6. Consider the system

$$\begin{aligned}\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} &= -\begin{pmatrix} 6.1x_1 \\ 6.9x_2 \end{pmatrix} + \begin{pmatrix} 0.8 & 1 \\ -1 & 0.75 \end{pmatrix}\begin{pmatrix} f_1(x_1) \\ f_2(x_2) \end{pmatrix} \\ &+ \begin{pmatrix} 1.1 & 0.5 \\ -1.1 & 1 \end{pmatrix}\begin{pmatrix} g_{1_1}(x_1, y_{1_1}(t-\tau_{1_1}) & g_{2_1}(x_2, y_{2_1}(t-\tau_{1_2})) \\ g_{1_2}(x_1, y_{1_2}(t-\tau_{2_1})) & g_{2_2}(x_2, y_{2_2}(t-\tau_{2_2})) \end{pmatrix} + \begin{pmatrix} I_1 \\ I_2 \end{pmatrix},\end{aligned}$$

$$\begin{pmatrix} y_{1_1}' \\ y_{1_2}' \end{pmatrix} = -\begin{pmatrix} 3.7y_{1_1} \\ 5.2y_{1_2} \end{pmatrix} + \begin{pmatrix} 1 & 0.5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} h_{1_1}(y_{1_1}(t-\tau_{1_1})) \\ h_{1_2}(y_{1_2}(t-\tau_{1_2})) \end{pmatrix} + \begin{pmatrix} J_{1_1}(t-\tau_{1_1}) \\ J_{1_2}(t-\tau_{1_2}) \end{pmatrix},$$

$$\begin{pmatrix} y_{2_1}' \\ y_{2_2}' \end{pmatrix} = -\begin{pmatrix} 6.2y_{1_1} \\ 6.4y_{1_2} \end{pmatrix} + \begin{pmatrix} 0.25 & 0.5 \\ 1 & -0.5 \end{pmatrix}\begin{pmatrix} h_{2_1}(y_{2_1}(t-\tau_{2_1})) \\ h_{2_2}(y_{2_2}(t-\tau_{2_2})) \end{pmatrix} + \begin{pmatrix} J_{2_1}(t-\tau_{2_1}) \\ J_{2_2}(t-\tau_{2_2}) \end{pmatrix}$$

with all response functions as in above example. Then $p_j = q_{i_k} = M_{1i_k} = M_{2i_k} = 1, i = 1, 2, k = 1, 2$.

(i) Now choosing $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{1}{4}$, one may notice that all the conditions of Theorem 3.1 are satisfied for the range of values of $0.217 < \eta_3 < 1.189$ (approximately). Hence, the equilibrium of the above system is globally asymptotically stable by virtue of Theorem 3.1 for all $\tau_{i_l} = 0$ and $\tau_i = 0$.

(ii) Again all the parametric conditions of Theorem 3.2 are satisfied for all delays $\tau_{i_k} \geq 0$ $\tau_{i_l} \geq 0$ and $\tau_i = 0$, $i = 1, 2$, and hence, the equilibrium solution is globally asymptotically stable by virtue of Theorem 3.2.

(iii) Choosing $J_{i_k}(x_i) = x_i$ for all $i, k = 1, 2$, we get α_{i_k} $k = 1, 2$. Then the parameters of the system satisfy all the conditions of Theorem 3.4 and hence, system tolerates all three types of delays involved.

4 Conclusion

In the present paper we have considered a cooperative and supportive neural network which is under influence of time delays both in processing of information with in the subgroup network and transmission of information from subgroup network to main network. Conditions on parameters are obtained so that the equilibrium is stable for any length of delays. Under these conditions the system behaves like delay independent system. However, it is also observed that the parameters are strained much for such stability. Hence conditions straining parameters less are welcome for more applicability of the network. Parametric conditions involving suitably restricted time delay parameters may be a better choice in this case. Our results in this direction will be reported soon. Another distinguishing feature of this paper is that the main components (x_i) of the system monitor the performance of the subcomponents (y_{i_k}) (work together attitude or parallel processing) unlike its earlier counter part. It is interesting to see how the system withstands if some of its subcomponents do not respond properly to the requirements of its main components. In other words, can the (x_i) converge even if some of the (y_{i_k}) do not converge or non cooperate? This will be a question of our future contention.

