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Asymptotic Stability Conditions for Some Classes of Mechanical Systems with Switched Nonlinear Force Fields

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Abstract: Certain classes of switched mechanical systems with nonlinear potential and dissipative forces are studied. By the use of the differential inequalities method and multiple Lyapunov functions, conditions on switching law guaranteeing the asymptotic stability of the trivial equilibrium position of the considered systems are obtained. An example and the results of a computer simulation are presented to demonstrate the effectiveness of the proposed approaches.

Keywords: *mechanical systems; switched force fields; asymptotic stability; multiple Lyapunov functions; differential inequalities; dwell-time.*

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1 Introduction

Stability of switched systems has attracted an increasing attention during last decades, mainly due to the numerous applications of these systems in engineering, technological processes, mechanics, population dynamics, chemistry and economics, see, e.g., [1, 7, 9, 10, 12, 16, 17, 20] and the references cited therein. A switched system is a particular kind of hybrid dynamical system that consists of a family of subsystems and a switching law determining at each time instant which subsystem is active.

There are two principal approaches to the stability analysis of switched systems. The first one is based on the constructing of a common Lyapunov function for the family of

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subsystems corresponding to a switched system [6, 12, 13, 19]. The existence of a such function guarantees the stability of the considered system for any admissible switching law. In the situations where we cannot prove the existence of a common Lyapunov function, the stability of a switched system can be provided by means of additional restrictions on the switching law (dwell-time approach) [8, 9, 13, 19, 21]. It is known that, under the suitable assumptions on the system investigated, the stability is ensured if the intervals between consecutive switching times are sufficiently large [13, 19]. However, it should be noted that these approaches are well-developed mostly for linear switched systems.

The problem of stability analysis of hybrid systems is especially difficult for mechanical systems with switched force fields. In numerous applications, mechanical systems are described by nonlinear differential equations of the second order. This results in the appearance of certain special properties of motions and essentially complicates the investigation of systems dynamics [2, 3, 9, 16]. In particular, well-known approaches developed for switched systems of general form might be inefficient or even inapplicable for mechanical systems, see [3].

In the present paper, certain classes of switched mechanical systems with nonlinear potential and dissipative forces are studied. By the use of the differential inequalities method and multiple Lyapunov functions, conditions on switching law guaranteeing the asymptotic stability of the trivial equilibrium position of the considered systems are obtained.

2 Statement of the Problem

Let the family of systems

$$\ddot{\mathbf{x}} + \mathbf{D}_s(\mathbf{x})\dot{\mathbf{x}} + \frac{\partial \Pi_s(\mathbf{x})}{\partial \mathbf{x}} = \mathbf{0}, \quad s = 1, \ldots, N, \tag{1}$$

be given. Here $\mathbf{x} \in \mathbb{R}^n$; $\Pi_s(\mathbf{x})$ are continuously differentiable for $\mathbf{x} \in \mathbb{R}^n$ homogeneous of the order $\mu+1$ functions, $\mu \geq 1$; entries of the matrices $\mathbf{D}_s(\mathbf{x})$ are continuous for $\mathbf{x} \in \mathbb{R}^n$ homogeneous of the order ν functions, $\nu > 0$. Systems from the family (1) are vector type Lienard equations, see [18]. They can be used for the modelling of mechanical systems with potential and essentially nonlinear velocity forces.

Switched system generated by the family (1) and a switching law σ is

$$\ddot{\mathbf{x}} + \mathbf{D}_\sigma(\mathbf{x})\dot{\mathbf{x}} + \frac{\partial \Pi_\sigma(\mathbf{x})}{\partial \mathbf{x}} = \mathbf{0}. \tag{2}$$

Here $\sigma = \sigma(t): \ [0, +\infty) \to \{1, \ldots, N\}$ is a piecewise constant function. Without loss of generality, consider the only case where the interval $(0, +\infty)$ contains the infinite number of switching instants. Let θ_i, $i = 1, 2, \ldots$, be the switching times, $0 < \theta_1 < \theta_2 < \ldots$, and $\theta_0 = 0$. Assume that the function $\sigma(t)$ is right-continuous, and the sequence $\{\theta_i\}_{i=0}^{\infty}$ is a minimal one ($\sigma(\theta_i) \neq \sigma(\theta_{i+1})$, $i = 0, 1, \ldots$). Hereinafter, we consider non Zeno sequences [12, 13], i.e., sequences that switch at most a finite number of times in any finite time interval. This kind of switching law is called admissible one.

Systems (1) and system (2) admit the trivial equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$. Assume that, for every system from the family (1), the equilibrium position is asymptotically stable. Let us determine conditions under which the equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ of switched system (2) is also asymptotically stable.

The problem of the construction of a common Lyapunov function for the family of systems of the form (1) was studied in [5, 15]. As it was mentioned in the Introduction, the existence of a such function guarantees the asymptotic stability of (2) for any admissible switching law.

In this paper, it is assumed that we failed to prove the existence of a common Lyapunov function for (1). We will look for conditions on switching law guaranteeing asymptotic stability of the equilibrium position.

It should be noted that such conditions were obtained in [4] for system (2) with constant matrices $\mathbf{D}_1, \ldots, \mathbf{D}_N$. The goal of the present paper is extension of the results of [4] to the case of essentially nonlinear velocity forces. We will assume that the forces $\mathbf{F}_s(\mathbf{x}, \dot{\mathbf{x}}) = -\mathbf{D}_s(\mathbf{x})\dot{\mathbf{x}}$, $s = 1, \ldots, N$, are dissipative ones and consider two types of such forces. It is worth mentioning that asymptotic stability conditions will depend not only on the type of the dissipative forces but also on the information available on the switching law.

3 The First Type of Dissipative Forces

3.1 Stability analysis via multiple Lyapunov functions

First, consider the case when the switching instants θ_i, $i = 1, 2, \ldots$, are given, while the order of switching between the systems from (1) might be unknown.

Let us impose additional restrictions on the functions $\Pi_1(\mathbf{x}), \ldots, \Pi_N(\mathbf{x})$ and the matrices $\mathbf{D}_1(\mathbf{x}), \ldots, \mathbf{D}_N(\mathbf{x})$.

Assumption 3.1 Functions $\Pi_1(\mathbf{x}), \ldots, \Pi_N(\mathbf{x})$ are positive definite.

Assumption 3.2 For any fixed $\mathbf{x} \neq \mathbf{0}$, the matrices $\mathbf{D}_s(\mathbf{x}) + \mathbf{D}_s^T(\mathbf{x})$, $s = 1, \ldots, N$, are positive definite.

Remark 3.1 Taking into account homogeneity of $\mathbf{D}_1(\mathbf{x}), \ldots, \mathbf{D}_N(\mathbf{x})$, we obtain, see [22], that Assumption 3.2 implies that the estimates

$$\mathbf{z}^T \mathbf{D}_s(\mathbf{x})\, \mathbf{z} \geq c_s\, \|\mathbf{x}\|^{\nu} \|\mathbf{z}\|^2, \qquad s = 1, \ldots, N,$$

hold for all $\mathbf{x}, \mathbf{z} \in \mathbb{R}^n$. Here $c_1, \ldots, c_N$ are positive constants, and $\|\cdot\|$ denotes the Euclidean norm of a vector.

Remark 3.2 It is known, see [18, 22], that if Assumptions 3.1 and 3.2 are fulfilled, then, for any system from the family (1), the equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ is asymptotically stable.

For every s in $\{1, \ldots, N\}$, choose a Lyapunov function for the s-th system from (1) in the form

$$V_s(\mathbf{x}, \dot{\mathbf{x}}) = \Pi_s(\mathbf{x}) + \frac{1}{2}\dot{\mathbf{x}}^T \dot{\mathbf{x}} - \gamma_{1s}\|\dot{\mathbf{x}}\|^{\beta-1}\mathbf{x}^T \dot{\mathbf{x}} + \gamma_{2s}\|\mathbf{x}\|^{k-1}\mathbf{x}^T \dot{\mathbf{x}}, \tag{3}$$

where $\gamma_{1s} > 0$, $\gamma_{2s} > 0$, $\beta \geq 1$, $k \geq 1$. Differentiating function (3) with respect to the s-th system, we obtain

$$\dot{V}_s\big|_{(s)} = -\gamma_{2s}(\mu+1)\|\mathbf{x}\|^{k-1}\Pi_s(\mathbf{x}) - \gamma_{1s}\|\dot{\mathbf{x}}\|^{\beta+1} - \dot{\mathbf{x}}^T \mathbf{D}_s(\mathbf{x})\dot{\mathbf{x}}$$

$$-\gamma_{1s}\mathbf{x}^T \frac{\partial\left(\|\dot{\mathbf{x}}\|^{\beta-1}\dot{\mathbf{x}}\right)}{\partial\dot{\mathbf{x}}}\left(-\frac{\partial\Pi_s(\mathbf{x})}{\partial\mathbf{x}}-\mathbf{D}_s(\mathbf{x})\dot{\mathbf{x}}\right)$$

$$+\gamma_{2s}\dot{\mathbf{x}}^T \frac{\partial\left(\|\mathbf{x}\|^{k-1}\mathbf{x}\right)}{\partial\mathbf{x}}\dot{\mathbf{x}}-\gamma_{2s}\|\mathbf{x}\|^{k-1}\mathbf{x}^T\mathbf{D}_s(\mathbf{x})\dot{\mathbf{x}}.$$

Hence, the estimates

$$a_{1s}\left(\|\dot{\mathbf{x}}\|^2+\|\mathbf{x}\|^{\mu+1}\right)-\left(\gamma_{1s}\|\mathbf{x}\|\|\dot{\mathbf{x}}\|^{\beta}+\gamma_{2s}\|\mathbf{x}\|^{k}\|\dot{\mathbf{x}}\|\right)\le V_s(\mathbf{x},\dot{\mathbf{x}})$$

$$\le a_{2s}\left(\|\dot{\mathbf{x}}\|^2+\|\mathbf{x}\|^{\mu+1}\right)+\left(\gamma_{1s}\|\mathbf{x}\|\|\dot{\mathbf{x}}\|^{\beta}+\gamma_{2s}\|\mathbf{x}\|^{k}\|\dot{\mathbf{x}}\|\right),$$

$$\dot{V}_s\big|_{(s)}\le -a_{3s}\left(\gamma_{2s}\|\mathbf{x}\|^{k+\mu}+\gamma_{1s}\|\dot{\mathbf{x}}\|^{\beta+1}+\|\mathbf{x}\|^{\nu}\|\dot{\mathbf{x}}\|^2\right)$$

$$+a_{4s}\left(\gamma_{1s}\|\mathbf{x}\|^{\mu+1}\|\dot{\mathbf{x}}\|^{\beta-1}+\gamma_{1s}\|\mathbf{x}\|^{\nu+1}\|\dot{\mathbf{x}}\|^{\beta}+\gamma_{2s}\|\mathbf{x}\|^{k-1}\|\dot{\mathbf{x}}\|^2+\gamma_{2s}\|\mathbf{x}\|^{k+\nu}\|\dot{\mathbf{x}}\|\right)$$

hold for $\mathbf{x},\dot{\mathbf{x}}\in\mathbb{R}^n$. Here $a_{1s},\dots,a_{4s}$ are positive constants.

By the use of generalized homogeneous functions properties [22], it is easy to verify that, if

$$k=\max\{\mu-\nu;\ \nu+1\},\qquad \beta=1+\max\left\{\frac{2\nu}{\mu+1};\ \frac{2(k-1)}{k+\mu-\nu}\right\},\tag{4}$$

then there exist positive numbers $\gamma_{11},\dots,\gamma_{1N}$, $\gamma_{21},\dots,\gamma_{2N}$, b_1, b_2, α and H such that the inequalities

$$b_1 r(\mathbf{x},\dot{\mathbf{x}})\le V_s(\mathbf{x},\dot{\mathbf{x}})\le b_2 r(\mathbf{x},\dot{\mathbf{x}}),\qquad s=1,\dots,N,\tag{5}$$

$$\dot{V}_s\big|_{(s)}\le -\alpha V_s^{1+\xi}(\mathbf{x},\dot{\mathbf{x}}),\qquad s=1,\dots,N,\tag{6}$$

are valid for $r(\mathbf{x},\dot{\mathbf{x}})<H$. Here $r(\mathbf{x},\dot{\mathbf{x}})=\|\dot{\mathbf{x}}\|^2+\|\mathbf{x}\|^{\mu+1}$, and $\xi=(k-1)/(\mu+1)$.