References

[1] Ben Omrane, I. and Chatti, A. Training a neural network using Hierarchical genetic algorithm for Modelling and Controlling a non-linear system of waterlevel regulation. *Nonlinear Dynamics and Systems Theory* **10** (1) (2010) 65–76.

[2] Caelli, T., Guan, L. and Wen, W. Modularity in neural computing. *Procs. of IEEE* **87** (1999) 1497–1518.

[3] Carpenter, G.A., Cohen, M.A. and Grossberg, S. Computing with neural networks. *Science* **235** (1987) 1226–1227.

[4] Gopalsamy, K. and He, X.Z., Delay independent stability in BAM networks, *IEEE Trans. on Neural Networks* **5** (1994) 998–1002.

[5] Hamza, A.A. and Rasheed, W.A.J. Cooperative neural network generalization model incorporating classification and association. *European J. Scientific Research* (2009) 639–648.

[6] Hopfield, J.J. Neural netwokrs and physical systems with emergent collective computational abilities. *Proc. Natl. Acad. Sci. USA* **79** (1982) 2554–2558.

[7] Kohonen, T. *Self Organization and Associative Memory*, third ed. Springer-Verlag, New York, 1989.

[8] Kohonen, T. Correlative Associate Memory. *IEEE Trans. Comput. C* **21** (1972) 353–359.

[9] Kohonen, T. *Associative Memory A System Theoretical Approach*. Springer-Verlag, New York, 1977.

[10] Kosko, B. *Neural Networks and Fuzzy SystemsA Dynamical Systems Approach to Machine Intelligence*. Prentice Hall of India, New Delhi, 1994.

[11] Kosko, B. Bidirectional Associative Memories. *IEEE Trans. Syst. Man Cybern. SMC* **18** (1988) 49–60.

[12] Luo, F.L. and Unbehauen, R. *Applied Neural Networks for Signal Processing*. Cambridge Univ. Press, Cambridge, 1997.

[13] Misra, B.B. and Dehuri, S. Functional link artificial neural network for classification task in data mining. *J. Computer Science* **3** (12) (2007) 948–955.

[14] Roy, S.S. Probability method of reliability for cooperative neural network. *Anale. Seria Informatica, Vol. VIII fasc.* **2** (2010) 160–168.

[15] Shapiro, A.F. and Jain, L.C. *Intelligent and Other Computational Techniques in Insurance: Theory and Applications*. World Scientific, Singapore, 2003.

[16] Sharma, M. and Verma, A. Wavelet Neural Network based adaptive tracking control for a class of uncertain non-linear sytems using reinforcement learning. *Nonlinear Dynamics and Systems Theory* **12** (3) (2012) 279–288.

[17] Simpson, P.K. *Artificial Neural SystemsFoundations, Paradigms, Applications and Implementations*. Pergamon Press, New York, 1989.

[18] Sree Hari Rao, V. and Naresh Kumar, M. Estimation of the parameters of an infectious disease model using neural networks. *Nonlinear Analysis:Real World Applications* **11** (3) (2010) 1810–1818.

[19] Sree Hari Rao, V. and Phaneendra, Bh.R.M. Global dynamics of bidirectional associative memory neural netwokrs involving transmission delays and dead zones. *Neural Networks* **12** (1999) 455–465.

[20] Sree Hari Rao, V. and Raja Sekhara Rao, P. Cooperative and Supportive Neural Networks. *Physics Letters A* **371** (2007) 101–110.

[21] Raja Sekhara Rao, P., Venkata Ratnam, K. and Lalitha, P. Estimation of inputs for a desired output of a cooperative and supportive neural network. *International Journal of Emerging Technologies in Computational and Applied Sciences* **9** (1) (2014) 99–105.