Find $\omega\ge 1$, such that

$$V_s(\mathbf{x},\dot{\mathbf{x}})\le \omega V_l(\mathbf{x},\dot{\mathbf{x}}),\qquad s,l=1,\dots,N,\tag{7}$$

for $r(\mathbf{x},\dot{\mathbf{x}})<H$.

Denote $h=\omega^{-\xi}$; $\tau_i=\theta_i-\theta_{i-1}$, $i=1,2,\dots$; $\psi(m,1)=0$, and $\psi(m,p)=\sum_{i=1}^{p-1}\tau_{m+i}h^{p-i}$ for $p=2,3,\dots$, $m=1,2,\dots$.

Theorem 3.1 *Let Assumptions 3.1 and 3.2 be fulfilled, and for family* (1) *the Lyapunov functions* $V_1(\mathbf{x},\dot{\mathbf{x}}),\dots,V_N(\mathbf{x},\dot{\mathbf{x}})$ *be constructed satisfying the estimates* (5), (6) *and* (7). *If*

$$\psi(m,p)\to+\infty\quad \textit{as}\quad p\to\infty\tag{8}$$

for any positive integer m, *then the equilibrium position* $\mathbf{x}=\dot{\mathbf{x}}=\mathbf{0}$ *of system* (2) *is asymptotically stable. In the case when the tendency* (8) *is uniform with respect to* $m=1,2,\dots$, *the equilibrium position is uniformly asymptotically stable.*

Proof. By the use of the partial Lyapunov functions $V_1(\mathbf{x},\dot{\mathbf{x}}),\dots,V_N(\mathbf{x},\dot{\mathbf{x}})$, construct the multiple Lyapunov function $V_{\sigma(t)}(\mathbf{x},\dot{\mathbf{x}})$ corresponding to the switching law $\sigma(t)$.

Choose $\varepsilon\in(0,H)$ and $t_0\ge 0$. Consider a solution $\mathbf{x}(t)$ of (2) with initial conditions satisfying the inequalities $0<r(\mathbf{x}(t_0),\dot{\mathbf{x}}(t_0))<\varepsilon$. Find the positive integer m such that $t_0\in[\theta_{m-1},\theta_m)$.

Assume that $r(\mathbf{x}(t),\dot{\mathbf{x}}(t)) < \varepsilon$ for $t \in [t_0,\tilde{t}]$. If $t_0 < \tilde{t} \le \theta_m$ then, integrating the corresponding differential inequality from (6), we obtain that the estimate

$$V^{-\xi}_{\sigma(\theta_{m-1})}(\mathbf{x}(\tilde{t}),\dot{\mathbf{x}}(\tilde{t})) \ge \alpha\xi(\tilde{t}-t_0) + V^{-\xi}_{\sigma(\theta_{m-1})}(\mathbf{x}(t_0),\dot{\mathbf{x}}(t_0)) \tag{9}$$

is valid.

In the case when $\tilde{t} \ge \theta_m$, there exists a positive integer p such that $\theta_{m+p-1} \le \tilde{t} < \theta_{m+p}$. It should be noted that $p \to \infty$ as $\tilde{t} \to +\infty$. Integrating successively the corresponding differential inequalities from family (6) on the intervals $[\theta_{m+p-1},\tilde{t}]$, $[\theta_{m+p-2},\theta_{m+p-1}]$, …, $[t_0,\theta_m]$ and taking into account inequalities (7), we obtain

$$\begin{aligned} V^{-\xi}_{\sigma(\theta_{m+p-1})}(\mathbf{x}(\tilde{t}),\dot{\mathbf{x}}(\tilde{t})) &\ge \alpha\xi\left(\tilde{t}-\theta_{m+p-1}\right) + V^{-\xi}_{\sigma(\theta_{m+p-1})}(\mathbf{x}(\theta_{m+p-1}),\dot{\mathbf{x}}(\theta_{m+p-1}))\\ &\ge hV^{-\xi}_{\sigma(\theta_{m+p-2})}(\mathbf{x}(\theta_{m+p-1}),\dot{\mathbf{x}}(\theta_{m+p-1})) + \alpha\xi\left(\tilde{t}-\theta_{m+p-1}\right) \ge \dots \\ \ge h^p V^{-\xi}_{\sigma(\theta_{m-1})}&(\mathbf{x}(t_0),\dot{\mathbf{x}}(t_0)) + \alpha\xi\left(\left(\tilde{t}-\theta_{m+p-1}\right) + \psi(m,p) + h^p\left(\theta_m - t_0\right)\right). \end{aligned} \tag{10}$$

From (5), (9) and (10) it follows that

$$r(\mathbf{x}(\tilde{t}),\dot{\mathbf{x}}(\tilde{t})) \le b_1^{-1}\Big(b_2^{-\xi} r^{-\xi}(\mathbf{x}(t_0),\dot{\mathbf{x}}(t_0)) + \alpha\xi\left(\tilde{t}-t_0\right)\Big)^{-\frac{1}{\xi}}$$

for $\tilde{t} \in [t_0,\theta_m)$, and

$$\begin{aligned} r(\mathbf{x}(\tilde{t}),\dot{\mathbf{x}}(\tilde{t})) &\le b_1^{-1}\Big(h^p b_2^{-\xi} r^{-\xi}(\mathbf{x}(t_0),\dot{\mathbf{x}}(t_0)) \\ &+\alpha\xi\big(\left(\tilde{t}-\theta_{m+p-1}\right) + \psi(m,p) + h^p\left(\theta_m - t_0\right)\big)\Big)^{-\frac{1}{\xi}} \end{aligned}$$

for $\tilde{t} \in [\theta_{m+p-1},\theta_{m+p})$, $p \ge 1$.

With the usage of these estimates the subsequent proof is similar to that of Theorem 1 in [4]. □

Corollary 3.1 *Let Assumptions 3.1 and 3.2 be fulfilled. If $\tau_i \to +\infty$ as $i \to \infty$, then the equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ of system* (2) *is uniformly asymptotically stable.*

Remark 3.3 It is a fairly well-known fact, see [13, 19], that for any family consisting of a finite number of linear time invariant asymptotically stable systems there exists a number $L > 0$ (dwell time), such that the corresponding switched system is also asymptotically stable providing that the intervals between consecutive switching times are not less than L. Theorem 3.1 does not permit to obtain a similar result for the family of nonlinear systems (1). If $\tau_i = L = \text{const} > 0$, $i = 1,2,\ldots$, then condition (8) is not fulfilled for any choice of L. However, for nonlinear switched system (2), a positive lower bound for the values of $\tau_1,\tau_2,\ldots$ can be found guaranteeing the practical stability [11] of the equilibrium position.

Corollary 3.2 *Let Assumptions 3.1 and 3.2 be fulfilled. Then there exists a positive number Δ, such that for any $\varepsilon > 0$ one can choose $L_1 > 0$ and $L_2 > 0$ satisfying the following condition: if $\tau_i \ge L_1$, $i = 1,2,\ldots$, and for a solution $\mathbf{x}(t)$ of* (2) *the inequalities $t_0 \ge 0$, $r(\mathbf{x}(t_0),\dot{\mathbf{x}}(t_0)) < \Delta$ are valid, then $r(\mathbf{x}(t),\dot{\mathbf{x}}(t)) < \varepsilon$ for all $t \ge t_0 + L_2$.*

3.2 Asymptotic stability conditions in the case of complete information on the switching law

Assume now that we know not only the switching instants θ_i, $i = 1, 2, \ldots$, but also the order of switching between the systems. Then another approach for the stability analysis can be used [4, 14]. Choose a system from family (1) and determine relationship between this system activity intervals and those of the remained systems under which it is possible to guarantee the asymptotic stability of the equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ of switched system (2).

Let (for definiteness) the first system from (1) be chosen. In the present subsection, instead of Assumption 3.1, we will use a weaker assumption.

Assumption 3.3 Function $\Pi_1(\mathbf{x})$ is positive definite.

Consider the Lyapunov function

$$V_1(\mathbf{x}, \dot{\mathbf{x}}) = \Pi_1(\mathbf{x}) + \frac{1}{2}\dot{\mathbf{x}}^T\dot{\mathbf{x}} - \gamma_{11}\|\dot{\mathbf{x}}\|^{\beta-1}\mathbf{x}^T\dot{\mathbf{x}} + \gamma_{21}\|\mathbf{x}\|^{k-1}\mathbf{x}^T\dot{\mathbf{x}},$$

where $\gamma_{11} > 0$, $\gamma_{21} > 0$, and the values of the parameters β and k are defined by the formulae (4).

Denote by $\dot{V}_1\big|_{(s)}$ the derivative of $V_1(\mathbf{x}, \dot{\mathbf{x}})$ with respect to the s-th system from (1), $s = 1, \ldots, N$. We obtain

$$\dot{V}_1\big|_{(s)} = -\gamma_{21}(\mu + 1)\|\mathbf{x}\|^{k-1}\Pi_s(\mathbf{x}) - \gamma_{11}\|\dot{\mathbf{x}}\|^{\beta+1} - \dot{\mathbf{x}}^T\mathbf{D}_s(\mathbf{x})\dot{\mathbf{x}}$$
$$-\gamma_{11}\mathbf{x}^T\frac{\partial\left(\|\dot{\mathbf{x}}\|^{\beta-1}\dot{\mathbf{x}}\right)}{\partial\dot{\mathbf{x}}}\left(-\frac{\partial\Pi_s(\mathbf{x})}{\partial\mathbf{x}} - \mathbf{D}_s(\mathbf{x})\dot{\mathbf{x}}\right)$$
$$+\gamma_{21}\dot{\mathbf{x}}^T\frac{\partial\left(\|\mathbf{x}\|^{k-1}\mathbf{x}\right)}{\partial\mathbf{x}}\dot{\mathbf{x}} - \gamma_{21}\|\mathbf{x}\|^{k-1}\mathbf{x}^T\mathbf{D}_s(\mathbf{x})\dot{\mathbf{x}} + \left(\frac{\partial\Pi_1(\mathbf{x})}{\partial\mathbf{x}}\right)^T\dot{\mathbf{x}} - \left(\frac{\partial\Pi_s(\mathbf{x})}{\partial\mathbf{x}}\right)^T\dot{\mathbf{x}}.$$

Let again $r(\mathbf{x}, \dot{\mathbf{x}}) = \|\dot{\mathbf{x}}\|^2 + \|\mathbf{x}\|^{\mu+1}$, $\xi = (k-1)/(\mu+1)$. It is easy to verify that if $\mu \geq 2\nu + 1$, Assumptions 3.2 and 3.3 are fulfilled, and values of γ_{11} and γ_{21} are sufficiently small, then there exists a number $H > 0$ such that the estimates

$$b_1 r(\mathbf{x}, \dot{\mathbf{x}}) \leq V_1(\mathbf{x}, \dot{\mathbf{x}}) \leq b_2 r(\mathbf{x}, \dot{\mathbf{x}}), \qquad \dot{V}_1\big|_{(s)} \leq \alpha_s V_1^{1+\xi}(\mathbf{x}, \dot{\mathbf{x}}), \quad s = 1, \ldots, N, \tag{11}$$

hold for $r(\mathbf{x}, \dot{\mathbf{x}}) < H$. Here b_1, b_2, $\alpha_1, \ldots, \alpha_N$ are constants with $b_1 > 0$, $b_2 > 0$, $\alpha_1 < 0$.

For given switching law $\sigma(t)$, define the auxiliary piecewise constant function $\eta(t)$ by the formula $\eta(t) = -\alpha_{\sigma(t)}$ for $t \geq 0$.

Theorem 3.2 *Let $\mu \geq 2\nu + 1$, Assumptions 3.2 and 3.3 be fulfilled, and for family* (1) *the Lyapunov function $V_1(\mathbf{x}, \dot{\mathbf{x}})$ be constructed satisfying the estimates* (11). *If*

$$\int_0^t \eta(\tau)\, d\tau \to +\infty \quad \text{as} \quad t \to +\infty, \tag{12}$$

then the equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ of system (2) *is asymptotically stable. In the case when*

$$\int_{t_0}^{t_0+t} \eta(\tau)\, d\tau \to +\infty \quad \text{as} \quad t \to +\infty \tag{13}$$

uniformly with respect to $t_0 \geq 0$, the equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ of system (2) *is uniformly asymptotically stable.*

Proof. For given switching law $\sigma(t)$, construct the function $\eta(t)$. Let the numbers $\varepsilon > 0$ and $t_0 \geq 0$ be chosen. Without loss of generality, assume that $\varepsilon < H$.

If (12) holds, then there exists a constant ρ_0, such that $\int_{t_0}^{t} \eta(\tau)\, d\tau \geq \rho_0$ for all $t \geq t_0$. Choose $\delta > 0$ satisfying the condition

$$(b_2\delta)^{-\xi} + \xi\rho_0 > (b_1\varepsilon)^{-\xi}\,.$$

Consider a solution $\mathbf{x}(t)$ of system (2), such that $0 < r(\mathbf{x}(t_0), \dot{\mathbf{x}}(t_0)) < \delta$. If $r(\mathbf{x}(t), \dot{\mathbf{x}}(t)) < \varepsilon$ for $t \in [t_0, \tilde{t})$, then the differential inequality

$$\dot{V}_1(\mathbf{x}(t), \dot{\mathbf{x}}(t)) \leq -\eta(t) V_1^{1+\xi}(\mathbf{x}(t), \dot{\mathbf{x}}(t)) \tag{14}$$

is valid for $t \in [t_0, \tilde{t}]$.

With the aid of estimate (14), it is easy to show that

$$\big(b_1 r(\mathbf{x}(\tilde{t}), \dot{\mathbf{x}}(\tilde{t}))\big)^{-\xi} \geq V_1^{-\xi}(\mathbf{x}(\tilde{t}), \dot{\mathbf{x}}(\tilde{t})) \geq V_1^{-\xi}(\mathbf{x}(t_0), \dot{\mathbf{x}}(t_0)) + \xi \int_{t_0}^{\tilde{t}} \eta(\tau)\, d\tau$$

$$\geq (b_2 r(\mathbf{x}(t_0), \dot{\mathbf{x}}(t_0)))^{-\xi} + \xi \int_{t_0}^{\tilde{t}} \eta(\tau)\, d\tau \geq (b_2\delta)^{-\xi} + \xi\rho_0 > (b_1\varepsilon)^{-\xi}\,.$$

Hence, $r(\mathbf{x}(t), \dot{\mathbf{x}}(t)) < \varepsilon$ for all $t \geq t_0$, and $r(\mathbf{x}(t), \dot{\mathbf{x}}(t)) \to 0$ as $t \to +\infty$.

If the tendency (13) is uniform with respect to $t_0 \geq 0$, then the number δ can be chosen independent of t_0, and $r(\mathbf{x}(t), \dot{\mathbf{x}}(t)) \to 0$ as $t - t_0 \to +\infty$ uniformly with respect to $t_0 \geq 0$. □

Remark 3.4 In the proof of Theorem 3.2, we did not use the positive definiteness property of functions $\Pi_2(\mathbf{x}), \ldots, \Pi_N(\mathbf{x})$. Hence, this theorem remains valid also in the case when the equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ is not asymptotically stable either for a part of systems numbered $2, \ldots, N$, or for all of these systems.

4 The Second Type of Dissipative Forces

Next, we will assume that in system (2) potential forces are switched, whereas dissipative forces are nonswitched, i.e., $\mathbf{D}_s(\mathbf{x}) = \mathbf{D}(\mathbf{x})$, $s = 1, \ldots, N$, where entries of the matrix $\mathbf{D}(\mathbf{x})$ are continuous for $\mathbf{x} \in \mathbb{R}^n$ homogeneous of the order ν functions, $\nu > 0$.

Moreover, we will impose an additional restriction on the structure of the matrix $\mathbf{D}(\mathbf{x})$.

Assumption 4.1 The matrix $\mathbf{D}(\mathbf{x})$ is represented in the form $\mathbf{D}(\mathbf{x}) = \partial \mathbf{G}(\mathbf{x})/\partial \mathbf{x}$, where components of the vector $\mathbf{G}(\mathbf{x})$ are continuously differentiable for $\mathbf{x} \in \mathbb{R}^n$ homogeneous of the order $\nu + 1$ functions, $\nu > 0$.

Then family (1) can be rewritten as follows

$$\dot{\mathbf{x}} = \mathbf{y} - \mathbf{G}(\mathbf{x}), \quad \dot{\mathbf{y}} = -\frac{\partial \Pi_s(\mathbf{x})}{\partial \mathbf{x}}, \quad s = 1, \ldots, N. \tag{15}$$

4.1 Stability analysis via multiple Lyapunov functions

As in the previous section, consider first the case when the switching instants θ_i, $i = 1, 2, \ldots$, are given, while the order of switching between the systems from (15) might be unknown.

Assumption 4.2 The functions $(\partial \Pi_s(\mathbf{x})/\partial \mathbf{x})^T \mathbf{G}(\mathbf{x})$, $s = 1, \ldots, N$, are positive definite.

Remark 4.1 The class of matrices $\mathbf{D}(\mathbf{x})$ defined by Assumption 3.2 differs from that defined by Assumptions 4.1 and 4.2.

Example 4.1 Let $\Pi_s(\mathbf{x}) = a_1^{(s)} x_1^{\mu+1} + \ldots + a_n^{(s)} x_n^{\mu+1}$, $s = 1, \ldots, N$. Here $\mathbf{x} = (x_1, \ldots, x_n)^T$, $\mu \geq 1$ is a rational with the odd numerator and denominator, and $a_i^{(s)}$ are positive coefficients, $i = 1, \ldots, n$; $s = 1, \ldots, N$. The functions $\Pi_1(\mathbf{x}), \ldots, \Pi_N(\mathbf{x})$ satisfy Assumption 3.1.

On the one hand, if $\mathbf{D}(\mathbf{x}) = \|\mathbf{x}\|^\nu \mathbf{A}$, where $\nu > 0$, and $\mathbf{A}$ is a constant matrix such that the matrix $\mathbf{A} + \mathbf{A}^T$ is positive definite, then Assumption 3.2 is fulfilled, whereas Assumption 4.1 is not fulfilled.

On the other hand, choose the matrix $\mathbf{D}(\mathbf{x})$ in the form $\mathbf{D}(\mathbf{x}) = \operatorname{diag}\{b_1 x_1^\nu, \ldots, b_n x_n^\nu\}$, where ν is a positive rational with the even numerator and the odd denominator, and b_i are positive constants, $i = 1, \ldots, n$. In this case Assumptions 4.1 and 4.2 are fulfilled (here $\mathbf{G}(\mathbf{x}) = \left(b_1 x_1^{\nu+1}, \ldots, b_n x_n^{\nu+1}\right)^T / (\nu + 1)$), whereas Assumption 3.2 is not fulfilled.

Remark 4.2 It is known, see [18, 22], that under Assumptions 3.1 and 4.2 any system from the family (15) admits the asymptotically stable zero solution.

For every $s \in \{1, \ldots, N\}$, construct a Lyapunov function for the s-th system from (15) by the formula

$$\hat{V}_s(\mathbf{x}, \mathbf{y}) = \Pi_s(\mathbf{x}) + \frac{1}{2}\mathbf{y}^T \mathbf{y} - \hat{\gamma}_s \|\mathbf{y}\|^{\lambda-1} \mathbf{x}^T \mathbf{y},$$

where $\hat{\gamma}_s > 0$, $\lambda \geq 1$. We obtain

$$\dot{\hat{V}}_s\Big|_{(s)} = -\left(\frac{\partial \Pi_s(\mathbf{x})}{\partial \mathbf{x}}\right)^T \mathbf{G}(\mathbf{x}) - \hat{\gamma}_s \|\mathbf{y}\|^{\lambda+1}$$

$$+\hat{\gamma}_s \|\mathbf{y}\|^{\lambda-1} \mathbf{y}^T \mathbf{G}(\mathbf{x}) + \hat{\gamma}_s \mathbf{x}^T \frac{\partial (\|\mathbf{y}\|^{\lambda-1} \mathbf{y})}{\partial \mathbf{y}} \frac{\partial \Pi_s(\mathbf{x})}{\partial \mathbf{x}}.$$

Hence, under Assumptions 3.1 and 4.2 the estimates

$$\hat{a}_{1s}\left(\|\mathbf{x}\|^{\mu+1} + \|\mathbf{y}\|^2\right) - \hat{\gamma}_s \|\mathbf{x}\| \|\mathbf{y}\|^\lambda \leq \hat{V}_s(\mathbf{x}, \mathbf{y}) \leq \hat{a}_{2s}\left(\|\mathbf{x}\|^{\mu+1} + \|\mathbf{y}\|^2\right) + \hat{\gamma}_s \|\mathbf{x}\| \|\mathbf{y}\|^\lambda,$$

$$\dot{\hat{V}}_s\Big|_{(s)} \leq -\left(\hat{a}_{3s} \|\mathbf{x}\|^{\mu+\nu+1} + \hat{\gamma}_s \|\mathbf{y}\|^{\lambda+1}\right) + \hat{a}_{4s} \hat{\gamma}_s \left(\|\mathbf{x}\|^{\nu+1} \|\mathbf{y}\|^\lambda + \|\mathbf{x}\|^{\mu+1} \|\mathbf{y}\|^{\lambda-1}\right)$$

hold for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Here $\hat{a}_{1s}, \ldots, \hat{a}_{4s}$ are positive constants.

It is easy to verify, see [22], that if

$$\lambda = \max\left\{1 + \frac{2\nu}{\mu+1}; \frac{\mu}{\nu+1}\right\}, \tag{16}$$

then there exist positive numbers $\hat{\gamma}_1, \ldots, \hat{\gamma}_N$, $\hat{b}_1$, $\hat{b}_2$, $\hat{\alpha}$ and $\hat{H}$ such that the inequalities

$$\hat{b}_1 r(\mathbf{x},\mathbf{y}) \le \hat{V}_s(\mathbf{x},\mathbf{y}) \le \hat{b}_2 r(\mathbf{x},\mathbf{y}), \qquad \dot{\hat{V}}_s\Big|_{(s)} \le -\hat{\alpha}\hat{V}_s^{1+\hat{\xi}}(\mathbf{x},\mathbf{y}), \qquad s = 1,\ldots,N, \tag{17}$$

are valid for $r(\mathbf{x},\mathbf{y}) < \hat{H}$. Here $\hat{\xi} = (\lambda - 1)/2$, and $r(\mathbf{x},\mathbf{y}) = \|\mathbf{x}\|^{\mu+1} + \|\mathbf{y}\|^2$.

Find $\hat{\omega} \ge 1$, such that

$$\hat{V}_s(\mathbf{x},\mathbf{y}) \le \hat{\omega}\hat{V}_l(\mathbf{x},\mathbf{y}), \qquad s,l = 1,\ldots,N, \tag{18}$$

for $r(\mathbf{x},\mathbf{y}) < \hat{H}$.

Denote $\hat{h} = \hat{\omega}^{-\hat{\xi}}$; $\hat{\psi}(m,1) = 0$, and $\hat{\psi}(m,p) = \sum_{i=1}^{p-1} \tau_{m+i}\hat{h}^{p-i}$ for $p = 2,3,\ldots$, $m = 1,2,\ldots$.

Theorem 4.1 *Let Assumptions 3.1, 4.1 and 4.2 be fulfilled, and for family* (15) *the Lyapunov functions* $\hat{V}_1(\mathbf{x},\mathbf{y}),\ldots,\hat{V}_N(\mathbf{x},\mathbf{y})$ *be constructed satisfying the estimates* (17) *and* (18). *If*

$$\hat{\psi}(m,p) \to +\infty \quad as \quad p \to \infty \tag{19}$$

for any positive integer m, *then the equilibrium position* $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ *of system* (2) *is asymptotically stable. In the case when the tendency* (19) *is uniform with respect to* $m = 1,2,\ldots$, *the equilibrium position is uniformly asymptotically stable.*

The proof of the theorem is similar to that of Theorem 3.1.

Remark 4.3 For Theorem 4.1, corollaries similar to Corollaries 3.1 and 3.2 can be formulated.

4.2 Asymptotic stability conditions in the case of complete information on the switching law

Assume now that we know not only the switching instants θ_i, $i = 1,2,\ldots$, but also the order of switching between the systems. Then for finding asymptotic stability conditions we can apply the approach considered in Subsection 3.2.

Choose the first system from the family (15). Instead of Assumption 4.2, we will use a weaker assumption.

Assumption 4.3 The function $(\partial\Pi_1(\mathbf{x})/\partial\mathbf{x})^T\, \mathbf{G}(\mathbf{x})$ is positive definite.

Let

$$\hat{V}_1(\mathbf{x},\mathbf{y}) = \Pi_1(\mathbf{x}) + \frac{1}{2}\mathbf{y}^T\mathbf{y} - \hat{\gamma}_1\|\mathbf{y}\|^{\lambda-1}\mathbf{x}^T\mathbf{y}.$$

Here $\hat{\gamma}_1 > 0$, and the value of the parameter λ is defined by the formula (16). Then

$$\dot{\hat{V}}_1\Big|_{(s)} = -\left(\frac{\partial\Pi_1(\mathbf{x})}{\partial\mathbf{x}}\right)^T \mathbf{G}(\mathbf{x}) - \hat{\gamma}_1\|\mathbf{y}\|^{\lambda+1} + \hat{\gamma}_1\|\mathbf{y}\|^{\lambda-1}\mathbf{y}^T\mathbf{G}(\mathbf{x})$$

$$+\hat{\gamma}_1\mathbf{x}^T\frac{\partial(\|\mathbf{y}\|^{\lambda-1}\mathbf{y})}{\partial\mathbf{y}}\frac{\partial\Pi_s(\mathbf{x})}{\partial\mathbf{x}} + \left(\frac{\partial\Pi_1(\mathbf{x})}{\partial\mathbf{x}}\right)^T\mathbf{y} - \left(\frac{\partial\Pi_s(\mathbf{x})}{\partial\mathbf{x}}\right)^T\mathbf{y}.$$

If $\mu \ge 2\nu + 1$, Assumptions 3.3 and 4.3 are fulfilled, and the value of $\hat{\gamma}_1$ is sufficiently small, then there exists a number $\hat{H} > 0$ such that the estimates

$$\hat{b}_1 r(\mathbf{x},\mathbf{y}) \le \hat{V}_1(\mathbf{x},\mathbf{y}) \le \hat{b}_2 r(\mathbf{x},\mathbf{y}), \qquad \dot{\hat{V}}_1\Big|_{(s)} \le \hat{\alpha}_s\hat{V}_1^{1+\hat{\xi}}(\mathbf{x},\mathbf{y}), \quad s = 1,\ldots,N, \tag{20}$$

hold for $r(\mathbf{x},\mathbf{y}) < \hat{H}$. Here $\hat{b}_1$, $\hat{b}_2$, $\hat{\alpha}_1,\dots,\hat{\alpha}_N$ are constants with $\hat{b}_1 > 0$, $\hat{b}_2 > 0$, $\hat{\alpha}_1 < 0$, and $\hat{\xi} = (\lambda - 1)/2$.

For given switching law $\sigma(t)$, define the auxiliary piecewise constant function $\hat{\eta}(t)$ by the formula $\hat{\eta}(t) = -\hat{\alpha}_{\sigma(t)}$ for $t \geq 0$.

Theorem 4.2 *Let $\mu \geq 2\nu + 1$, Assumptions 3.3, 4.1 and 4.3 be fulfilled, and for family* (15) *the Lyapunov function $\hat{V}_1(\mathbf{x},\mathbf{y})$ be constructed satisfying the estimates* (20). *If*

$$\int_0^t \eta(\tau)\,d\tau \to +\infty \quad as \quad t \to +\infty,$$

then the equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ of system (2) *is asymptotically stable. In the case when*

$$\int_{t_0}^{t_0+t} \eta(\tau)\,d\tau \to +\infty \quad as \quad t \to +\infty$$

uniformly with respect to $t_0 \geq 0$, the equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ of system (2) *is uniformly asymptotically stable.*

The proof of the theorem is similar to that of Theorem 3.2.

Remark 4.4 As well as Theorem 3.2, Theorem 4.2 remains valid in the case when the zero solution is not asymptotically stable either for a part of systems from the family (15) numbered $2,\dots,N$, or for all of these systems.

5 A Numerical Example

Let family (1) consist of two systems

$$\begin{cases} \ddot{x}_1 + \sqrt{x_1^2 + x_2^2}\,(\dot{x}_1 + a_s\dot{x}_2) + c_s x_1^3 = 0, \\ \ddot{x}_2 + \sqrt{x_1^2 + x_2^2}\,(-\dot{x}_1 + b_s\dot{x}_2) + d_s x_2^3 = 0, \qquad s = 1,2, \end{cases} \tag{21}$$

where a_s, b_s, c_s, d_s are constant coefficients. Thus, we have $n = 2$, $\mathbf{x} = (x_1, x_2)^T$, $N = 2$, $\nu = 1$, $\mu = 3$, $\Pi_s(\mathbf{x}) = (c_s x_1^4 + d_s x_2^4)/4$, and

$$\mathbf{D}_s(\mathbf{x}) = \sqrt{x_1^2 + x_2^2}\,\begin{pmatrix} 1 & a_s \\ -1 & b_s \end{pmatrix}, \qquad s = 1,2.$$

The results of a numerical simulation are presented in Figs. 1–4, where for solutions of switched systems generated by the family (21) and four types of switching law the dependence of the coordinate x_1 on time is shown. The initial conditions of solutions are determined by the formulae

$$t_0 = 0, \quad x_1(0) = -0.03, \quad x_2(0) = 0.05, \quad \dot{x}_1(0) = 0.02, \quad \dot{x}_2(0) = 0.04.$$

First, the following values of coefficients were chosen: $a_1 = 0.9$, $b_1 = 0.3$, $c_1 = 1$, $d_1 = 10$, $a_2 = 0.8$, $b_2 = 0.1$, $c_2 = 10$, $d_2 = 1$. In this case Assumptions 3.1 and 3.2 are fulfilled, and both systems admit the asymptotically stable equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$.

Fig. 1 corresponds to a switching law satisfying the conditions of Theorem 3.1. Here $\tau_{2i-1} = 5$, $\tau_{2i} = 5^i$, and the first system from the family (21) is active on the intervals

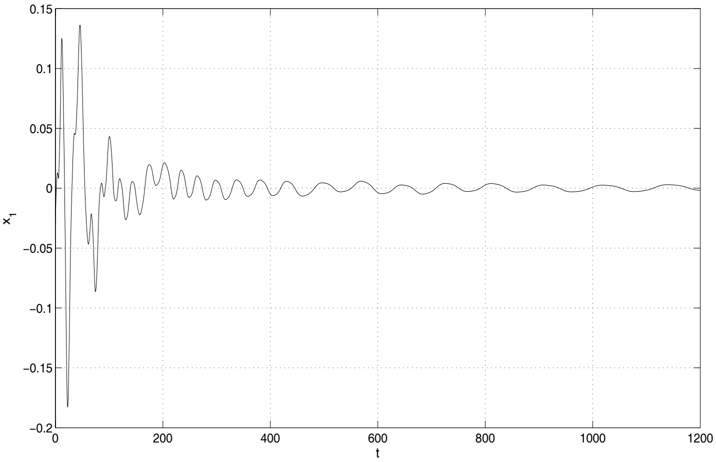

Figure 1: Switching between two asymptotically stable systems (asymptotically stable equilibrium position).

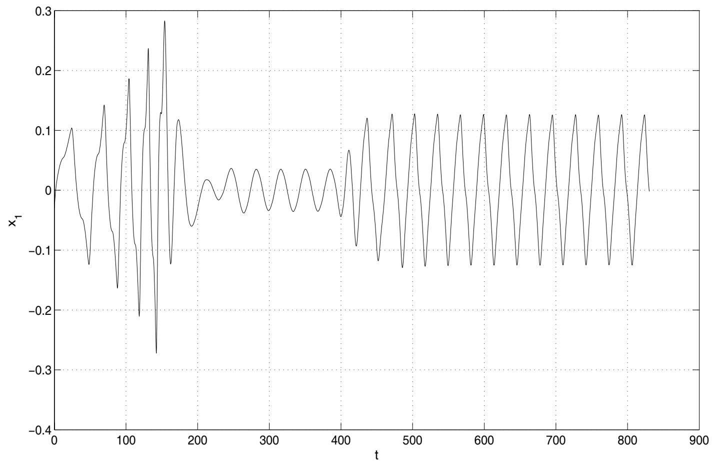

Figure 2: Switching between two asymptotically stable systems (unstable equilibrium position).

$[\theta_{2i-1}, \theta_{2i})$, whereas the second one is active on the intervals $[\theta_{2i-2}, \theta_{2i-1})$, $i = 1, 2, \ldots$. For such switching law the equilibrium position is asymptotically stable.

Fig. 2 demonstrates that there exist switching laws for which the equilibrium position is unstable. Here switching from the first system to the second one occurs when $\dot{x}_2 = 0$ and $\dot{x}_1 \neq 0$, whereas switching from the second system to the first one occurs when $\dot{x}_1 = 0$. Moreover, in order to avoid Zeno type switching signal, the following additional restriction is imposed: $\tau_i \geq 4$, $i = 1, 2, \ldots$.

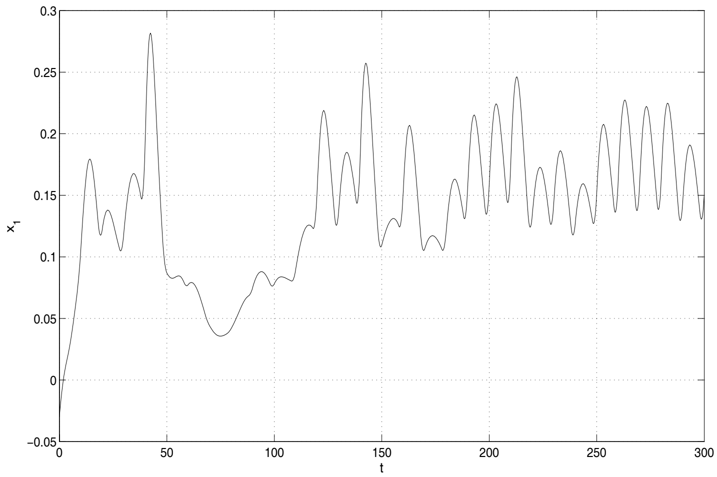

Figure 3: Switching between asymptotically stable and unstable systems (unstable equilibrium position).

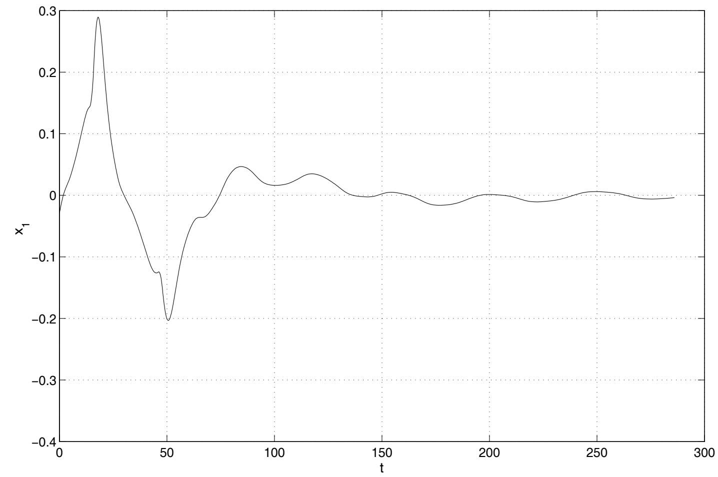

Figure 4: Switching between asymptotically stable and unstable systems (asymptotically stable equilibrium position).

Next, consider the case when $a_1 = 0.9$, $b_1 = 0.3$, $c_1 = 1$, $d_1 = 10$, $a_2 = 0.8$, $b_2 = 0.1$, $c_2 = -10$, $d_2 = -1$. Then the equilibrium position of the first system from the family (21) is asymptotically stable, and the equilibrium position of the second system is unstable. For such values of coefficients Assumptions 3.2 and 3.3 are fulfilled.

Let $\tau_{2i-1} = 2\chi$, $\tau_{2i} = 2$, where χ is a positive parameter, the first system from the family (21) be active on the intervals $[\theta_{2i-2}, \theta_{2i-1})$, and the second one be active on the intervals $[\theta_{2i-1}, \theta_{2i})$, $i = 1, 2, \ldots$. The results of numerical simulation show that if $\chi = 4$, then the equilibrium position of the corresponding switched system is unstable (see Fig. 3), whereas if $\chi = 7$, then the equilibrium position is asymptotically stable (see Fig. 4).

6 Conclusion

In the present paper, certain classes of switched mechanical systems with nonlinear dissipative and potential forces are studied. By the application of the multiple Lyapunov functions approach and the dwell time approach, we found the restrictions on the switching law guaranteeing the asymptotic stability of the trivial equilibrium position.

The obtained results can be used for the design of switched controllers providing the asymptotic stability and the practical stability of equilibrium positions for nonlinear mechanical systems.

The interesting direction for further research is the extension of the obtained results to the case when switched nonlinear dissipative forces depend on velocities and are independent of coordinates. Moreover, the impact of gyroscopis and nonconservative forces on the considered systems may be studied.

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